EE306001 Probability, Fall 2012

Quiz #10, Problems and Solutions

Some probability density function you may need.

(a) Normal with parameters μ, σ^2:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \ \forall\, x \in \mathbb{R}.$$

(b) Exponential with parameter λ:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Prob. 1: The number of years a radio functions is exponentially distributed random variable with parameter $\lambda = \frac{1}{8}$. If Jones buys a used radio, what is the probability that it will be working after an additional 8 years?

Solution: Let X be the life time of radio. According to the problem, X is exponential with parameter $\lambda = \frac{1}{8}$. Assume that the radio have been used for y years when Jones buys it. The probability that it will be working after an additional 8 years is

$$\begin{aligned} P(X > t + 8 | X > t) =& P(X > 8), \qquad \text{since exponentially distributed r.v. is memoryless.} \\ =& \int_8^\infty \frac{1}{8} e^{-(1/8)x} dx \\ =& - e^{-(1/8)x} \,|_8^\infty = e^{-1}. \end{aligned}$$

Prob. 2: The median of a continuous random variable having distribution function F is that value m such that $F(m) = \frac{1}{2}$. Find the median of X if X is

(a) uniformly distributed over (a, b);

(b) normal with parameters μ, σ^2;

(c) exponential with rate λ.

Solution: Let $f(x)$ be the p.d.f. of X.

(a) For uniformly distributed X over (a, b), we have

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise.} \end{cases}$$

Since $F(a) = 0$, $F(b) = 1$ and $F(x)$ is monotonic increasing function, We have $m > a$. So

$$F(m) = \int_{-\infty}^{m} f(x)dx = \int_{a}^{m} \frac{1}{b-a}dx = \frac{m-a}{b-a} = \frac{1}{2}$$
$$\Rightarrow m = a + \frac{b-a}{2}.$$

(b) Since

$$F(\mu) = \int_{-\infty}^{\mu} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx = \int_{\mu}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-\mu)^2/2\sigma^2} dy,$$

where $y = 2\mu - x$, and

$$\int_{-\infty}^{\mu} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx + \int_{\mu}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-\mu)^2/2\sigma^2} dy = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx = 1.$$

We have

$$F(\mu) = \frac{1}{2}.$$

Therefore $m = \mu$.

(c)Since $F(m) = \frac{1}{2} > 0$, we have $m > 0$.

$$F(m) = \int_{-\infty}^{m} f(x)dx = \int_{0}^{m} \lambda e^{-\lambda x} dx = 1 - e^{-\lambda m} = \frac{1}{2}.$$
$$\Rightarrow m = \frac{\ln 2}{\lambda}.$$

Prob. 3: Let X be a continuous random variable having cumulative distribution function F. Define the random variable Y by $Y = F(X)$. Show that Y is uniformly distributed over $(0, 1)$.

Solution: Since $F_X(x)$ is a distribution function, $F_X(x) \in [0, 1]$ for all $x \in \mathbb{R}$ so that $Y = F_X(X)$ takes one of the values in $[0, 1]$. Thus for $y \geq 1$, $F_Y(y) = P(Y \leq y) = 1$ and for $y < 0$, $F_Y(y) = P(Y \leq y) = 0$. Now for $0 \leq y < 1$,

$$F_Y(y) = P(Y \leq y) = P(0 \leq Y \leq y) = P(F_X(X) \in [0, y]) = P(X \in F_X^{-1}([0, y])).$$

Note that

$$F_X^{-1}([0, y]) = \{x \in \mathbb{R} | F_X(x) \leq y\} = (-\infty, z]$$

for some $z \in \mathbb{R}$, since $F_X(x)$ is a monotone continuous function on $\mathbb{R}$. Note that $F_X(z) = y$. Thus

$$F_Y(y) = P(X \in F^{-1}([0, y])) = P(X \leq z) = F_X(z) = y$$

and then Y is uniformly distributed over $(0, 1)$.