EE306001, Probability, Fall 2012

Quiz #1, Problems and Solutions

Prob. 1: A system is comprised of 4 components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector (x_1, x_2, x_3, x_4), where x_i is equal to 1 if component i is working and is 0 if component i is failed.

(a) What are the sample space S and the σ-algebra $\mathcal{F}$? How many outcomes in S? How many events in $\mathcal{F}$?

(b) Suppose the system will work if the components 1 and 2 are both working, or if components 1, 3, and 4 are all working. Let W be the event that the system will work. Specify all the outcomes in W.

(c) Let A be the event that components 3 and 4 are both failed, write out all the outcomes in the event AW.

(d) We assume that all outcomes are equally probable. What is the probability that the system will work? You have to specify the probability function P.

Solution.

(a) The sample space,

$$\begin{aligned} S &= \{(x_1, x_2, x_3, x_4) \mid x_i = 0, 1 \text{ for } i = 1, 2, 3, 4\} \\ &= \{(0, 0, 0, 0), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), \dots, (1, 1, 1, 1)\} \end{aligned}$$

and there are $|S| = 2^4 = 16$ outcomes in the sample space. The σ-algebra $\mathcal{F}$ is the power set of S, i.e.,

$$\begin{aligned} \mathcal{F} &= 2^S \\ &= \text{ the collection of all subset of } S \\ &= \{\emptyset, \{(0, 0, 0, 0)\}, \{(1, 0, 0, 0)\}, \dots, \{(0, 0, 0, 0), (1, 0, 0, 0)\}, \dots, S\} \end{aligned}$$

and there are $|\mathcal{F}| = 2^{|S|} = 2^{16}$ events in $\mathcal{F}$.

(b) Let E_1 be the event that components 1 and 2 are working; and E_2 be the event that components 1, 3, and 4 are working, that is,

$$\begin{aligned} E_1 &= \{(1, 1, x_3, x_4) \mid x_i = 0, 1; i = 3, 4\} \\ E_2 &= \{(1, x_2, 1, 1) \mid x_2 = 0, 1\}. \end{aligned}$$

Then,

$$\begin{aligned} W &= E_1 \cup E_2 \\ &= \{(1, 1, 0, 0), (1, 1, 0, 1), (1, 1, 1, 0), (1, 1, 1, 1)\} \cup \{(1, 0, 1, 1), (1, 1, 1, 1)\} \\ &= \{(1, 1, 0, 0), (1, 1, 0, 1), (1, 1, 1, 0), (1, 1, 1, 1), (1, 0, 1, 1)\}. \end{aligned}$$

(c) Since the event

$$\begin{aligned} A &= \{(x_1, x_2, 0, 0) \mid x_i = 0, 1, \text{ for } i = 1, 2\} \\ &= \{(0,0,0,0), (0,1,0,0), (1,0,0,0), (1,1,0,0)\}, \end{aligned}$$

the event

$$\begin{aligned} AW =& \{(0,0,0,0), (0,1,0,0), (1,0,0,0), (1,1,0,0)\} \\ & \cap \{(1,1,0,0), (1,1,0,1), (1,1,1,0), (1,1,1,1), (1,0,1,1)\} \\ =& \{(1,1,0,0)\}. \end{aligned}$$

(d) Since all outcomes are equally probable, the probability function $P : \mathcal{F} \to \mathbb{R}$ can be

$$P(\{\omega\}) = \frac{1}{|S|}, \qquad \text{for } \omega \in S.$$

Therefore, the probability that the system will work is

$$P(W) = \sum_{\omega \in W} P(\omega) = \frac{|W|}{|S|} = \frac{5}{16}.$$

Prob. 2: Prove that

$$\left(\cup_{i=1}^{\infty} E_i\right) F = \cup_{i=1}^{\infty} E_i F.$$

Solution. If $x \in \left(\cup_{i=1}^{\infty} E_i\right) F$, then $x \in \cup_{i=1}^{\infty} E_i$ and $x \in F$, i.e., $x \in E_j$ for some j, $1 \le j < \infty$ and $x \in F$. Therefore $x \in E_j \cap F = E_j F$, and then $x \in \cup_{i=1}^{\infty} E_i F$. We have shown that $(\cup_{i=1}^{\infty} E_i) F \subset \cup_{i=1}^{\infty} E_i F$.

On the other hand, if $x \in \cup_{i=1}^{\infty} E_i F$, then $x \in E_k F = E_k \cap F$ for some k, $1 \le k < \infty$. That is, $x \in E_k$ and $x \in F$. Therefore $x \in \cup_{i=1}^{\infty} E_k$ and $x \in F$, i.e., $x \in (\cup_{i=1}^{\infty} E_k) \cap F = (\cup_{i=1}^{\infty} E_k) F$. We have shown that $\cup_{i=1}^{\infty} E_i F \subset (\cup_{i=1}^{\infty} E_i) F$.

Prob. 3: Urn A contains 2 black and 2 white balls, whereas urn B contains 1 black and 2 red balls. If a ball is randomly selected from each urn, what is the probability that two balls have different color? (You have to specify the sample space S, the σ-algebra and the probability function P.)

Solution. Model Urn A as a set contains 4 elements,

$$A = \{b1, b2, w1, w2\},$$

where $b1$, $b2$ are black and $w1$, $w2$ are white. Similarly, model Urn B as a set with 3 elements,

$$B = \{b, r1, r2\},$$

where b is black and $r1$, $r2$ are red.

The sample space

$$\begin{aligned} S =& A \times B \\ =& \{(b1, b), (b2, b), (w1, b), (w2, b), \\ & (b1, r1), (b2, r1), (w1, r1), (w2, r1), \\ & (b1, r2), (b2, r2), (w1, r2), (w2, r2)\}. \end{aligned}$$

The $\sigma-$algebra $\mathcal{F}$ is the power set of S, i.e.,

$$\begin{aligned}\mathcal{F} =& 2^S \\ =& \{\emptyset, \{(b1,b)\}, \{(b2,b)\}, ..., \{(b1,b),(b2,b),(w1,r2)\}, ..., S\}.\end{aligned}$$

Since all outcomes are equally probable, the probability function $P : \mathcal{F} \rightarrow \mathbb{R}$ can be

$$P(\{\omega\}) = \frac{1}{|S|} = \frac{1}{12}, \qquad \text{for } \omega \in S.$$

The event E considered in this problem is

$$\begin{aligned}&\{(a,b) \in A \times B | \text{a and b have different color}\} \\ =& \{(w1,b),(w2,b),(b1,r1),(b2,r1),(w1,r1), \\ &(w2,r1),(b1,r2),(b2,r2),(w1,r2),(w2,r2)\}.\end{aligned}$$

The probability of this event is

$$\begin{aligned}P(E) =& \sum_{\omega \in E} P(\{\omega\}) \\ =& \frac{|E|}{12} \\ =& \frac{10}{12} = \frac{5}{6}.\end{aligned}$$

Note: For two sets A and B, the set $A \times B \triangleq \{(a,b) | \forall a \in A, b \in B\}$.