EE306001, Probability, Fall 2012

Hw #2, Solutions

P. 52, Prob. 25: Let the set A contain all possible outcomes of rolling a pair of dice,

$$A = \{(d_1, d_2) \in (\mathbb{Z}, \mathbb{Z}) | 1 \leq d_1, d_2 \leq 6\}.$$

Define a function $f : A \to \mathbb{R}$ such that

$$f((d_1, d_2)) = d_1 + d_2,\ \forall (d_1, d_2) = A$$

Consider a probability space $(S, \mathcal{F}, P)$. The sample space S contains all outcomes of rolling a pair of dice infinitily times. That is,

$$S = A \times A \times A \times \cdots = A^{\infty}.$$

Define $S(a_1, a_2, ..., a_n)$ as the event that the results of first n times of rolling are $a_1, a_2, ..., a_n$ respectively, $a_i \in A$, $\forall 1 \leq i \leq n$. That is

$$S(a_1, a_2, ..., a_n) = \{(s_1, s_2, ..., s_n, ...) \in S | s_1 = a_1, s_2 = a_2, ..., s_n = a_n\}.$$

$\mathcal{F}$ is the σ-field generated by the events $S(a_1, a_2, ..., a_n)$, $n \geq 1$, $a_1, ..., a_n \in A$.
The probability function P is assigned as

$$P(S(a_1, a_2, ..., a_n)) = (\frac{1}{|A|})^n = \frac{1}{36^n}.$$

Note: You can try to verify whether this probability space satisfies all three axioms of probability space.

Define the event we want as

$$E = \{5 \text{ occurs first}\},$$

and the event

$$E_n = \{\text{a 5 occurs on the } n\text{th roll and no 5 or 7 occurs on the first } n-1 \text{ rolls.}\}$$

Since

$$E_n \subseteq E,\ \forall n \geq 1.$$

Thus we have

$$\bigcup_{n=1}^{\infty} E_n \subseteq E.$$

On the other hand, for all $s \in S$ and $s \in E$, we can find $k \geq 1$ such that $s \in E_k$. Thus we also have

$$E \subseteq \bigcup_{n=1}^{\infty} E_n.$$

So we have

$$E = \bigcup_{n=1}^{\infty} E_n.$$

Besides,

$$E_i \cap E_j = \emptyset,\ \forall i \neq j.$$

Hence the probability

$$P(E) = P(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} P(E_n).$$

Now we compute the probability

$$\begin{aligned}
P(E_n) =& P(\bigcup_{f(b_1),...,f(b_{n-1})\neq 7 \text{ or } 5, \text{ and } f_{b_n}=5} S(b_1, ..., b_n)) \\
=& \sum_{f(b_1),...,f(b_{n-1})\neq 7 \text{ or } 5, \text{ and } f(b_n)=5} P(S(b_1, ..., b_n)) \\
=& (36-6-4)^{n-1} \times 4 \times (\frac{1}{36})^n \\
=& 26^{n-1} \times 4 \times (\frac{1}{36})^n.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
P(E) =& \sum_{n=1}^{\infty} P(E_n) \\
=& \sum_{n=1}^{\infty} 26^{n-1} \times 4 \times (\frac{1}{36})^n \\
=& \frac{4}{36} \sum_{n=1}^{\infty} (\frac{26}{36})^{n-1} \\
=& \frac{4}{36} \times \frac{1}{1-\frac{26}{36}} \\
=& \frac{2}{5}.
\end{aligned}$$

P. 52, Prob. 30: Let school A has 9 players and school B has 8 players.Model these two school as two sets

$$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\},$$

and

$$B = \{1, 2, 3, 4, 5, 6, 7, 8\}.$$

W.o.l.g., we can let Rebecca be the element 1 in A and Elise be the element 1 in B. The teams chose from A and B is model as

$$T_A = \{(a_1, a_2, a_3, a_4) | a_i \in A,\ a_i \neq a_j,\ \forall 1 \leq i, j \leq 4\},$$

and

$$T_B = \{(b_1, b_2, b_3, b_4) | b_i \in B,\ b_i \neq b_j,\ \forall 1 \leq i, j \leq 4\}.$$

The size of T_A and T_B are

$$|T_A| = 9 \times 8 \times 7 \times 6 = 3024,$$

and

$$|T_B| = 8 \times 7 \times 6 \times 5 = 1680.$$

Define the sample space as the all possible match of T_A and T_B, that is

$$S = T_A \times T_B.$$

Then the size of S will be

$$|S| = |T_A| \times |T_B| = 3024 \times 1680 = 5080320.$$

Define the σ-field as

$$\mathcal{F} = 2^S,$$

and the probability function $P : \mathcal{F} \to \mathbb{R}$ is defined as

$$P(E) = \frac{|E|}{|S|},\ \forall\ E \in \mathcal{F}.$$

(a) Define

$$E = \{\text{Rebecca and Elise will be paired}\}.$$

The size of E is

$$|E| = \binom{4}{1} \times (8 \times 7 \times 6) \times (7 \times 6 \times 5) = 282240.$$

Then

$$P(E) = \frac{|E|}{|S|} = \frac{282240}{5080320} = \frac{1}{18}.$$

(b) Define

$$E_1 = \{\text{Rebecca and Elise will be choosen to represent their schools}\},$$

and

$$E_2 = \{\text{Rebecca and Elise will be paired}\}.$$

So the event

$$\begin{aligned} E =& \{\text{Rebecca and Elise will be chosen to represent their schools} \\ & \quad \text{but will not play each other}\} \\ =& E_1 - E_2. \end{aligned}$$

The size of E is

$$\begin{aligned} |E| =& |E_1| - |E_2| \\ =& \binom{8}{3} \times 4! \times \binom{7}{3} \times 4! - 282240 \\ =& 1128960 - 282240 \\ =& 846720. \end{aligned}$$

Then

$$P(E) = \frac{|E|}{|S|} = \frac{846720}{5080320} = \frac{1}{6}.$$

(c) Define

$$E_1 = \{\text{Rebecca will be chosen to represent her school}\},$$

and

$$E_2 = \{\text{Elise will be chosen to represent her school}\}.$$

The event

$$\begin{aligned} E =& \{\text{either Rebecca or Elise will be chosen to represent her school}\} \\ =& E_1 \cup E_2. \end{aligned}$$

The size of E is

$$\begin{aligned} |E| =& |E_1 + E_2| \\ =& |E_1| + |E_2| - |E_1 \cap E_2| \\ =& \binom{8}{3} \times 4! \times \binom{8}{4} \times 4! + \binom{9}{4} \times 4! \times \binom{7}{3} \times 4! \\ & - \binom{8}{3} \times 4! \times \binom{7}{3} \times 4! \\ =& 2257920 + 2540160 - 1128960 \\ =& 3669120. \end{aligned}$$

Then

$$P(E) = \frac{|E|}{|S|} = \frac{3669120}{5080320} = \frac{13}{18}.$$

P. 52, Prob. 50:

Let $S \triangleq \{\text{All possible hands in a bridge game}\}$, then

$$|S| = \binom{52}{13, 13, 13, 13} = \frac{52!}{13!\,13!\,13!\,13!}.$$

Now we define the collection of all events $\mathcal{F} = 2^S$, and the probability function $\mathcal{P} : \mathcal{F} \to \mathbb{R}$ by

$$\mathcal{P}(E) = \frac{|E|}{|S|},\ \forall\ E \in \mathcal{F}.$$

Let $E_\spadesuit = \{\text{I have 5 spades and my partner has the other 8 spades}\}$, then

$$|E_\spadesuit| = \binom{13}{5}\binom{8}{8}\binom{39}{8, 5, 13, 13},$$

where $\binom{13}{5}$ means I get 5 of all the 13 spade cards randomly, $\binom{8}{8}$ means my partner gets 8 of the remain 8 spade cards, and $\binom{39}{8, 5, 13, 13}$ is the partition of the last 39 cards , which is without spades. Thus

$$\mathcal{P}(E_\spadesuit) = \frac{|E|}{|S|} = \frac{\binom{13}{5}\binom{8}{8}\binom{39}{8, 5, 13, 13}}{\binom{52}{13, 13, 13, 13}} \approx 2.6 \times 10^{-6}$$

You can consult Example 5f, 5g and 5h in P.36-37 to solve this problem.

P. 55, Prob. 19:

Let $S \triangleq \{$The arrangement of n red and m blue balls$\}$, then

$$|S| = \binom{n+m}{n, m} = \frac{(n+m)!}{n!\, m!}$$

Now we define the collection of all events $\mathcal{F} = 2^S$, and the probability function $\mathcal{P} : \mathcal{F} \to \mathbb{R}$ by

$$\mathcal{P}(E) = \frac{|E|}{|S|}, \ \forall\, E \in \mathcal{F}.$$

Let $E_k = \{r$ red balls are withdrawn after k balls are withdrawn$\}$, Note that the event E_k means that $(r-1)$ red balls are withdrawn when $(k-1)$ balls are withdrawn, and the k-th withdrawl is a red ball. Hence

$$|E_k| = \binom{k-1}{r-1}\binom{n+m-k}{n-r}.$$

Therefore

$$\mathcal{P}(E_k) = \frac{|E_k|}{|S|}.$$