EE306001, Probability, Fall 2012

Hw #1, Solutions

P. 51, Prob. 15: Note that each card can be represented by its suit-denomination pair (s, d), where $s \in S \triangleq \{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$ and $d \in D \triangleq \{1, 2, \ldots, 13\}$. Therefor the sample space

$$\Omega = \{\{(s_1, d_1), (s_2, d_2), (s_3, d_3), (s_4, d_4), (s_5, d_5)\} \mid s_i \in S, d_i \in D, i = 1, 2, \ldots, 5\},$$

in other words, each sample point $\omega \in \Omega$ is a set consisting of (s_1, d_1), (s_2, d_2), (s_3, d_3), (s_4, d_4), (s_5, d_5), where $s_i \in S, d_i \in D, i = 1, 2, \ldots, 5$. The sample space Ω is finite, therefore the σ-algebra $\mathcal{F}$ can be defined as the power set of Ω, i.e.,

$$\mathcal{F} = 2^{\Omega}.$$

Since all outcomes are equally likely, the probability function can be

$$P(E) = \sum_{\omega \in E} P(\{\omega\}) = \sum_{\omega \in E} \frac{1}{|\Omega|} = \frac{|E|}{|\Omega|}, \text{ for } E \in \mathcal{F}.$$

(a) Let E_1 be the event that a hand is a flush, then

$$E_1 = \{\{(s_1, d_1), (s_2, d_2), (s_3, d_3), (s_4, d_4), (s_5, d_5)\} \mid s_1 = s_2 = \cdots = s_5, \\ s_i \in S, d_i \in D, i = 1, 2, \ldots, 5\}.$$

and $|E_1| = 4 \times \binom{13}{5}$, thus

$$P(E_1) = \frac{|E_1|}{|\Omega|} = \frac{4 \times \binom{13}{5}}{\binom{52}{5}}.$$

(b) Let E_2 be the event that one is dealt a pair, then

$$E_2 = \{\{(s_1, d_1), (s_2, d_2), (s_3, d_3), (s_4, d_4), (s_5, d_5)\} \mid s_i \in S, d_i \in D, i = 1, 2, \ldots, 5 \\ \text{exactly two of } d_1, d_2, \ldots, d_5 \text{ are equal}\}.$$

Let F_i be the event that the denomination of the pair is equal to i, $i = 1, 2, \ldots, 13$, then $E_2 = \uplus_{i=1}^{13} F_i$, where $\uplus$ denotes disjoint union. The number of outcomes in F_1 is equal to

$$\binom{4}{2} \cdot \frac{12\binom{4}{1}11\binom{4}{1}10\binom{4}{1}}{3!},$$

where $\binom{4}{2}$ is the number of combinations of 1's from four suits and $\frac{12\binom{4}{1}11\binom{4}{1}13\binom{4}{1}}{3!}$ the number of combinations of the other distinct denominations of the hand. Therefore

$$P(E_2) = \frac{|E_2|}{|\Omega|} = \frac{13 \cdot |F_1|}{|\Omega|} = \frac{13}{\binom{52}{5}} \binom{4}{2} \cdot \frac{12\binom{4}{1}11\binom{4}{1}10\binom{4}{1}}{3!}.$$

(c) Let E_3 be the event that one is dealt two pairs, then

$$E_3 = \{\{(s_1, d_1), (s_2, d_2), (s_3, d_3), (s_4, d_4), (s_5, d_5)\} \mid s_i \in S, d_i \in D, i = 1, 2, \ldots, 5 \\ \text{exactly two of } d_1, d_2, \ldots, d_5 \text{ are equal to } d' \\ \text{and exactly two of } d_1, d_2, \ldots, d_5 \text{ are equal to } d'', d' \neq d''\}.$$

Applying the similar reasoning in part (b), we know that the number of outcomes of the event E_3 is

$$|E_3| = \frac{13\binom{4}{2}12\binom{4}{2}}{2!}(52 - 8),$$

thus,

$$P(E_3) = \frac{|E_3|}{|\Omega|} = \frac{13\binom{4}{2}12\binom{4}{2}}{2!\binom{52}{5}}(52 - 8).$$

(d) Let E_4 be the event that one is dealt three of a kind, then

$$E_4 = \{\{(s_1, d_1), (s_2, d_2), (s_3, d_3), (s_4, d_4), (s_5, d_5)\} \mid s_i \in S, d_i \in D, i = 1, 2, \ldots, 5 \\ \text{exactly three of } d_1, d_2, \ldots, d_5 \text{ are equal}\}.$$

Applying the similar reasoning in part (b), we know that the number of outcomes of the event E_4 is

$$|E_4| = \frac{13\binom{4}{3}12\binom{4}{1}11\binom{4}{1}}{2!},$$

thus,

$$P(E_4) = \frac{|E_4|}{|\Omega|} = \frac{13\binom{4}{3}12\binom{4}{1}11\binom{4}{1}}{2!\binom{52}{5}}.$$

(e) Let E_5 be the event that one is dealt three of a kind, then

$$E_5 = \{\{(s_1, d_1), (s_2, d_2), (s_3, d_3), (s_4, d_4), (s_5, d_5)\} \mid s_i \in S, d_i \in D, i = 1, 2, \ldots, 5 \\ \text{exactly four of } d_1, d_2, \ldots, d_5 \text{ are equal}\}.$$

Applying the similar reasoning in part (b), we know that the number of outcomes of the event E_5 is

$$|E_5| = 13 \cdot (52 - 4),$$

thus,

$$P(E_5) = \frac{|E_5|}{|\Omega|} = \frac{13 \cdot (52 - 4)}{\binom{52}{5}}.$$

P. 52, Prob. 27: In order to define a probability space in which all outcomes are equally likely, we consider the sample space

$$S = \{(c_1, c_2, \ldots, c_{10}) \mid c_i = b, r\},$$

where b and r denote black and red, respectively; the σ-algebra $\mathcal{F} = 2^S$; and the probability function $P(\{\omega\}) = 1/|S|$. It is clear that $|S| = \binom{10}{7}$, by considering the number of ways that we put 7 black balls into 10 positions.

Let W be the event that A selects the first red ball and E_i be the event that the outcome of the ith selection is the first red ball, $i = 1, 3, 5, 7$; we can see that

$$\begin{aligned} E_1 &= \{(r, c_2, \ldots, c_{10}) \mid c_i = b, r, i = 2, 3, \ldots, 10\} \\ E_3 &= \{(b, b, r, c_4, \ldots, c_{10}) \mid c_i = b, r, i = 4, 5, \ldots, 10\} \\ E_5 &= \{(b, b, b, b, r, c_6, \ldots, c_{10}) \mid c_i = b, r, i = 6, 7, \ldots, 10\} \\ E_7 &= \{(b, b, b, b, b, b, r, c_8, \ldots, c_{10}) \mid c_i = b, r, i = 8, 9, 10\} \end{aligned}$$

Furthermore, we have that

$$W = E_1 \uplus E_3 \uplus E_5 \uplus E_7.$$

Now,

$$|E_1| = \binom{9}{7}\binom{2}{2}, \quad |E_3| = \binom{7}{5}\binom{2}{2}, \quad |E_5| = \binom{5}{3}\binom{2}{2}, \quad |E_7| = \binom{3}{2}.$$

Therefore,

$$\begin{aligned} P(W) &= P(E_1 \uplus E_3 \uplus E_5 \uplus E_7) \\ &= \frac{1}{|S|}(|E_1| + |E_3| + |E_5| + |E_7|) \\ &= \frac{1}{\binom{10}{7}}\left(\binom{9}{7} + \binom{7}{5} + \binom{5}{3} + \binom{3}{2}\right) \\ &= \frac{7}{12}. \end{aligned}$$

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Hw #10, Solutions

P. 53, Prob. 37: Let

$$Q = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

be the set that contains all 10 problems, and

$$A = \{x \in 2^Q \mid |x| = 5\}$$

be the set of all possible combination of problems in exam, and

$$B = \{x \in 2^Q \mid |x| = 7\}$$

be the set of all possible combination of problems that a student has figured out. The sample space

$$S = A \times B.$$

The σ-algebra

$$\mathcal{F} = 2^S.$$

Assign probability function P as

$$P(\{\omega\}) \equiv \frac{1}{|S|} = \frac{1}{|A| \times |B|} = \frac{1}{\binom{10}{5}\binom{10}{7}} = \frac{1}{30240}, \ \forall \omega \in S.$$

(a) Let

$$E_1 = \{\text{student will answer all 5 problems correctly}\}$$

$$\begin{aligned} P(E_1) =& \sum_{\omega \in E} P(\{\omega\}) = \frac{|E_1|}{30240} \\ =& \frac{\binom{10}{5}\binom{5}{2}}{30240} = \frac{2520}{30240} = \frac{1}{12}. \end{aligned}$$

(b) Let

$$E_2 = \{\text{student will answer 4 of the problems correctly}\},$$

and

$$\begin{aligned} E_3 =& \{\text{student will answer at least 4 of the problems correctly}\} \\ =& E_1 \cup E_2. \end{aligned}$$

Then,

$$\begin{aligned} P(E_3) =& P(E_1 \cup E_2) = P(E_1) + P(E_2) \\ =& \sum_{\omega \in E_1} P(\{\omega\}) + \sum_{\omega \in E_2} P(\{\omega\}) = \frac{1}{12} + \frac{|E_2|}{30240} \\ =& \frac{1}{12} + \frac{\binom{10}{5}\binom{5}{4}\binom{5}{3}}{30240} \\ =& \frac{1}{12} + \frac{12600}{30240} = \frac{1}{2}. \end{aligned}$$

P. 54, Exer. 20: Assume there exit a probability sapce $(S, \mathcal{F}, P)$, s.t. $S = \{s_1, s_2, \ldots, s_n, \ldots\}$, $\mathcal{F} = 2^S$, and

$$P(\{\omega\}) = a, \ \forall \omega \in S, \ a > 0.$$

Then,

$$\begin{aligned} P(S) =& P(\cup_{i=1}^{\infty}\{s_i\}) \\ =& \sum_{i=1}^{\infty} P(\{s_i\}) && \text{by axiom 3 of probability} \\ =& \sum_{i=1}^{\infty} a = \infty, \end{aligned}$$

which is contradiction to axiom 2 of probability.

Therefore, not all points in an experiment whose sample space consists of a countably infinite number of points can be equally likely.

To show that all points have a positive probability of occurring, consider a probability space $(S, \mathcal{F}, P_1)$, where S and $\mathcal{F}$ are the same with above. Assign

$$P_1(\{s_i\}) = \frac{1}{2^i}, \ \forall i \in N.$$

And

$$P_1(E) = \sum_{s\in E} P_1(\{s\}),\ \forall E \in \mathcal{F}.$$

(i) Since

$$\begin{aligned} P_1(S) =& \sum_{s\in S} P_1(\{s\}) \\ =& \sum_{i=1}^{\infty} P_1\{s_i\} = \sum_{i=1}^{\infty} \frac{1}{2^i} \\ =& \frac{1/2}{1-1/2} = 1, \end{aligned}$$

axiom 2 is satisfied.

(ii) Since $P_1(E) \geq 0$, $\forall E \in \mathcal{F}$, by definition, and

$$P_1(E) = \sum_{s\in E} P_1(\{s\}) = \sum_{s\in E} P_1\{s\} \leq \sum_{s\in S} P_1\{s\} = 1,$$

axiom 1 is satisfied.

(iii) If $E_1, E_2, \ldots$ are mutually exclusive events in $\mathcal{F}$, then

$$\begin{aligned} P_1(\cup_{i=1}^{\infty} E_i) =& \sum_{s\in \cup_{i=1}^{\infty} E_i} P_1(\{s_i\}) \\ =& \sum_{s\in E_1} P_1(\{s\}) + \sum_{s\in E_2} P_1(\{s\}) + \cdots + \sum_{s\in E_n} P_1(\{s\}) + \cdots \\ =& P_1(E_1) + P_1(E_2) + \cdots + P_1(E_n) + \cdots \\ =& \sum_{i=1}^{\infty} P_1(E_i). \end{aligned}$$

Therefore axiom 3 is satisfied.

By (i), (ii) and (iii), we know that the probability space $(S, \mathcal{F}, P_1)$ is well defined. So there exist a probability space that all points have a positive probability of occurring.