EE565000 Stochastic Process
Homework #4   2008
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1. Since $\mathcal{E}$ is well-defined,

$$|\,\mathcal{E}(X)| = |\,\mathcal{E}(X^+) - \mathcal{E}(X^-)| \le \mathcal{E}(X^+) + \mathcal{E}(X^-) = \mathcal{E}(|X|).$$

2. X is an r.v. with finite expectation.

   (a) We assume that $\mathcal{E}(|X|) > 0$.

   (i) Since $1_{(X\in B)}|X|$ is a non-negative r.v., $\mathcal{E}(1_{(X\in B)}) \ge 0$ and by Corollary 1.4.10, $\mathcal{E}(1_{(X\in B)}) < +\infty$, we have

$$0 \le |\mu|(B) = \frac{\mathcal{E}(1_{(X\in B)}|X|)}{\mathcal{E}(|X|)} < +\infty.$$

   (ii) Let $B_1, B_2 \ldots$ be disjoint Borel sets, by Theorem 1.4.9,

$$|\mu|(\cup_{n=1}^{\infty} B_n) = \frac{\mathcal{E}(1_{(X\in \cup_{n=1}^{\infty} B_n)} \cdot |X|)}{\mathcal{E}(|X|)} = \frac{\sum_{n=1}^{\infty} \mathcal{E}(1_{(X\in B_n)} \cdot |X|)}{\mathcal{E}(|X|)} = \sum_{n=1}^{\infty} |\mu|(B_n).$$

   (iii)

$$|\mu|(R) = \frac{\mathcal{E}(1_{(X\in R)} \cdot |X|)}{\mathcal{E}(|X|)} = \frac{\mathcal{E}(|X|)}{\mathcal{E}(|X|)} = 1.$$

   Hence, $|\mu|$ is a probability measure on $(R, \mathcal{B})$.

   (b) We first assume that $\mathcal{E}(|X|) > 0$. From (a), $|\mu|$ is a probability measure, by the monotone property,

$$B_n \downarrow \phi \;\; \Rightarrow \; |\mu|(B_n) = \frac{\mathcal{E}(1_{(X\in B_n)} \cdot |X|)}{\mathcal{E}(|X|)} \to |\mu|(\phi) = 0 \text{ as } n \to \infty,$$

   i.e.

$$\lim_{n\to\infty} \mathcal{E}(1_{(X\in B_n)} \cdot |X|) = 0.$$

   By Exercise 1 in above,

$$0 \le \lim_{n\to\infty} |\mathcal{E}(1_{(X\in B_n)} \cdot X)| \le \lim_{n\to\infty} \mathcal{E}(|1_{(X\in B_n)} \cdot X|) = \lim_{n\to\infty} \mathcal{E}(1_{(X\in B_n)} \cdot |X|) = 0,$$

   which implies

$$\lim_{n\to\infty} \mathcal{E}(1_{(X\in B_n)} \cdot X) = 0.$$

   If $\mathcal{E}(|X|) = 0$, by Exercise 1 in above,

$$0 \le |\mathcal{E}(1_{(X\in B_n)} \cdot X)| \le \mathcal{E}(1_{(X\in B_n)} \cdot |X|) \le \mathcal{E}(|X|) = 0$$

   i.e.

$$\mathcal{E}(1_{(X\in B_n)} \cdot X) = 0, \;\; \forall\, n \;\Rightarrow \lim_{n\to\infty} \mathcal{E}(1_{(X\in B_n)} \cdot X) = 0.$$

(c) Let $\{ t_n \}$ be a sequence of real numbers s.t. $t_1 \le t_2 \le t_3, \dots$ and $t_n \to \infty$ as $n \to \infty$. Since $B_n = (t_n, \infty) \downarrow \phi$, by (b), we have $\lim_{n\to\infty} \mathcal{E}(1_{(X\in B_n)} \cdot X) = 0$. Moreover,

$$\mathcal{E}(1_{(X\in B_n)} \cdot X) = \mathcal{E}(1_{(X>t_n)} \cdot X) \ge \mathcal{E}(1_{(X>t_n)} \cdot t_n)$$

$$\Rightarrow\ 0 \le \liminf_n \mathcal{E}(1_{(X>t_n)} \cdot t_n) \le \limsup_n \mathcal{E}(1_{(X>t_n)} \cdot t_n) \le \limsup_n \mathcal{E}(1_{(X\in B_n)} \cdot X) = 0.$$

Since $\{ t_n \}$ is an arbitrary increasing sequence s.t. $t_n \to \infty$ as $n \to \infty$, we have

$$0 = \lim_{t\to\infty} \mathcal{E}(1_{(X>t)} \cdot t) = \lim_{t\to\infty} (t \cdot \mathcal{E}(1_{(X>t)})) = \lim_{t\to\infty} (t \cdot \mathcal{P}(X > t)).$$

Now, consider $Y = -X$. Then as proved in above,

$$\lim_{t\to\infty} t \cdot \mathcal{P}(Y > t) = 0$$

Let $t = -s$. Then,

$$0 = \lim_{s\to-\infty} (-s)\mathcal{P}(-X > -s) = -\lim_{s\to-\infty} s \cdot \mathcal{P}(X < s),$$

and we have

$$\lim_{t\to-\infty} t \cdot \mathcal{P}(X < t) = 0.$$

With similar argument, we have

$$\lim_{t\to\infty} t \cdot \mathcal{P}(X \ge t) = \lim_{t\to-\infty} t \cdot \mathcal{P}(X \le t)$$

3. (a) Let X be a non-negative simple r.v. belonging to $\{ \Lambda_j ; x_j \}_{j=1}^n$. Then

$$\mathcal{E}(X) = \sum_{j=1}^n x_j \mathcal{P}(\Lambda_j).$$

With loss of generality, we assume $0 \le x_1 < x_2 < \dots < x_n$, then

$$\begin{aligned}
\mathcal{E}(X) =& \sum_{j=1}^n x_j \mathcal{P}(\Lambda_j) \\
=& x_1 + (x_2 - x_1)\mathcal{P}(\Omega \setminus \Lambda_1) + \dots + (x_n - x_{n-1})\mathcal{P}(\Omega \setminus \cup_{i=1}^{n-1} \Lambda_i) \\
=& x_1 + \int_{[x_1,x_2)} \mathcal{P}(X > t)dt + \dots + \int_{[x_n-1,x_n)} \mathcal{P}(X > t)dt \\
=& \int_{[0,x_1)} \mathcal{P}(X > t)dt + \int_{[x_1,x_2)} \mathcal{P}(X > t)dt + \dots \\
&+ \int_{[x_{n-1},x_n)} \mathcal{P}(X > t)dt + \int_{[x_n,\infty)} \mathcal{P}(X > t)dt. \\
=& \int_0^\infty \mathcal{P}(X > t)dt \\
=& \int_0^\infty (1 - F(t))dt.
\end{aligned}$$

(b) Since $\{ X_n, n \geq 1 \}$ is a sequence of monotone increasing non-negative r.v.'s and is assumed to converge to X, we have

$$(X_n > t) \uparrow (X > t).$$

This is because

i. if $\omega \in (X_n > t)$. i.e, $X_n(\omega) > t$, then

$$t < X_n(\omega) \leq X_{n+1}(\omega), \quad \text{i.e.,} \quad \omega \ \in (X_{n+1} > t),$$

which shows that $(X_n > t) \subseteq (X_{n+1} > t)$;

ii. similarly, since $X_n \leq X \ \ \forall\, n$, $(X_n > t) \subseteq (X > t) \ \ \forall\, n$ so that

$$\cup_{n=1}^{\infty}(X_n > t) \subseteq (X > t);$$

iii. if $\omega \in (X > t)$, i.e., $t < X(\omega) = \lim_{n\to\infty} X_n(\omega)$, then there exist N_0 s.t. $X_n(\omega) > t \ \ \forall\, n \geq N_0$. In particular, $\omega \in (X_{N_0} > t)$, which shows that

$$(X > t) \subseteq \cup_{n=1}^{\infty}(X_n > t).$$

Now, by the monotone probability of probability measure , we have

$$1 - F_n(t) = \mathcal{P}(X_n > t) \uparrow \mathcal{P}(X > t) = 1 - F(t).$$

(c) For a non-negative r.v. X, let $\{ X_n, n \geq 1 \}$ be an increasing sequence of non-negative simple random variables defined as

$$X_n(\omega) = \begin{cases} \frac{i}{2^n}, & \text{if } \frac{i}{2^n} \leq X(\omega) < \frac{i+1}{2^n} \text{ and } 0 \leq i < n2^n, \\ n, & \text{otherwise.} \end{cases}$$

Then by Lemma 1.4.6, 3(a), 3(b) and the monotone converge theorem,

$$\begin{aligned} \mathcal{E}(X) &= \lim_{n\to\infty} \mathcal{E}(X_n) \\ &= \lim_{n\to\infty} \int_0^{\infty} (1 - F_n(t))dt \\ &= \int_0^{\infty} (1 - F(t))dt, \end{aligned}$$

where $1 - F(t), 1 - F_n(t)$ are Borel measurable function (probability distribution functions are Borel measurable) and we use the monotone convergence theorem for non-negative Borel measurable function in the last equality.

4. Since $\mathcal{E}(X)$ is well-defined and by problem 3,

$$\begin{aligned} \mathcal{E}(X) &= \mathcal{E}(X^+) - \mathcal{E}(X^-) \\ &= \int_0^{\infty} \mathcal{P}(X^+ > t)dt - \int_0^{\infty} \mathcal{P}(X^- > t)dt \\ &= \int_0^{\infty} \mathcal{P}(X > t)dt - \int_0^{\infty} \mathcal{P}(-X > t)dt \\ &= \int_0^{\infty} \mathcal{P}(X > t)dt - \int_{-\infty}^{0} \mathcal{P}(X < t)dt \\ &= \int_0^{\infty} (1 - F(t))dt - \int_{-\infty}^{0} F(t^-)dt. \end{aligned}$$

5. Since $f(x)$ is continuous,

$$\begin{aligned}
F(t^-) =& \lim_{s\uparrow t} F(s) = \lim_{s\uparrow t} \mathcal{P}(X \le s) = \mathcal{P}(X < t) \\
=& \int_{(-\infty<x<t)} f(x)dx = \int_{(-\infty<x\le t)} f(x)dx = \mathcal{P}(X \le t) \\
=& F(t).
\end{aligned}$$

Then by Exercise 4,

$$\begin{aligned}
\mathcal{E}(X) &= \int_0^\infty \mathcal{P}(X > t)dt - \int_{-\infty}^0 \mathcal{P}(X < t)dt \\
&= \int_0^\infty (1 - F(t))dt - \int_{-\infty}^0 F(t)dt.
\end{aligned}$$

And let $u = 1 - F(t), du = -f(t)dt, v = t, dv = dt$, then

$$\int_0^\infty (1 - F(t))dt = [(1 - F(t)) \cdot t] \Big|_0^\infty - \int_0^\infty (-f(t) \cdot t)\, dt.$$

Similarly, let $u = F(t), du = f(t)dt, v = t, dv = dt$, then

$$\int_{-\infty}^0 F(t)dt = [(F(t)) \cdot t] \Big|_{-\infty}^0 - \int_{-\infty}^0 (f(t) \cdot t)\, dt.$$

Thus,

$$\begin{aligned}
\mathcal{E}(X) =& \int_0^\infty (1 - F(t))dt - \int_{-\infty}^0 F(t)dt \\
=& [(1 - F(t)) \cdot t] \Big|_0^\infty - \int_0^\infty (-f(t) \cdot t)\, dt - \left([(F(t)) \cdot t] \Big|_{-\infty}^0 - \int_{-\infty}^0 (f(t) \cdot t)\, dt \right) \\
=& [\mathcal{P}(X > t) \cdot t] \Big|_0^\infty - \int_0^\infty (-f(t) \cdot t)\, dt - \left([(\mathcal{P}(X \le t)) \cdot t] \Big|_{-\infty}^0 - \int_{-\infty}^0 (f(t) \cdot t)\, dt \right) \\
=& \lim_{t\to\infty} t \cdot \mathcal{P}(X > t) - \int_0^\infty (-f(t) \cdot t)\, dt + \lim_{t\to-\infty} t \cdot \mathcal{P}(X \le t) + \int_{-\infty}^0 (f(t) \cdot t)\, dt \\
=& \int_{-\infty}^\infty tf(t)dt,
\end{aligned}$$

by Exercise 2(c) in above.

6.

$$\begin{aligned}
\mathcal{E}(X) &= \int_{-\infty}^{\infty} x f(x) dx \\
&= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{(x-\eta)^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} (y+\eta) \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{y^2}{2\sigma^2}} dy \\
&= \int_{-\infty}^{\infty} y \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{y^2}{2\sigma^2}} dy + \eta \\
&= - \frac{1}{\sqrt{2\pi}\sigma} \sigma^2 e^{-\frac{x^2}{2\sigma^2}} \Big|_{-\infty}^{\infty} + \eta \\
&= \eta.
\end{aligned}$$

7. (a) (i) We assume that X is a non-negative r.v. and $\mathcal{S}(X)$ is the set of all non-negative simple r.v.'s Z s.t. $0 \le Z \le X$. Since $X = 0$ w.p.1, there exist a $\Lambda \in \mathcal{F}$ s.t. $\mathcal{P}(\Lambda) = 1$ and $X(\omega) = 0 \ \ \forall\, \omega \in \Lambda$. Then for each $Z \in S(X)$

$$0 \le Z(\omega) \le X(\omega) = 0 \ \ \forall\, \omega \in \Lambda$$

and

$$Z(\omega) = 0 \ \ \forall\, \omega \in \Lambda.$$

Then Z belongs to a weighted finite partition $\{\, \Lambda_j; z_j \,\}_{j=1}^{n}$ with $\Lambda \subseteq \Lambda_1, Z_1 = 0$, and $\mathcal{P}(\Lambda_j) = 0 \ \ \forall\, j > 1$. Then

$$\mathcal{E}(Z) = \sum_{j=0}^{n} z_j \mathcal{P}(\Lambda_j) = 0 \cdot \mathcal{P}(\Lambda_1) + \sum_{j=2}^{n} Z_j \cdot 0 = 0$$

$$\Rightarrow \quad \mathcal{E}(X) = \sup_{Z \in \mathcal{S}(X)} \mathcal{E}(Z) = 0.$$

(ii) Let X be a generic r.v.. If $X = 0$ w.p.1, then

$$X^+ = 0 \ \text{ w.p.1 and } X^- = 0 \ \text{ w.p.1.}$$

Thus

$$\mathcal{E}(X^+) = 0 \ \text{ and } \ \mathcal{E}(X^-) = 0$$

as shown in (i). This shows that $\mathcal{E}(X)$ is well-defined and

$$\mathcal{E}(X) = \mathcal{E}(X^+) - \mathcal{E}(X^-) = 0.$$

(b) Since $X = Y$ w.p.1, we have $X^+ = Y^+$ w.p.1 and $X^- = Y^-$ w.p.1.
Let $Z = Y^+ - X^+$. Then

$$Z^+ + X^+ = Y^+ + Z^- \Rightarrow \mathcal{E}(Z^+) + \mathcal{E}(X^+) = \mathcal{E}(Y^+) + \mathcal{E}(Z^-)$$

by Theorem 1.4.7. Since $Z = 0$ w.p.1 and by part(a), $\mathcal{E}(Z) = \mathcal{E}(Z^+)\ \mathcal{E}(Z^-) = 0$, we have

$$\mathcal{E}(X^+) = \mathcal{E}(Y^+).$$

Similarly, we have

$$\mathcal{E}(X^-) = \mathcal{E}(Y^-).$$

Thus if $\mathcal{E}(X)$ is well-defined, then $\mathcal{E}(Y)$ is well-defined and

$$\mathcal{E}(Y) = \mathcal{E}(Y^+) - \mathcal{E}(Y^-) = \mathcal{E}(X^+) - \mathcal{E}(X^-) = \mathcal{E}(X).$$

8. By Exercise 3,

$$\mathcal{E}(X) = \int_0^\infty \mathcal{P}(X > t)dt = \sum_{n=1}^{\infty} \int_{[n-1,n)} \mathcal{P}(X > t)dt \geq \sum_{n=1}^{\infty} \mathcal{P}(X \geq n),$$

since $(X > t) \supseteq (X \geq n) \ \ \forall t \in [n-1, n), n = 1, 2 \cdots$. On the other hand,

$$\begin{aligned}
\mathcal{E}(X) &= \int_0^\infty \mathcal{P}(X > t)dt \\
&= \sum_{n=1}^{\infty} \int_{[n-1,n)} \mathcal{P}(X > t)dt \\
&\leq \sum_{n=1}^{\infty} \mathcal{P}(X \geq n-1) \quad \text{since} (X > t) \subseteq (X \geq n-1) \forall \ n-1 \leq t < n \\
&= \sum_{n=0}^{\infty} \mathcal{P}(X \geq n) \\
&= 1 + \sum_{n=1}^{\infty} \mathcal{P}(X \geq n),
\end{aligned}$$

since $(X \geq 0) = \Omega$.