EE565000 Stochastic Process
Homework #3   2008
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1. i. $\forall\ B \in \mathcal{B}^2$, $\mu_{X,Y}(B) \equiv \mathcal{P}((X,Y)^{-1}(B)) \geq 0$.

   ii. If $B_j, j = 1, 2, \dots$ are mutually disjoint events in $\mathcal{F}$, then $(X,Y)^{-1}(B_j)$, $j = 1, 2, \dots$ are also mutually disjoint. Therefore,

$$\begin{aligned}
\mu_{X,Y}(\cup_{j=1}^{\infty} B_j) =& \mathcal{P}((X,Y)^{-1}(\cup_{j=1}^{\infty} B_j)) \\
=& \mathcal{P}(\cup_{j=1}^{\infty} (X,Y)^{-1}(B_j)) \\
=& \sum_{j=1}^{\infty} \mathcal{P}((X,Y)^{-1}(B_j)) \\
=& \sum_{j=1}^{\infty} \mu_{X,Y}(B_j).
\end{aligned}$$

   iii.

$$\mu_{X,Y}(R^2) = \mathcal{P}((X,Y)^{-1}(R^2))) = \mathcal{P}(\Omega) = 1.$$

2. (a) i Since $f_{(X,Y)}(x,y)$ is a non-negative integrable function on $\mathbb{R}^2$, $f_X(x) \equiv \int_R f_{(X,Y)}(x,y)dx$ is a non-negative function.

   ii We have to show that $F_X(x) = \int_{-\infty}^{x} f_X(s)ds \ \ \forall x \in \mathbb{R}$. Note that

$$\begin{aligned}
F_X(x) =& \mathcal{P}(X \leq x) \\
=& \mathcal{P}(X \leq x, Y \in \mathbb{R}) \\
=& \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{X,Y}(s,t)dtds \\
=& \int_{-\infty}^{x} \left(\int_{\mathbb{R}} f_{X,Y}(s,t)dt \right) ds \\
=& \int_{-\infty}^{x} f_X(s)ds.
\end{aligned}$$

   By i and ii, we have $f_X(x)$ is a density function of X. Similarly, $f_Y(y)$ is a density function of Y.

   (b)

$$\begin{aligned}
f_X(x) =& \int_R f_{X,Y}(x,y)dy \\
=& \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\frac{x^2}{\sigma_X^2}} \int_R e^{-\frac{1}{2(1-\rho^2)}\left(\frac{y^2}{\sigma_Y^2} - \frac{2\rho xy}{\sigma_X\sigma_Y}\right)} dy \\
=& \frac{1}{\sqrt{2\pi}\sigma_X} \left(e^{-\frac{1}{2(1-\rho^2)}\frac{x^2}{\sigma_X^2}} \right) \left(e^{\frac{1}{2(1-\rho^2)}\frac{\rho^2 x^2}{\sigma_X^2}} \right) \int_R \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}\sigma_Y} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{y}{\sigma_Y} - \frac{\rho x}{\sigma_X}\right)^2} dy \\
=& \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{x^2}{2\sigma_X^2}}.
\end{aligned}$$

Similarly, $f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{y^2}{2\sigma_Y^2}}$.

3. $X_1, X_2, \ldots, X_n$ are r.v.'s taking values on the set $\{0, 1\}$. We can partition Ω into $X^{-1}(0) = E_i$ and $X_i^{-1}(1) = E_i^c$. Let $\mathcal{G}_i = \{E_i, E_i^c\}$, and $\mathcal{S}$ be the collection of subsets $A = \cap_{i=1}^n F_i$,where $F_i \in \mathcal{G}_i$. Then $\mathcal{S}$ is a partition of Ω and the σ-algebra $\mathcal{F}(X_1, X_2, \ldots, X_n)$ is a collection of all possible unions of sets in $\mathcal{S}$.

4. An uncountable partition, $R = \cup_{r \in R} \{r\}$.
   A countable example, $R = (-\infty, 0] \cup (0, \infty)$.
   Yes, both partitions are Borel measurable.

5. (a) $\inf_n a_n = -1$, $\sup_n a_n = 1$, $\liminf_n a_n = -1$, $\limsup_n a_n = 1$,
   $E = \{1, -1\}$.

   (b) $\inf_n a_n = -e^{-1}$, $\sup_n a_n = e^{-1}$, $\liminf_n a_n = -e^{-1}$, $\limsup_n a_n = e^{-1}$,
   $E = \{e^{-1}, -e^{-1}\}$.

   (c) $\inf_n a_n = -\infty$, $\sup_n a_n = \infty$, $\liminf_n a_n = -\infty$, $\limsup_n a_n = \infty$,
   $E = \{-\infty, \infty\}$.

   (d) $\inf_n a_n = \sup_n a_n = \liminf_n a_n = \limsup_n a_n = 0$,
   $E = \{0\}$.

   (e) $\inf_n a_n = -\infty$, $\sup_n a_n = \infty$, $\liminf_n a_n = -\infty$, $\limsup_n a_n = \infty$,
   $E = \{-\infty, \infty\}$.

   (f) $\inf_n a_n = 0$, $\sup_n a_n = \frac{2}{3}$, $\liminf_n a_n = 0$, $\limsup_n a_n = \frac{2}{3}$,
   $E = \{0, \frac{1}{3}, \frac{2}{3}\}$.

6. Given an $\omega \in \Omega$, there is one and only one set $\Lambda_k \in \{\Lambda_j, j \geq 0\}$ such that $\omega \in \Lambda_k$. Thus $\sup_j 1_{\Lambda_j}(\omega) = 1$ and $\inf_j 1_{\Lambda_j}(\omega) = 0, \forall\ \omega$, and then $\sup_j 1_{\Lambda_j} = 1_\Omega = 1$ and $\inf_j 1_{\Lambda_j} = 1_\phi = 0$.

   Since $\cup_{j \geq k} \Lambda_j \downarrow \phi$ as $k \to \infty$, $\limsup_j 1_{\Lambda_j} = \inf_l \sup_{j \geq l} 1_{\Lambda_j} = 1_\phi = 0$
   and $\liminf_j 1_{\Lambda_j} = \sup_k \inf_{j \geq k} 1_{\Lambda_j} = 1_\phi = 0$.