EE565000 Stochastic Process
Homework #2   2008
TA: Chien-Tien Wu

1. (a) If $x \leq y$, we have $\mu((-\infty, x]) \leq \mu((-\infty, y])$. Hence $F(x) \leq F(y)$ for $x \leq y$
   (b) We only need to prove that if $x_n \downarrow y$, then
$$\lim_{x\to\infty} F(x_n) = F(y).$$
   Since $(-\infty, x_n] \downarrow (-\infty, y]$, by monotone property of probability measure
$$\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \mu((-\infty, x_n]) = \mu((-\infty, y]) = F(y).$$
   (c) $(-\infty, -n] \downarrow \phi$ and $(\infty, n] \uparrow \Omega$, by monotone property of probability measure
$$\lim_{x\to-\infty} F(x) = \lim_{n\to\infty} \mu((-\infty, -n]) = \mu(\phi) = 0,$$
$$\lim_{x\to\infty} F(x) = \lim_{n\to\infty} \mu((-\infty, n]) = \mu(\Omega) = 1.$$

2. i. Let
$$F(x) = \begin{cases} 0, & x \leq 0, \\ x, & 0 < x \leq 1, \\ 1, & 1 < x. \end{cases}$$
   The corresponding probability measure μ is defined as $\mu(I) = \int_{I\cap[0.1]} 1dx$ for I $\in \mathcal{B}$.
   ii. Let $\mathcal{F}(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$, then $\mu(I) = \int_I \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$ is the corresponding probability measure on $(R, \mathcal{B})$.

3. ($\Rightarrow$)
   Suppose $A \in \mathcal{F}$,
$$\{\,\omega : 1_A(\omega) \leq x\,\} = \begin{cases} \Omega, & x \geq 1, \\ A^c, & 0 \leq x < 1, \\ \phi, & x < 0, \end{cases}$$
   which means $\{\,\omega : 1_A(\omega) \leq x\,\} \in \mathcal{F}$, $\forall x \in R$. By theorem 1.3.4, 1_A is an r.v. on $(\Omega, \mathcal{F})$.
   ($\Leftarrow$)
   If 1_A is an random variable, $1_A^{-1}(\{\,1\,\}) = A \in \mathcal{F}$.

4. (a) $f^{-1}(1, 5, 25) = [1, 2) \cup [5, 6) \cup [25, 26)$.
   (b) $\forall B \in \mathcal{B}$
$$\begin{aligned} f^{-1}(B) =& f^{-1}((B \cap \mathbb{Z}^c) \cup (B \cap \mathbb{Z})) \\ =& f^{-1}(B \cap \mathbb{Z}^c) \cup f^{-1}(B \cap \mathbb{Z}) \\ =& \bigcup_{n \in Z_0 \subset \mathbb{Z}} [n, n+1) \in \mathcal{B} \end{aligned}$$
   $\Rightarrow f(x)$ is a Borel measurable function from $(R, \mathcal{B})$ into $(R, \mathcal{B})$.

5. Since $\Lambda \in \mathcal{F}(X)$, there exists $B \in \mathcal{B}$ such that $\Lambda = X^{-1}(B)$.Since B is an event in $(\mathbb{R}, \mathcal{B})$, by problem 3, the indicator function 1_B of B is a Borel measurable function from $(\mathbb{R}, \mathcal{B})$ to $(\mathbb{R}, \mathcal{B})$. We let $f(x) = 1_B(x)$. Then $1_\Lambda(\omega) = 1_B(X(\omega)) = f(X(\omega))$,i.e. $1_\Lambda = f(X)$.

6. (i) $\forall B \in \mathcal{B}$, $\mu_X(B) \equiv \mathcal{P}(X^{-1}(B)) \geq 0$.

   (ii) If $A_j, j = 1, 2, \ldots$are mutually disjoint Borel set in $\mathcal{B}$, then $X^{-1}(A_j)$ are also disjoint events in $\mathcal{F}$, and

$$\begin{aligned}\mu_X(\cup_{j=1}^{\infty} A_j) =& \mathcal{P}(X^{-1}(\cup_{j=1}^{\infty} A_j))\\ =& \mathcal{P}(\cup_{j=0}^{\infty} X^{-1}(A_j))\\ =& \sum_{j=1}^{\infty} \mathcal{P}(X^{-1}(A_j))\\ =& \sum_{j=1}^{\infty} \mu_x(A_j).\end{aligned}$$

   (iii)

$$\begin{aligned}\mu_X(R) =& \mathcal{P}(X^{-1}(R))\\ =& \mathcal{P}(\Omega)\\ =& 1.\end{aligned}$$

   By (i), (ii), (iii) in above, μ_X is a probability measure on $(R, \mathcal{B})$.

7. Let $Y = f(X_1, X_2)$. $\forall A \in \mathcal{B}, Y^{-1}(A) = (X_1, X_2)^{-1} \circ f^{-1}(A)$. Since f is a Borel measurable function, $f^{-1}(A) \in \mathcal{B}^2$. By problem 8(a) which will be proved later, we know that $Y^{-1}(A) = (X_1, X_2)^{-1} \circ f^{-1}(A)$ is an event, i.e $Y^{-1}(A) \in \mathcal{F}$. Thus $Y = f(X_1, X_2)$ is an r.v. on $(\Omega, \mathcal{F})$.

8. (a) (i.) Let $W = I_1 \times I_2 \in R^2$.

$$\begin{aligned}(X, Y)^{-1}(W) =& \{\, \omega \in \Omega \mid (X(\omega), Y(\omega)) \in W \,\}\\ =& \{\, \omega \in \Omega \mid X(\omega) \in I_1, Y(\omega) \in I_2 \,\}\\ =& \{\, \omega \in \Omega \mid X(\omega) \in I_1 \,\} \cap \{\, \omega \in \Omega \mid Y(\omega) \in I_2 \,\}\\ =& X^{-1}(I_1) \cap Y^{-1}(I_2).\end{aligned}$$

   Since $X^{-1}(I_1) \in \mathcal{F}$ and $Y^{-1}(I_2) \in \mathcal{F}$, $(X, Y)^{-1}(W) \in \mathcal{F}$ for $W \in R^2$.

   (ii.) Let $(X, Y)^{-1}(B) \in \mathcal{F}$ for a subset B of R^2.

$$\begin{aligned}& \omega \in (X, Y)^{-1}(B^c)\\ \Leftrightarrow\ & (X(\omega), Y(\omega)) \in B^c\\ \Leftrightarrow\ & \omega \notin (X, Y)^{-1}(B)\\ \Leftrightarrow\ & \omega((X, Y)^{-1}(B))^c,\end{aligned}$$

   which implies

$$(X, Y)^{-1}(B) = ((X, Y)^{-1}(B))^c \in \mathcal{F}.$$

(iii.) If $(X,Y)^{-1}(B_n)$ is an event for each of a countable number of subsets B_n of R^2, $n = 1, 2, \ldots$, then

$$\begin{aligned}(X,Y)^{-1}(\cup_n B_n) &= \{\, \omega \in \Omega \mid (X(\omega), Y(\omega)) \in \cup_n B_n \,\} \\ &= \cup_n \{\, \omega \in \Omega \mid (X(\omega), Y(\omega)) \in B_n \,\} \\ &= \cup_n (X,Y)^{-1}(B_n) \in \mathcal{F}.\end{aligned}$$

(iv.) Let $\mathcal{G}$ be the collection of all subsets A of R^2 s.t. $(X,Y)^{-1}(A) \in \mathcal{F}$. By (ii) and (iii) (similar to the proof of theorem (1.3.4)), $\mathcal{G}$ is a σ-algebra of subsets of R^2. By (i), $\mathcal{G}$ contains all 2-cells of R^2. Since $\mathcal{B}^2$ is generated by all 2-cells, we have $\mathcal{B}^2 \subseteq \mathcal{G}$. Thus for any two-dimensional Borel set B in $\mathcal{B}^2$, $(X,Y)^{-1}(B) \in \mathcal{F}$.

(b) The collection $\mathcal{C}$ of all events $(X,Y)^{-1}(B)$ for all $B \in \mathcal{B}^2$ is a σ-algebra by similar proof in (ii) and (iii) in part(a).
Since $\mathcal{B}^2$ is generated by 2-cells $\{\, I_1 \times I_2 \,\}$, $\mathcal{C}$ is generated by $(X,Y)^{-1}(I_1 \times I_2)$. Then

$$\begin{aligned}(X,Y)^{-1}(I_1 \times I_2) &= \{\, \omega \in \Omega \mid (X(\omega), Y(\omega)) \in I_1 \times I_2 \,\} \\ &= \{\, \{\, \omega \in \Omega \mid X(\omega) \in I_1 \,\} \cap \{\, \omega \in \Omega \mid Y(\omega) \in I_2 \,\} \,\}.\end{aligned}$$

$$\Rightarrow \mathcal{C} \subseteq \mathcal{F}(X,Y).$$

On the other hand, since $\mathcal{F}(X)$ is generated by $X^{-1}(I_1)$ and $\mathcal{F}(Y)$ is generated by $Y^{-1}(I_2)$, $\mathcal{F}(X,Y)$ is generated by $X^{-1}(I_1)$ and $Y^{-1}(I_2)$ for all intervals I_1 and I_2. Then

$$\begin{aligned}X^{-1}(I_1) &= X^{-1}(I_1) \cap Y^{-1}(\mathbb{R}) \\ &= (X,Y)^{-1}(I_1 \times \mathbb{R}) \\ &\in \mathcal{C}.\end{aligned}$$

Similarly,

$$\begin{aligned}Y^{-1}(I_2) &= Y^{-1}(I_2) \cap X^{-1}(\mathbb{R}) \\ &= (X,Y)^{-1}(\mathbb{R} \times I_2) \\ &\in \mathcal{C}.\end{aligned}$$

$$\Rightarrow \mathcal{F}(X,Y) \subseteq \mathcal{C}.$$

Thus the collection $\mathcal{C}$ of all events $(X,Y)^{-1}(B)$ is the smallest σ-algebra $\mathcal{F}(X,Y)$ generated by the union of $\mathcal{F}(X)$ and $\mathcal{F}(Y)$.

9. (a) Since $a_n = (1 + \frac{1}{n}) \sin n\pi = 0$, $\sup_n a_n = \inf_n a_n = 0$.
   (b) $\sup_n a_n = 1$ and $\inf_n a_n = -1$.