EE 565000
Homework Assignment #4

Fall Semester, 2008

Due Date: October 16, 2008

In the lecturenotes, we have proved the monotone convergence theorem for non-negative r.v.'s. There is another version of the monotone convergence theorem for non-negative Borel measurable functions from $(R, \mathcal{B})$ to $(R, \mathcal{B})$ which will be used in this homework.

Theorem (Monotone convergence theorem for non-negative Boreal measurable functions) Let $\{f_n(t), n \geq 1\}$ be a sequence of monotone increasing (extended-valued) non-negative Borel measurable functions, $0 \leq f_1(t) \leq f_2(t) \leq \ldots \leq +\infty$. Let $f(t)$ be the limit of $f_n(t)$, i.e., $f(t) = \lim_{n\to\infty} f_n(t)$. Then

$$\int_{-\infty}^{\infty} f(t)dt = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(t)dt.$$

$\square$

If mentioned, $(\Omega, \mathcal{F}, \mathcal{P})$ is assumed to be a probability space.

1. (10%) Let X be a r.v. such that $\mathcal{E}(X)$ is well-defined. Please show that $|\mathcal{E}(X)| \leq \mathcal{E}(|X|)$. (Hint: use Eq. (1.11) on page 16 of the lecturenotes.)

2. Let X be a r.v. with finite expectation, i.e., $\mathcal{E}(|X|) < \infty$. Define a set function from the collection $\mathcal{B}$ of all Borel sets of R to R as follows:

$$|\mu|(B) \triangleq \frac{\mathcal{E}(1_{(X\in B)} \cdot |X|)}{\mathcal{E}(|X|)}, \ \forall\ B \in \mathcal{B}.$$

   (a) (7%) Please show that $|\mu|$ is a probability measure on the Borel measurable space $(R, \mathcal{B})$. (Hint: use Theorem 1.4.9 and Corollary 1.4.10.)

   (b) (7%) Let $\{B_n, n \geq 1\}$ be a monotone decreasing sequence of Borel sets of R such that $B_n \downarrow \emptyset$. Please show that $\lim_{n\to\infty} \mathcal{E}(1_{(X\in B)} \cdot X) = 0$, i.e., in abstract integral form,

$$\lim_{n\to\infty} \int_{(X\in B_n)} X(\omega)\mathcal{P}(d\omega) = 0,$$

   (Hint: use Exercise 1 in above and the monotone property of the p.m. $|\mu|$.)

   (c) (6%) Please show that $\lim_{t\to\infty} t \cdot \mathcal{P}(X > t) = 0$ and $\lim_{t\to-\infty} t \cdot \mathcal{P}(X \leq t) = 0$.

3. Let X be a non-negative r.v. with probability distribution function $F(x)$ over R. Please show that

$$\mathcal{E}(X) = \int_0^{\infty} \mathcal{P}(X > t)dt = \int_0^{\infty} (1 - F(t))dt \tag{1}$$

   in the following three steps:

(a) (7%) Show Eq. (1) for non-negative simple r.v.'s.

(b) (7%) Show that if $\{X_n, n \geq 1\}$ is a monotone increasing non-negative r.v.'s with probability distribution functions $F_n(t), n \geq 1$, then $(1-F_n(t)) \uparrow (1-F(t)$. (Hint: use the monotone property of probability measure.)

(c) (6%) Use the monotone convergent theorem both for non-negative r.v.'s and for non-negative Boreal measurable functions to complete the proof of Eq. (1) for general non-negative r.v. X.

4. (10%) Let X be a r.v. with probability distribution function $F(x)$ over R. Please show that if $\mathcal{E}(X)$ is well-defined, then

$$\mathcal{E}(X) = \int_0^\infty \mathcal{P}(X > t)dt - \int_{-\infty}^0 \mathcal{P}(X \leq t)dt = \int_0^\infty (1 - F(t))dt - \int_{-\infty}^0 F(t)dt.$$

(Hint: use Eq. (1.11) on page 16 of the lecturenotes and Exercise 3 in above.)

5. (10%) Let X be a r.v. with finite expectation and a continuous probability density function $f(x)$ over R. Please show that

$$\mathcal{E}(X) = \int_{-\infty}^\infty x f(x)dx.$$

(Hint: use integration by parts and Exercise 2(c) in above.)

6. (10%)Let X be a r.v. with a probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\eta)^2/(2\sigma^2)}.$$

Please find $\mathcal{E}(X)$.

7. Two (extended-valued) random variables X and Y are said to be equal with probability one (w.p.1) if $\mathcal{P}(X = Y) = 1$.

(a) (5%) Please show that if $X = 0$ w.p.1, then $\mathcal{E}(X)$ is well-defined and $\mathcal{E}(X) = 0$. (Hint: you may deal with non-negative r.v. first and then a generic r.v. latter.)

(b) (5%) Let $X = Y$ w.p.1. Please show that if $\mathcal{E}(X)$ is well-defined, then $\mathcal{E}(Y)$ is well-defined and $\mathcal{E}(X) = \mathcal{E}(Y)$.

8. (10%) Let X be an (extended-valued) non-negative random variable. Please show that

$$\sum_{n=1}^\infty \mathcal{P}(X \geq n) \leq \mathcal{E}(X) \leq 1 + \sum_{n=1}^\infty \mathcal{P}(X \geq n).$$

(Hint: use Exercise 3.)