EE 565000
Homework Assignment #3

Fall Semester, 2008

Due Date: October 9, 2008

If mentioned, $(\Omega, \mathcal{F}, \mathcal{P})$ is assumed to be a probability space.

1. (10%) Given two real-valued random variables $X, Y : \Omega \to R$, define a set function $\mu_{X,Y}$ on all Borel sets in R^2 as follows:

$$\mu_{X,Y}(B) \equiv \mathcal{P}((X,Y)^{-1}(B)), \ \forall\ B \in \mathcal{B}^2.$$

(cf. Exercise #8 of Homework #2.) Please show that $\mu_{X,Y}$ is a probability measure on the Borel measurable space $(R^2, \mathcal{B}^2)$ ($\mu_{X,Y}$ is called the probability measure induced by r.v.'s X, Y and $(R^2, \mathcal{B}^2, \mu_{X,Y})$ becomes to a probability space). Note that the distribution function $F_{X,Y}(x,y) \equiv \mu_{X,Y}((-\infty, x] \times (-\infty, y])$, $x, y \in R$, corresponding to the probability measure $\mu_{X,Y}$ is called the joint probability distribution function of r.v.'s X and Y. (Indeed, $F_{X,Y}(x,y)$ and $\mu_{X,Y}$ relate to each other uniquely similar to the relation between $F_X(x)$ and μ_X as stated in Theorem 1.2.3 of the lecturenotes.)

2. Let X and Y be two real-valued random variables. Let $\mu_{X,Y}$ be the probability measure on $(R^2, \mathcal{B}^2)$ induced by X and Y and $F_{X,Y}(x,y)$ the joint probability distribution function of X and Y as in the above exercise. X and Y are said to have a joint probability density function $f_{X,Y}(x,y)$ on R^2 if $f_{X,Y}(x,y)$ is a non-negative integrable function on R^2 such that

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t)\,ds\,dt, \quad \forall\ x, y.$$

   (a) (10%) Please show that $f_X(x) \equiv \int_R f_{X,Y}(x,y)dy$ and $f_Y(y) \equiv \int_R f_{X,Y}(x,y)dx$ are probability density functions of X and Y respectively.

   (b) (10%)Suppose that

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right)\right\}$$

   is a joint probability density function of X and Y, where $\sigma_X > 0$, $\sigma_Y > 0$, $0 < \rho < 1$. Please find $f_X(x)$ and $f_Y(x)$.

3. (10%) Suppose that $X_1, X_2, \ldots, X_n$ be n r.v.'s taking values on the subset $\{0, 1\}$ of integers. Please find the σ-algebra $\mathcal{F}(X_1, X_2, \ldots, X_n)$ generated by $X_1, X_2, \ldots, X_n$, i.e., the smallest σ-algebra on Ω which contains $\mathcal{F}(X_i)$ for $i = 1, 2, \ldots, n$.

4. (10%) Please find an uncountable partition and a countable partition of the real line R. Are these partitions Borel measurable?

5. Please find sup, inf, lim sup, lim inf and the set E of all subsequential limits of each of the following sequences:

(a) (5%) $a_n = (-1)^n$;

(b) (5%) $a_n = (-1 + \frac{1}{n})^n$;

(c) (5%) $a_n = (-1)^n n$;

(d) (5%) $a_n = (1 + \frac{1}{n}) \sin n\pi$;

(e) (5%) $a_n = n \sin \frac{n\pi}{3}$;

(f) (5%) $a_n = \frac{n}{3} - \lfloor \frac{n}{3} \rfloor$.

6. (20%) Let $\{\Lambda_j, j \geq 1\}$ be a countable measurable partition of Ω. Then $\{1_{\Lambda_j}, j \geq 1\}$ is a sequence of r.v.'s. Please find $\sup_j 1_{\Lambda_j}$, $\inf_j 1_{\Lambda_j}$, $\limsup_j 1_{\Lambda_j}$ and $\liminf_j 1_{\Lambda_j}$.