EE 565000
Homework Assignment #2

Fall Semester, 2008

Due Date: October 2, 2008

If mentioned, $(\Omega, \mathcal{F}, \mathcal{P})$ is assumed to be a probability space.

1. Let μ be a probability measure on the Borel measurable space $(R, \mathcal{B})$ and define a point function $F(x)$ of x in R as

$$F(x) \equiv \mu((-\infty, x]), \ \forall \ x \in R.$$

   (Please see page 5 of the lecturenotes.) Please show that

   (a) (5%) $F(x)$ is monotonely increasing, i.e. $F(x) \leq F(y)$ if $x \leq y$;

   (b) (5%) it is right-continuous, i.e. $\lim_{x \to y^+} F(x) = F(y)$, over R;

   (c) (5%) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$. (Hint: Use the monotone property of probability measure.)

   $F(x)$ is called the distribution function of the p.m. μ.

2. (10%) Please construct two probability measures on the Borel measurable space $(R, \mathcal{B})$ by using two different distribution functions as stated in Theorem 1.2.3 of the lecturenotes.

3. (10%) The indicator function $1_A(\omega)$ of a subset A of the sample space Ω is defined as $1_A(\omega) = 1$ if $\omega \in A$ and $1_A(\omega) = 0$ otherwise. Please show that A is an event if and only if 1_A is a random variable.

4. The stairwise function is defined as $f(x) = n$, for $x \in [n, n+1)$ for all integers n.

   (a) (5%) What is the set $f^{-1}(\{1, 5, 25\})$?

   (b) (5%) Please show that $f(x)$ is Borel measurable. (Note: You may not simply use Theorem 1.3.5 in the lecturenotes.)

5. (10%) Let $\mathcal{F}(X)$ be the σ-algebra induced by a r.v. X and Λ be a member in $\mathcal{F}(X)$. Please show that the indicator function 1_Λ of Λ is a function $f(X)$ of X for some Borel measurable function $f(\cdot)$.

6. (10%) Given a r.v. $X : \Omega \to R$, define a set function μ_X on all Borel sets in R as follows:

$$\mu_X(B) \equiv \mathcal{P}(X^{-1}(B)), \ \forall \ B \in \mathcal{B}.$$

   Please show that μ_X is a probability measure on the Borel measurable space $(R, \mathcal{B})$ (μ_X is called the probability measure induced by r.v. X and $(R, \mathcal{B}, \mu)$ becomes to a probability space). Note that the distribution function $f_X(x) \equiv \mu_X((-\infty, x])$, $x \in R$, corresponding to the probability measure μ_X is called the probability distribution function of the r.v. X.

7. (10%) Please show Theorem 1.3.7 for $n = 2$ on page 8 of the lecturenotes.

8. Let X and Y be two real-valued random variables and B a two-dimensional Borel set in R^2, i.e. B in $\mathcal{B}^2$.

   (a) (10%) Please show that the set

   $$(X, Y)^{-1}(B) \equiv \{\omega \in \Omega | (X(\omega), Y(\omega)) \in B\}$$

   is an event, i.e a member of $\mathcal{F}$, in four steps:

   i. Show that $(X, Y)^{-1}(W)$ is an event for any 2-cell W in R^2.
   ii. Show that if $(X, Y)^{-1}(B)$ is an event for a subset B of R^2, then $(X, Y)^{-1}(B^c)$ is also an event.
   iii. Show that if $(X, Y)^{-1}(B_n)$ is an event for each of a countable number of subsets B_n, $n = 1, 2 \ldots$, of R^2, then $(X, Y)^{-1}(\cup_n B_n)$ is also an event.
   iv. Show that $(X, Y)^{-1}(B)$ is an event for any B in $\mathcal{B}^2$.

   (b) (10%) Please also show that the collection of all events $(X, Y)^{-1}(B)$ for all $B \in \mathcal{B}^2$ is just the smallest σ-algebra $\mathcal{F}(X, Y)$ generated by the union of $\mathcal{F}(X)$ and $\mathcal{F}(Y)$.

9. Please find sup and inf of each of the following sequences:

   (a) (2%) $a_n = (1 + 1/n) \sin n\pi$;

   (b) (3%) $a_n = \sin n$.