EE 565000
Homework Assignment #1

Fall Semester, 2008

Due Date: September 25, 2008

If mentioned, $(\Omega, \mathcal{F}, \mathcal{P})$ is assumed to be a probability space.

1. (10%) Let $\Omega = \{a, b, c, d, e\}$ and $\Lambda = \{1, 2, 3\}$ be two sets. Let X be a mapping from Ω into Λ, described explicitly as

$$X(a) = X(c) = 1, X(b) = X(e) = 2, X(d) = 3.$$

The inverse mapping X^{-1} of X can be regarded as a set function from the power set 2^Λ into the power set 2^Ω. Please describe X^{-1} explicitly as we have done for X in above.

2. (10%) Please prove Theorem 1.1.5.

3. (10%) Please show that the collection $\mathcal{T} = \{\emptyset, \Omega\}$ of subsets of Ω is a σ-algebra. And please show that $\mathcal{T}$ is the smallest possible σ-algebra on Ω, i.e., $\mathcal{T} \subseteq \mathcal{F}$ for any σ-algebra on a sample space Ω.

4. (10%) Please prove Theorem 1.1.7.

5. Consider a sample space $S = \{a, b, c, d\}$.
   (a) (5%) Please list all possible σ-algebras on S.
   (b) (5%) What is the σ-algebra generated by the collection $\{\{a\}, \{c, d\}\}$ of subsets of S.
   (c) (Challenge problem for extra bonus) If Ω is a sample space of n elements, how many σ-algebras can we construct on Ω?

6. (10%) Please show that if Ω is a finite sample space, then an algebra on Ω must be a σ-algebra on Ω.

7. Please find the σ-algebras in the set R of all real numbers generated by each of the following collections of subsets of R:
   (a) (5%) $\{[-1, 15], (10, 20)\}$;
   (b) (5%) $\{\{n\} | n \in N\}$, where N is the set of all positive integers.

8. Please show that the σ-algebra $\mathcal{B}$ of all Borel sets in R can be generated by any of the following collection $\mathcal{G}$:
   (a) (5%) $\mathcal{G}$ consists of all open rays $(-\infty, a)$ to the left in R, where $a \in R$.
   (b) (5%) $\mathcal{G}$ consists of all closed rays $(-\infty, a]$ to the left in R, where $a \in R$.

9. (10%) Please prove the following properties of the probability measure $\mathcal{P}$. Let E, E_1, E_2 be events in $\mathcal{F}$. Then

   (a) $\mathcal{P}(E) \leq 1$.

   (b) $\mathcal{P}(\emptyset) = 0$.

   (c) $\mathcal{P}(E^c) = 1 - \mathcal{P}(E)$.

   (d) $\mathcal{P}(E_1 \cup E_2) + \mathcal{P}(E_1 \cap E_2) = \mathcal{P}(E_1) + \mathcal{P}(E_2)$.

   (e) $E_1 \subseteq E_2 \Rightarrow \mathcal{P}(E_1) \leq \mathcal{P}(E_2)$.

10. (10%) What is a discrete measurable space $(\Omega, \mathcal{F})$? What is a discrete probability space $(\Omega, \mathcal{F}, \mathcal{P})$? Please construct two discrete probability spaces, one with a finite sample space and the other with a countably infinite sample space.