Fuzzy Differential Games for Nonlinear Stochastic Systems: Suboptimal Approach

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***Abstract*—A fuzzy differential game theory is proposed to solve the n-person (or n-player) nonlinear differential noncooperative game and cooperative game (team) problems, which are not easily tackled by the conventional methods. In this paper, both noncooperative and cooperative quadratic differential games are considered. First, the nonlinear stochastic system is approximated by a fuzzy model. Based on the fuzzy model, a fuzzy controller is proposed to deal with the noncooperative differential game in the sense of Nash equilibrium strategies or with the cooperative game in the sense of Pareto-optimal strategies. Using a suboptimal approach, the outcomes of the fuzzy differential games for both the noncooperative and the cooperative cases are parameterized in terms of an eigenvalue problem. Since the state variables are usually unavailable, a suboptimal fuzzy observer is also proposed in this study to estimate the states for these differential game problems. Finally, simulation examples are given to illustrate the design procedures and to indicate the performance of the proposed methods.**

***Index Terms*—Cooperative game, fuzzy differential game, noncooperative game.**

I. INTRODUCTION

LARGE-SCALE systems are often controlled by more than one controller or decision maker with each using an individual strategy. These controllers may operate in a group as a team with a common objective function or in a conflicting manner with multiple-objective functions as a game [1]. Differential game theory has been widely applied to multiperson decision making problems, stimulated by a vast number of applications, including those in economics, management, communication networks, power networks, and in the design of complex engineering systems. In this situation, many decision makers are present or many possible conflicting objectives should be taken into account in order to reach some form of optimality [2], [3]. Typically, n-person (or n-player) differential games are divided into two classes: a noncooperative type of game in the sense of Nash and a cooperative one in the sense of Pareto. In the noncooperative game with n players, each participant pursues an individual goal which may partly conflict with others. The n players in the cooperative game work together and act as one player seeking their maximum common profit. In this paper, both noncooperative and cooperative differential game problems are considered.

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In the nonlinear n-person differential game problems, one needs to solve n-simultaneous Hamilton–Jacobi–Bellman (HJB) equations, which are all nonlinear partial differential equations [2]. At present, it is very difficult to solve the nonlinear n-person differential game problems, except for very special cases. For this reason, it is not easy to apply nonlinear n-person differential game theory to address the practical problems. The purpose of this work is to find a simple and feasible method to deal with the general problem of nonlinear n-person differential games so the results can be applied in a practical setting.

Recently, fuzzy models have been used to efficiently approximate nonlinear systems [5]–[7]. In this paper, in order to avoid solving n-simultaneous HJB equations, the Takagi–Sugeno fuzzy model [5] is employed to approximate the nonlinear stochastic dynamic systems in the nonlinear differential game problem. Therefore, the n-person nonlinear differential game problem is transformed to a n-person fuzzy differential game problem. Based on the fuzzy model, the n-person fuzzy differential game problems are characterized in terms of a minimization problem subject to some Riccati-like inequalities.

Since the state variables are not all available in practice, a state estimation algorithm is needed to estimate the state variables for the control design. In this study, a suboptimal fuzzy observer is proposed to estimate the states for controller design in these quadratic fuzzy differential game problems when state variables are unavailable. Using a separation method, the solution of the observer-based fuzzy differential game problem is also characterized in terms of a minimization problem subject to some Riccati-like inequalities.

Solving the minimization problem subject to some Riccati-like inequalities in n-person fuzzy differential game is still a challenging task. Fortunately, using the techniques of Schur complements, certain form of Riccati-like inequalities can be transformed into equivalent linear matrix inequalities (LMIs) [9], [10]. Therefore, the fuzzy differential game problems are reduced to solving the minimization problem subject to LMIs, which is known as an eigenvalue problem (EVP) [9]. The EVP can be solved very efficiently by convex optimization techniques using interior-point methods with the aid of a toolbox in Matlab [11].

The paper is organized as follows: the problem formulation is presented in Section II, while fuzzy observer combined with the fuzzy control for both noncooperative and cooperative games are described in Section III. In Section IV, simulation examples are provided to demonstrate the design procedures and indicate the performance of the proposed methods. Finally, concluding remarks are made in Section V.

II. Problem Formulation

Consider the following nonlinear stochastic system:

$$\begin{aligned}\dot{x}(t) &= f_1(x(t), u_1(t), \ldots, u_n(t)) + g_1(x(t))w(t)\\ y(t) &= f_2(x(t)) + g_2(x(t))v(t)\end{aligned} \tag{1}$$

where $x(t) = (x_1(t), \ldots, x_m(t))^T$ denotes state variables, $u(t) = (u_1(t), \ldots, u_n(t))^T$ denotes control inputs of n players, $y(t) = (y_1(t), \ldots, y_r(t))^T$ denotes output of the system, and external disturbance $w(t)$ and measurement noise $v(t)$ are assumed to be uncorrelated, zero-mean, white noises with identity power spectrum density matrices without loss of generality.

We assume that the action of the kth controller ($k \in \{1, 2, \ldots, n\}$) is determined by a control policy $u_k(t)$ and denote the class of all such policies for the kth controller by U_k, i.e., $u_k(t) \in U_k$.

For the noncooperative game of the nonlinear stochastic system (1), the individual cost to be minimized by the kth controller (or player) $u_k(t)$ is [2]

$$\begin{aligned}J_k^{NC}(u_k) = \lim_{t_f \to \infty} \frac{1}{t_f} E\Big\{ & x^T(t_f)Sx(t_f)\\ &+ \int_0^{t_f} (x^T(t)Qx(t) + u_k^T(t)R_k u_k(t))\, dt \Big\}\end{aligned} \tag{2}$$

where E denotes expectation, $S > 0, Q = Q^T > 0$ and $R_k > 0$ for $k = 1, 2, \ldots, n$.

The solution for noncooperative game problem in (2) is the Nash equilibrium. In other words, we seek a multipolicy $u^*(t) = (u_1^*(t), \ldots, u_n^*(t))^T$ that no controller has incentive to deviate from, i.e., [3]

$$J_k^{NC}(u^*(t)) = \inf_{u_k(t) \in U_k} J_k^{NC}([u_k(t) \,|\, u_{-k}^*(t)]) \tag{3}$$

where $[u_k(t) \,|\, u_{-k}^*(t)]$ is the policy obtained when for each $j \neq k$, player j uses policy $u_j^*(t)$, and player k uses $u_k(t)$, i.e.,

$$\begin{aligned}[u_k(t) \,|\, u_{-k}^*(t)] = [u_k(t) \,|\, & u_1^*(t), \ldots\\ & u_{k-1}^*(t), u_{k+1}^*(t), \ldots, u_n^*(t)].\end{aligned} \tag{4}$$

For an n-person differential game, an n-tuple of strategies provides a feedback Nash equilibrium solution for the noncooperative differential game.

On the other hand, for the cooperative game (i.e., team), the common cost to be minimized is [2], [4]

$$\begin{aligned}J^C(u) &= \lim_{t_f \to \infty} \frac{1}{t_f} E\Big\{ x^T(t_f)Sx(t_f)\\ &\quad + \int_0^{t_f} \Big(x^T(t)Qx(t) + \sum_{k=1}^{n} u_k^T(t)R_k u_k(t)\Big)\, dt \Big\}\\ &= \lim_{t_f \to \infty} \frac{1}{t_f} E\Big\{ x^T(t_f)Sx(t_f)\\ &\quad + \int_0^{t_f} [x^T(t)Qx(t) + u^T(t)Ru(t)]\, dt \Big\}\end{aligned} \tag{5}$$

where $S > 0, Q = Q^T > 0, R = \text{diag}(R_1, R_2, \ldots, R_n)$ and $R_k > 0$ for $k = 1, 2, \ldots, n$.

For an n-person cooperative differential game, we seek a cooperative strategies $u^*(t)$ to provide a feedback Pareto-optimal solution for the cooperative differential game in (5), i.e.,

$$\begin{aligned}J^C(u^*) = \inf_{u(t)} \lim_{t_f \to \infty} \frac{1}{t_f} E\Big\{ & x^T(t_f)Sx(t_f)\\ &+ \int_0^{t_f} [x^T(t)Qx(t) + u^T(t)Ru(t)\, dt] \Big\}.\end{aligned} \tag{6}$$

The fuzzy linear model is described by fuzzy If–Then rules and will be employed here to deal with the differential game control design problem for nonlinear stochastic systems. The ith rule of the fuzzy linear model for the nonlinear stochastic system in (1) is of the following form [6], [7], [12]:

Plant Rule i:

If $z_1(t)$ is F_{i1} and $\cdots$ and $z_g(t)$ is F_{ig}

$$\begin{aligned}\text{Then } \dot{x}(t) &= A_i x(t) + \sum_{k=1}^{n} (B_{ik} u_k(t)) + G_i w(t)\\ y(t) &= C_i x(t) + D_i v(t)\end{aligned} \tag{7}$$

for $i = 1, 2, \ldots, L$ where F_{ij} is the fuzzy set, $A_i \in R^{m\times m}, B_{ik} \in R^{m\times 1}, G_i \in R^{m\times p}, C_i \in R^{r\times m}, D_i \in R^{r\times q}$; L is the number of If–Then rules; $z_1(t), z_2(t), \ldots, z_g(t)$ are the premise variables.

Assumption: (A_i, B_i) are controllable and (A_i, C_i) are observable for $i = 1, 2, \ldots, L$. ■

The the fuzzy system is inferred as follows [6], [7], [12]:

$$\dot{x}(t) = \sum_{i=1}^{L} h_i(z(t))[A_i x(t) + \sum_{k=1}^{n} B_{ik} u_k(t) + G_i w(t)] \tag{8}$$

$$y(t) = \sum_{i=1}^{L} h_i(z(t))[C_i x(t) + D_i v(t)] \tag{9}$$

where

$$\begin{aligned}h_i(z(t)) &= \frac{\mu_i(z(t))}{\sum_{i=1}^{L} \mu_i(z(t))}\\ \mu_i(z(t)) &= \prod_{j=1}^{g} F_{ij}(z_j(t))\\ z(t) &= [z_1(t), z_2(t), \ldots, z_g(t)]\end{aligned} \tag{10}$$

and where $F_{ij}(z_j(t))$ is the grade of membership of $z_j(t)$ in F_{ij}.

The normalized membership functions in (10) satisfy

$$\sum_{i=1}^{L} h_i(z(t)) = 1 \tag{11}$$

where $h_i(z(t)) \in [0, 1]$ [13].

Suppose the following fuzzy controller of the kth player is employed to deal with the above fuzzy control system design

Control Rule i:

If $z_1(t)$ is F_{i1} and $\cdots$ and $z_g(t)$ is F_{ig}

$$\text{Then } u_k(t) = -K_{ik} x(t) \tag{12}$$

for $i = 1, 2, \ldots, L$, and $k = 1, 2, \ldots, n$.

Hence, the fuzzy controller is given by

$$u_k(t) = -\sum_{i=1}^{L} h_i(z(t))(K_{ik}x(t)) \tag{13}$$

where the control parameters K_{ik} (for $i = 1, 2, \ldots, L$, and $k = 1, 2, \ldots, n$) are to be specified later to achieve the desired control purpose.

In this paper, we define $\sum_{k=1}^{n} B_{ik}u_k = B_i u$ where $B_i = [B_{i1} \;\; B_{i2} \;\; \cdots \;\; B_{in}]; B_{ik} \in R^{m\times 1}$ for $i = 1, 2, \ldots, L$, and $k = 1, 2, \ldots, n$, and $u(t)$ is represented as follows:

$$\begin{aligned} u(t) &= [u_1(t), u_2(t), \ldots, u_n(t)]^T \\ &= -\sum_{i=1}^{L} h_i(z(t)) \left(\begin{bmatrix} K_{i1} \\ \vdots \\ K_{in} \end{bmatrix} x(t) \right) \\ &= -\sum_{i=1}^{L} h_i(z(t)) \left(\begin{bmatrix} k_{i11} & \cdots & k_{i1m} \\ \vdots & \ddots & \vdots \\ k_{in1} & \cdots & k_{inm} \end{bmatrix} x(t) \right) \\ &= -\sum_{i=1}^{L} h_i(z(t))(K_i x(t)) \end{aligned} \tag{14}$$

where

$$\begin{aligned} u_k(t) &= -\sum_{i=1}^{L} h_i(z(t))[K_{ik}x(t)] \\ &= -\sum_{i=1}^{L} h_i(z(t))([k_{ik1} \quad k_{ik2} \quad \cdots \quad k_{ikm}]x(t)) \end{aligned} \tag{15}$$

and

$$K_i = \begin{bmatrix} K_{i1} \\ \vdots \\ K_{in} \end{bmatrix} = \begin{bmatrix} k_{i11} & \cdots & k_{i1m} \\ \vdots & \ddots & \vdots \\ k_{in1} & \cdots & k_{inm} \end{bmatrix} \tag{16}$$

for $i = 1, 2, \ldots, L$ and $k = 1, 2, \ldots, n$.

Substituting (13) into (9), the fuzzy control system is obtained as follows:

$$\dot{x}(t) = \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(z(t))h_j(z(t)) \times [(A_i - B_iK_j)x(t) + G_iw(t)]. \tag{17}$$

III. Differential Games Via Combined Fuzzy Observer and Control

In practice, state variables are not all available. For this situation, we need to estimate the state vector $x(t)$ from the output $y(t)$ for state feedback control. Suppose the following fuzzy observer is proposed to deal with the state estimation for the nonlinear stochastic system (7).

Observer Rule i :

If $z_1(t)$ is F_{i1} and $\cdots$ and $z_g(t)$ is F_{ig}

Then $\dot{\hat{x}}(t) = [A_i\hat{x}(t) + B_iu(t) + L_i(y(t) - \hat{y}(t))]$ (18)

where L_i is the observer gain for the ith observer rule and is specified later to achieve the desired control purpose and $\hat{y}(t) = \sum_{i=1}^{L} h_i(z(t))(C_i\hat{x}(t))$.

The overall fuzzy observer is represented as follows:

$$\dot{\hat{x}}(t) = \sum_{i=1}^{L} h_i(z(t))\{A_i\hat{x}(t) + B_iu(t) + L_i[y(t) - \hat{y}(t)]\} \tag{19}$$

and the fuzzy observer-based controller is modified by

$$u(t) = -\sum_{i=1}^{L} h_i(z(t))[K_i\hat{x}(t)]. \tag{20}$$

Then, the augmented system is of the following form:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} \left\{ \sum_{i=1}^{L} h_i(z(t))[A_ix(t) + B_iu(t) + G_iw(t)] \right\} \\ \left\{ \sum_{i=1}^{L} h_i(z(t))[A_i\hat{x}(t) + B_iu(t) + L_i(y(t) - \hat{y}(t))] \right\} \end{bmatrix}. \tag{21}$$

Let us denote the estimation error as

$$\tilde{x}(t) = x(t) - \hat{x}(t). \tag{22}$$

By differentiating (22) and after some manipulation, we get

$$\dot{\tilde{x}}(t) = \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(z(t))h_j(z(t)) \times [(A_i - L_iC_j)\tilde{x}(t) + G_iw(t) - L_iD_jv(t)] \tag{23}$$

The design purpose in this study is to specify the fuzzy control in (13) and the fuzzy observer in (19) to achieve noncooperative control performance in (3) and cooperative control performance in (6), respectively.

A. Fuzzy Noncooperative Game Design:

Let us consider the noncooperative performance index in (2) at first. The design purpose of the noncooperative control is to specify the control gain K_{ik} and the estimator gain L_i (for $i = 1, 2, \ldots, L$) such that the individual cost function in (2) is minimized for the noncooperative fuzzy game problem. We now use the well-known relation [14], [15]

$$E\{E\{x(t)\,|\,Y(t)\}\} = E\{x(t)\} \tag{24}$$

to describe (2) in a form more suitable for the analysis to follow where

$$Y(t) = \{y(\tau)\,|\,0 \le \tau \le t)\}.$$

Equations (2) and (24) imply that

$$\begin{aligned} J_k^{NC}(u_k) = \lim_{t_f\to\infty} \frac{1}{t_f} E\Big\{ & E\left\{x^T(t_f)Sx(t_f)\,|\,Y(t)\right\} \\ &+ \int_0^{t_f} E\{x^T(t)Qx(t)\,|\,Y(t)\}dt \Big\} \\ &+ E\left\{ \int_0^{t_f} u_k^T(t)R_ku_k(t)\,dt \right\}. \end{aligned} \tag{25}$$

By the fact that

$$E\left\{x^T(t)Qx(t)\right\} = \bar{x}^T(t)Q\bar{x}(t) + \mathrm{tr}\left\{QE\left\{[x-\bar{x}][x-\bar{x}]^T\right\}\right\} = \bar{x}^T(t)Q\bar{x}(t) + \mathrm{tr}\left\{Q\mathrm{cov}[x(t)]\right\} \quad (26)$$

where $\bar{x}(t) = E\{x(t)\}$, (25) can be rewritten as follows [14], [15]:

$$\begin{aligned} J_k^{NC}(u_k) &= \lim_{t_f\to\infty}\frac{1}{t_f}E\left\{\hat{x}^T(t_f)S\hat{x}(t_f) + \mathrm{tr}(S\Sigma_{\tilde{x}}(t_f)) + \int_0^{t_f}\left[\hat{x}^T(t)Q\hat{x}(t) + u_k^T(t)R_k u_k(t)\right]\,dt\right\} \\ &\quad + \lim_{t_f\to\infty}\frac{1}{t_f}E\left\{\int_0^{t_f}\mathrm{tr}(QE\{[x-\hat{x}] \times [x-\hat{x}]^T\,|Y(t)\})\,dt\right\} \\ &= \lim_{t_f\to\infty}\frac{1}{t_f}E\left\{\hat{x}^T(t_f)S\hat{x}(t_f) + \int_0^{t_f}\left[\hat{x}^T(t)Q\hat{x}(t) + u_k^T(t)R_k u_k(t)\right]\,dt\right\} \\ &\quad + \lim_{t_f\to\infty}\frac{1}{t_f}\left\{\mathrm{tr}(S\Sigma_{\tilde{x}}(t_f)) + \int_0^{t_f}\mathrm{tr}(Q\Sigma_{\tilde{x}}(t))\,dt\right\} \\ &= J_k^{NC1}(u_k) + J_k^{NC2}(\tilde{x}) \end{aligned} \quad (27)$$

where $\hat{x}(t) = E\{x(t)|Y(t)\}$ [8], [15] and $\Sigma_{\tilde{x}}(t) = E\{[x-\hat{x}][x-\hat{x}]^T\} = \mathrm{cov}(\tilde{x}(t),\tilde{x}(t))$ [8]

$$J_k^{NC1}(u_k) = \lim_{t_f\to\infty}\frac{1}{t_f}E\left\{\hat{x}^T(t_f)S\hat{x}(t_f) + \int_0^{t_f}\left(\hat{x}^T(t)Q\hat{x}(t) + u_k^T(t)R_k u_k(t)\right)\,dt\right\} \quad (28)$$

and

$$J_k^{NC2}(\tilde{x}) = \lim_{t_f\to\infty}\frac{1}{t_f}\left\{\mathrm{tr}(S\Sigma_{\tilde{x}}(t_f)) + \int_0^{t_f}\mathrm{tr}(Q\Sigma_{\tilde{x}}(t))\,dt\right\}. \quad (29)$$

Observe that $J_k^{NC2}(\tilde{x})$ depends on the observer gain L_i only. Therefore, the minimization for $J_k^{NC}(u_k)$ can be done by minimizing $J_k^{NC2}(\tilde{x})$ first and then minimizing $J_k^{NC1}(u_k)$.

First, we work on the estimator gain L_i (for $i = 1, 2, \ldots, L$) such that $J_k^{NC2}(\tilde{x})$ (for $k = 1, 2, \ldots, n$) is minimized. The matrix differential equation for $\Sigma_{\tilde{x}}(t)$ can be determined as follows:

$$\dot{\Sigma}_{\tilde{x}}(t) = \sum_{i=1}^{L}\sum_{j=1}^{L}h_i(z(t))h_j(z(t))\left[(A_i - L_iC_j)\Sigma_{\tilde{x}}(t) + \Sigma_{\tilde{x}}(A_i - L_iC_j)^T + G_iG_i^T + L_iD_jD_j^TL_i^T\right]. \quad (30)$$

For a steady-state solution

$$\Sigma_{\tilde{x}}(t) = \mathrm{cov}(\tilde{x}(t),\tilde{x}(t)) = \tilde{P} \quad (31)$$

for all $t \geq 0$ where $\tilde{P}$ is a symmetry positive–semidefinite constant matrix.

Hence

$$\dot{\Sigma}_{\tilde{x}}(t) = 0 \quad (32)$$

and

$$\sum_{i=1}^{L}\sum_{j=1}^{L}h_i(z(t))h_j(z(t))\left[(A_i - L_iC_j)\tilde{P} + \tilde{P}(A_i - L_iC_j)^T + G_iG_i^T + L_iD_jD_j^TL_i^T\right] = 0. \quad (33)$$

Therefore, the optimal performance for $J_k^{NC2}(\tilde{x})$ is obtained as

$$J_k^{NC2}(\tilde{x}) = \mathrm{tr}(Q\tilde{P}). \quad (34)$$

Note that a sufficient condition for (33) implies that

$$(A_i - L_iC_j)\tilde{P} + \tilde{P}(A_i - L_iC_j)^T + G_iG_i^T + L_iD_jD_j^TL_i^T = 0. \quad (35)$$

If the observer parameters are chosen as follows:

$$L_i = \tilde{P}C_i^T$$

we obtain

$$A_i\tilde{P} + \tilde{P}A_i^T - \tilde{P}C_i^TC_j\tilde{P} - \tilde{P}C_j^TC_i\tilde{P} + G_iG_i^T + \tilde{P}C_i^TD_jD_j^TC_i\tilde{P} = 0 \quad (36)$$

for $i, j = 1, 2, \ldots, L$.

Next, we work on the control gain K_{ik} such that $J_k^{NC1}(u_k)$ (for $k = 1, 2, \ldots, n$) is minimized. From the stochastic Hamilton–Jacobi–Bellman equation, we define

$$V_k(\hat{x}(t), t) = E\left\{\hat{x}^T(t_f)S\hat{x}(t_f) + \int_t^{t_f}\left[\hat{x}(\tau)^TQ\hat{x}(\tau) + u_k^T(\tau)R_ku_k(\tau)\right]\,d\tau|\hat{x}(t)\right\}. \quad (37)$$

The stochastic Hamilton–Jacobi–Bellman equation then implies that

$$\frac{\partial V_k}{\partial t} + \hat{x}^TQ\hat{x} + u_k^TR_ku_k + \left(\frac{\partial V_k}{\partial \hat{x}}\right)^T\left\{\sum_{i=1}^{L}h_i(z(t))\left[A_i\hat{x}(t) + B_iu(t)\right]\right\} + \frac{1}{2}\mathrm{tr}\left(\sum_{i=1}^{L}\sum_{j=1}^{L}h_i(z(t))h_j(z(t)) \times \left(L_iD_jD_j^TL_i^T\frac{\partial^2 V_k}{\partial \hat{x}^2}\right)\right) = 0 \quad (38)$$

with endpoint condition $V_k(\hat{x}(t_f), t_f) = \hat{x}^T(t_f)S\hat{x}(t_f)$. Assuming that a solution of the above equation is of the following form:

$$V_k(\hat{x}(t), t) = \hat{x}^T(t)P_k^*(t)\hat{x}(t) + \eta_k(t) \quad (39)$$

For a steady-state solution, let $P_k^*(t) = P_k^*$ for all $t \geq 0$. Substituting (39) into (38), we obtain

$$\begin{aligned} &\sum_{i=1}^{L} h_i(z(t)) \left\{ \hat{x}^T Q \hat{x} + u_k^T R_k u_k + (2P_k^* \hat{x})^T A_i x \right. \\ &\quad \left. + (2P_k^* \hat{x})^T B_i u \right\} \\ &\quad = \sum_{i=1}^{L} h_i(z(t)) \left\{ \hat{x}^T Q \hat{x} + u_k^T R_k u_k + (P_k^* \hat{x})^T A_i \hat{x} \right. \\ &\qquad \left. + (A_i \hat{x})^T P_k^* \hat{x} + (P_k^* \hat{x})^T B_i u + (B_i u)^T P_k^* \hat{x} \right\} \\ &\quad = 0. \end{aligned} \tag{40}$$

Then, (40) can be rewritten as

$$\begin{aligned} &\sum_{i=1}^{L} h_i(z(t)) \Bigg\{ \hat{x}^T Q \hat{x} + u_k^T R_k u_k + (P_k^* \hat{x})^T A_i \hat{x} \\ &\quad + (A_i \hat{x})^T P_k^* \hat{x} + \hat{x}^T P_k^* \left(\sum_{l=1, l \neq k}^{n} (B_{il} u_l^*) + B_{ik} u_k \right) \\ &\quad + \left(\sum_{l=1, l \neq k}^{n} (B_{il} u_l^*) + B_{ik} u_k \right)^T P_k^* \hat{x} \Bigg\} \\ &\quad = \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t)) \left\{ \hat{x}^T \left[Q + P_k^* A_i + A_i^T P_k^* \right. \right. \\ &\qquad \left. \left. - \frac{1}{2} \left(P_k^* B_{ik} R_k^{-1} B_{jk}^T P_k^* + P_k^* B_{jk} R_k^{-1} B_{ik}^T P_k^* \right) \right] \hat{x} \right\} \\ &\qquad + \hat{x}^T P_k^* \left(\sum_{l=1, l \neq k}^{n} (B_{il} u_l^*) \right) + \left(\sum_{l=1, l \neq k}^{n} (B_{il} u_l^*) \right)^T P_k^* \hat{x} \\ &\qquad + u_k^T R_k u_k + \hat{x}^T P_k^* \left(\sum_{i=1}^{L} h_i(z(t)) (B_{ik} u_k) \right) \\ &\qquad + \left(\sum_{i=1}^{L} h_i(z(t)) (B_{ik} u_k) \right)^T P_k^* \hat{x} \\ &\qquad + \frac{1}{2} \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t)) \left[\hat{x}^T (P_k^* B_{ik} R_k^{-1} B_{jk}^T P_k^* \right. \\ &\qquad \left. + P_k^* B_{jk} R_k^{-1} B_{ik}^T P_k^*) \hat{x} \right] = 0. \end{aligned} \tag{41}$$

By the fact that $\sum_{i=1}^{L} h_i(z(t)) = \sum_{j=1}^{L} h_j(z(t))$, (41) can be rewritten as

$$\begin{aligned} &\sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t)) \left\{ \hat{x}^T \left[Q + P_k^* A_i + A_i^T P_k^* \right. \right. \\ &\quad \left. \left. - \frac{1}{2} \left(P_k^* B_{ik} R_k^{-1} B_{jk}^T P_k^* + P_k^* B_{jk} R_k^{-1} B_{ik}^T P_k^* \right) \right] \hat{x} \right\} \\ &\quad + \hat{x}^T P_k^* \left(\sum_{l=1, l \neq k}^{n} (B_{il} u_l^*) \right) \\ &\quad + \left(\sum_{l=1, l \neq k}^{n} (B_{il} u_l^*) \right)^T P_k^* \hat{x} \\ &\quad + \left(R_k u_k + \sum_{i=1}^{L} h_i(z(t)) (B_{ik} P_k^* \hat{x}) \right)^T R_k^{-1} \\ &\quad \times \left(R_k u_k + \sum_{j=1}^{L} h_j(z(t)) (B_{jk} P_k^* \hat{x}) \right) = 0. \end{aligned} \tag{42}$$

Observe that if we let

$$u_k = - \sum_{i=1}^{L} h_i(z(t)) (K_{ik}^* \hat{x}) \tag{43}$$

be denoted as u_k^*, where $K_{ik}^* = R_k^{-1} B_{ik}^T P_k^*$ for $i = 1, 2, \ldots, L$ and $k = 1, 2, \ldots, n$, then by substituting (43) into (42) we obtain

$$\begin{aligned} &\sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t)) h_j(z(t)) \Bigg\{ \hat{x}^T \Bigg[Q + P_k^* A_i + A_i^T P_k^* \\ &\quad - \frac{1}{2} \left(P_k^* B_{ik} R_k^{-1} B_{jk}^T P_k^* + P_k^* B_{jk} R_k^{-1} B_{ik}^T P_k^* \right) \\ &\quad - P_k^* \left(\sum_{l=1, l \neq k}^{n} \left(B_{il} R_l^{-1} B_{jl}^T P_l^* \right) \right) \\ &\quad - \left(\sum_{l=1, l \neq k}^{n} \left(B_{il} R_l^{-1} B_{jl}^T P_l^* \right) \right)^T P_k^* \Bigg] \hat{x} \Bigg\} = 0. \end{aligned} \tag{44}$$

A sufficient condition for (44) implies that

$$\begin{aligned} &P_k^* A_i + A_i^T P_k^* - \frac{1}{2} \left(P_k^* B_{ik} R_k^{-1} B_{jk}^T P_k^* \right. \\ &\quad \left. + P_k^* B_{jk} R_k^{-1} B_{ik}^T P_k^* \right) + Q \\ &\quad - P_k^* \left(\sum_{l=1, l \neq k}^{n} \left(B_{il} R_l^{-1} B_{jl}^T P_l^* \right) \right) \\ &\quad - \left(\sum_{l=1, l \neq k}^{n} \left(B_{il} R_l^{-1} B_{jl}^T P_l^* \right) \right)^T P_k^* = 0. \end{aligned} \tag{45}$$

Therefore, the optimal performance for $J_k^{NC1}(u^*)$ is obtained as

$$\begin{aligned} J_k^{NC1}(u^*) &= \lim_{t_f \to \infty} \frac{1}{t_f} \{ V_k(\hat{x}(0), 0) \} \\ &= \lim_{t_f \to \infty} \frac{1}{t_f} \Bigg\{ \int_0^{t_f} \operatorname{tr} \Bigg(\sum_{i=1}^{L} h_i(z(\tau)) \sum_{j=1}^{L} h_j(z(\tau)) \\ &\qquad \times \left[L_i D_j D_j^T L_i^T P_k^* \right] d\tau \Bigg) \Bigg\}. \end{aligned} \tag{46}$$

Furthermore, the noncooperative optimal performance is obtained as

$$\begin{aligned} J_k^{NC}(u^*(t)) = \operatorname{tr}(Q \tilde{P}) + \lim_{t_f \to \infty} \frac{1}{t_f} \Bigg\{ \int_0^{t_f} \operatorname{tr} \Bigg(\sum_{i=1}^{L} h_i(z(\tau)) \\ \times \sum_{j=1}^{L} h_j(z(\tau)) \left(L_i D_j D_j^T L_i^T P_k^* \right) \Bigg) d\tau \Bigg\}. \end{aligned} \tag{47}$$

In general, it is very difficult to get common solutions $\tilde{P}$ from a set of Riccati-like equations defined in (36). The following suboptimal solution is dealt with this problem. From (34), we get the upper bound of $J_k^{NC2}(\tilde{x})$ as

$$J_k^{NC2}(\tilde{x}) = \mathrm{tr}(Q\tilde{P}) < \mathrm{tr}(Q\hat{P}) \tag{48}$$

for any $\hat{P} = \hat{P}^T > 0$ such that

$$A_i\hat{P} + \hat{P}A_i^T - \hat{P}C_i^T C_j\hat{P} - \hat{P}C_j^T C_i\hat{P} + G_iG_i^T + \hat{P}C_i^T D_j D_j^T C_i\hat{P} < 0. \tag{49}$$

There are many feasible solutions for $\hat{P}$ in (49), a solution which minimizes the upper bound $\mathrm{tr}(Q\hat{P})$ is the suboptimal solution for $\tilde{P}$ in (36).

With $\hat{W} = \hat{P}^{-1}$, we get

$$\hat{W}A_i + A_i^T\hat{W} - C_i^T C_j - C_j^T C_i + \hat{W}G_iG_i^T\hat{W} + C_i^T D_j D_j^T C_i < 0. \tag{50}$$

By the Schur complements [9], (50) is equivalent to the following LMIs:

$$\begin{bmatrix} \begin{pmatrix} \hat{W}A_i + A_i^T\hat{W} - C_i^T C_j \\ -C_j^T C_i + C_i^T D_j D_j^T C_i \end{pmatrix} & \hat{W}G_i \\ G_i^T\hat{W} & -I \end{bmatrix} < 0 \tag{51}$$

for $i, j = 1, 2, \ldots, L$.

In other words, we seek the estimator gain $L_i = \hat{P}C_i^T = \hat{W}^{-1}C_i^T$ (for $i = 1, 2, \ldots, L$) such that $\mathrm{tr}(Q\hat{W}^{-1})$ is minimized subject to (51). Since Q is symmetric positive, there exists a symmetric $\tilde{Q}$ such that $Q = \tilde{Q}\tilde{Q}^T$, i.e., $\tilde{Q} = Q^{1/2}$. We obtain $\mathrm{tr}(Q\hat{W}^{-1}) = \mathrm{tr}(\tilde{Q}\tilde{Q}^T\hat{W}^{-1}) = \mathrm{tr}(\tilde{Q}^T\hat{W}^{-1}\tilde{Q})$. Consider a new matrix variable $\Phi > \tilde{Q}^T\hat{W}^{-1}\tilde{Q}$, then $\mathrm{tr}(\Phi) > \mathrm{tr}(\tilde{Q}^T\hat{W}^{-1}\tilde{Q})$. Also, $\Phi > \tilde{Q}^T\hat{W}^{-1}\tilde{Q}$ is equivalent to

$$\begin{bmatrix} \Phi & \tilde{Q}^T \\ \tilde{Q} & \hat{W} \end{bmatrix} > 0. \tag{52}$$

Therefore, the suboptimal fuzzy observer can be obtained by solving the following EVP:

$$\begin{aligned} &\min_{\hat{W}} \quad \mathrm{tr}(\Phi) \\ &\text{subject to} \quad \hat{W} = \hat{W}^T > 0 \quad \begin{bmatrix} \Phi & \tilde{Q}^T \\ \tilde{Q} & \hat{W} \end{bmatrix} > 0 \\ &\qquad\qquad \text{and (51).} \end{aligned} \tag{53}$$

Similarly, it is very difficult to solve P_k^* from the Riccati-like equations in (45). By the same argument as before, we can take a suboptimal approach for P_k^*. From (46) and (45), we get

$$J_k^{NC1}(u^*) < \lim_{t_f\to\infty}\frac{1}{t_f}\left\{\int_0^{t_f}\mathrm{tr}\left[\sum_{i=1}^L h_i(z(\tau)) \times \sum_{j=1}^L h_j(z(\tau)) \times \left(L_iD_jD_j^TL_i^TP_k\right)\right]d\tau\right\} \tag{54}$$

for any $P_k = P_k^T > 0$ such that

$$P_kA_i + A_i^TP_k - \frac{1}{2}\left(P_kB_{ik}R_k^{-1}B_{jk}^TP_k + P_kB_{jk}R_k^{-1}B_{ik}^TP_k\right) - P_k\left(\sum_{l=1,l\neq k}^n\left(B_{il}R_l^{-1}B_{jl}^TP_l\right)\right) - \left(\sum_{l=1,l\neq k}^n\left(B_{il}R_l^{-1}B_{jl}^TP_l\right)\right)^T P_k + Q < 0 \tag{55}$$

i.e., the suboptimal solution is to find a P_k from all feasible solutions of the inequality constraints in (55) such that the upper bound of $J_k^{NC1}(u^*)$ in the right-hand side of (54) is the smallest one.

With $W_k = P_k^{-1}$, (55) is equivalent to

$$A_iW_k + W_kA_i^T - \frac{1}{2}\left(B_{ik}R_k^{-1}B_{jk}^T + B_{jk}R_k^{-1}B_{ik}^T\right) - \left(\sum_{l=1,l\neq k}^n\left(B_{il}R_l^{-1}B_{jl}^TP_l\right)\right)W_k - W_k\left(\sum_{l=1,l\neq k}^n\left(B_{il}R_l^{-1}B_{jl}^TP_l\right)\right)^T + W_kQW_k < 0. \tag{56}$$

By the Schur complements [9], (56) is equivalent to the following LMIs:

$$\begin{bmatrix} G & W_k \\ W_k & -Q^{-1} \end{bmatrix} < 0 \tag{57}$$

where

$$G = A_iW_k + W_kA_i^T - \frac{1}{2}\left(B_{ik}R_k^{-1}B_{jk}^T + B_{jk}R_k^{-1}B_{ik}^T\right) - \left(\sum_{l=1,l\neq k}^n\left(B_{il}R_l^{-1}B_{jl}^TP_l\right)\right)W_k - W_k\left(\sum_{l=1,l\neq k}^n\left(B_{il}R_l^{-1}B_{jl}^TP_l\right)\right)^T$$

for $i, j = 1, 2, \ldots, L$ and $k, l = 1, 2, \ldots, n$.

Note that $\mathrm{tr}(L_iD_jD_j^TL_i^TW_k^{-1}) = \mathrm{tr}(D_j^TL_i^TW_k^{-1}L_iD_j)$. Consider a new matrix variable $\Psi_k > D_j^TL_i^TW_k^{-1}L_iD_j$, then $\mathrm{tr}(\Psi_k) > \mathrm{tr}(D_j^TL_i^TW_k^{-1}L_iD_j)$. Also, $\Psi_k > D_j^TL_i^TW_k^{-1}L_iD_j$ is equivalent to

$$\begin{bmatrix} \Psi_k & D_j^TL_i^T \\ L_iD_j & W_k \end{bmatrix} > 0. \tag{58}$$

The suboptimal solution W_k can be solved by minimizing the upper bound $\mathrm{tr}(\Psi_k)$ and can be found by solving the following minimization problem:

$$\begin{aligned} &\min_{W_k} \quad \mathrm{tr}(\Psi_k) \\ &\text{subject to} \quad W_k = W_k^T > 0, \text{(58) and (57)} \end{aligned} \tag{59}$$

for $i, j = 1, 2, \ldots, L$ and $k, l = 1, 2, \ldots, n$.

Although the Nash equilibrium is a natural solution concept for the noncooperative game problem, its computation might yet require more effort. Thus, it is natural to investigate iterative scheme for the determination of Nash equilibrium for (59). Consider the following updating algorithm [2]:

$$\begin{aligned} W_1^{(q+1)} &= \min_{W_1} J_1^{NC1}(W_1, P_2^{(q)}, \ldots, P_k^{(q)}, \ldots, P_N^{(q)}) \\ &\vdots \\ W_k^{(q+1)} &= \min_{W_k} J_k^{NC1}(P_1^{(q)}, P_2^{(q)}, \ldots, W_k, \ldots, P_N^{(q)}) \\ &\vdots \\ W_N^{(q+1)} &= \min_{W_N} J_N^{NC1}(P_1^{(q)}, P_2^{(q)}, \ldots, P_k^{(q)}, \ldots, W_N) \end{aligned} \tag{60}$$

where $W_k^{(q+1)} = (P_k^{(q+1)})^{-1}$.

To realize the above updating algorithm, we can solve the following minimization problem iteratively:

$$\begin{aligned} &\min_{W_k^{(q)}} \quad \mathrm{tr}(\Psi_k^{(q)}) \\ &\text{subject to} \quad W_k^{(q)} = (W_k^{(q)})^T > 0, \text{ (58) and (59)} \end{aligned} \tag{61}$$

where $q = 1, 2, \ldots, q_f$ (q is increased by one after each iteration) and P_l in (57) is replaced by $P_l^{(q-1)}$ and $P_l^{(0)}$ is a starting choice for player l ($l = 1, \ldots, n$ and $l \neq k$). The procedure is repeated until all $|\mathrm{tr}(\Psi_k^{(q_f)}) - \mathrm{tr}(\Psi_k^{(q_f-1)})| < \epsilon$ (for $k = 1, \ldots, n$) is satisfied where ϵ is a small value. Therefore, the suboptimal $W_k = W_k^{(q_f)}$ ($k = 1, \ldots, n$). And the initial $P_l^{(0)}$ ($l = 1, \ldots, n$) can be obtained as follows. Note that, with $Y_{lk} = P_l W_k$ (change of variables), (57) is equivalent to

$$\begin{bmatrix} H & W_k \\ W_k & -Q^{-1} \end{bmatrix} < 0 \tag{62}$$

where

$$H = A_i W_k + W_k A_i^T - \frac{1}{2}\left(B_{ik} R_k^{-1} B_{jk}^T + B_{jk} R_k^{-1} B_{ik}^T\right) - \left(\sum_{l=1, l\neq k}^{n} \left(B_{il} R_l^{-1} B_{jl}^T Y_{lk}\right)\right) - \left(\sum_{l=1, l\neq k}^{n} \left(B_{il} R_l^{-1} B_{jl}^T Y_{lk}\right)\right)^T.$$

We can solve the initial P_k^* ($k = 1, \ldots, n$) from the following minimization problem, denoted as $P_k^{(0)}$:

$$\begin{aligned} &\min_{\{W_k^{(0)}, Y_{lk}\}} \quad \mathrm{tr}(\Psi_k) \\ &\text{subject to} \quad W_k^{(0)} = (W_k^{(0)})^T > 0, \text{ (58) and (62)} \end{aligned} \tag{63}$$

and the initial $\mathrm{tr}(\Psi_k)$ obtained from (63) is denoted as $\mathrm{tr}(\Psi_k^{(0)})$ and $P_k^{(0)} = (W_k^{(0)})^{-1}$ for ($k = 1, \ldots, n$). Obviously, the initial solutions $P_k^{(0)}$ for ($k = 1, \ldots, n$) are not Nash equilibrium since they are solutions obtained for that the player k uses his best policy when for each $j \neq k$, player j does not use their best policy, i.e.,

$$J_k^{NC1}\left([u_k^*(t) | u_{-k}(t)]\right) = \min_{u_k} J_k^{NC1}\left([u_k(t) | u_{-k}(t)]\right).$$

Therefore, from (54), we get

$$J_k^{NC1}(u^*) < \mathrm{tr}\left(\Psi_k^{(q_f)}\right). \tag{64}$$

Therefore, by solving the iterative EVP in (61), a suboptimal solution can be obtained. In this situation, the value of $\mathrm{tr}(\Psi_k^{(q_f)})$ approaches its optimal value $J_k^{NC1}(u^*)$.

Based on the analysis above, we obtain the following result.

Theorem 1: In the noncooperative fuzzy differential game with fuzzy observer of (19), if the observer parameters is chosen as

$$L_i^{\mathrm{sub}} = \hat{W}^{-1} C_i^T \tag{65}$$

for $i = 1, 2, \ldots, L$ where $\hat{W} = \hat{W}^T > 0$ is common solution of the EVP in (53) and if the fuzzy control law

$$u_k^{\mathrm{sub}}(t) = -\sum_{i=1}^{L} h_i(z(t))\left(K_{ik}^{\mathrm{sub}} \hat{x}(t)\right) \tag{66}$$

is employed with

$$K_{ik}^{\mathrm{sub}} = R_k^{-1} B_{ik}^T (W_k)^{-1} \tag{67}$$

for $i = 1, 2, \ldots, L$ and $k = 1, 2, \ldots, n$ where $R_k = R_k^T > 0$ is a weighting matrix and $W_k = (W_k)^T > 0$ can be obtained by solving the EVP in (61) then the fuzzy observer (19) is suboptimal and $u_k^{\mathrm{sub}}(t)$ is suboptimal fuzzy control action of the kth player for the noncooperative control performance in (2).

Proof: Based on the previous analysis, the proof is immediately followed. ■

B. Fuzzy Cooperative Game Design

The design purpose of the cooperative control is to specify the control gain K_i and the estimator gain L_i (for $i = 1, 2, \ldots, L$) such that the common cost function in (5) is minimized for the cooperative fuzzy game problem. By the same argument as above, (5) can be rewritten as follows:

$$\begin{aligned} J^C(u) &= \lim_{t_f \to \infty} \frac{1}{t_f} E\left\{\hat{x}^T(t_f) S \hat{x}(t_f) + \int_0^{t_f} \left(\hat{x}^T(t) Q \hat{x}(t) + u^T(t) R u(t)\right) dt\right\} \\ &\quad + \lim_{t_f \to \infty} \frac{1}{t_f}\left\{\mathrm{tr}(S \Sigma_{\tilde{x}}(t_f)) + \int_0^{t_f} \mathrm{tr}(Q \Sigma_{\tilde{x}}(t))\, dt\right\} \\ &= J^{C1}(u) + J^{C2}(\tilde{x}) \end{aligned} \tag{68}$$

where

$$J^{C1}(u) = \lim_{t_f \to \infty} \frac{1}{t_f} E\left\{\hat{x}^T(t_f) S \hat{x}(t_f) + \int_0^{t_f} \left(\hat{x}^T(t) Q \hat{x}(t) + u^T(t) R u(t)\right) dt\right\}$$

is related to the fuzzy controller and

$$J^{C2}(\tilde{x}) = \lim_{t_f \to \infty} \frac{1}{t_f}\left\{\mathrm{tr}(S \Sigma_{\tilde{x}}(t_f)) + \int_0^{t_f} \mathrm{tr}(Q \Sigma_{\tilde{x}}(t))\, dt\right\}$$

is related to fuzzy observer.

For the observer part, it is the same as that in noncooperative case. For the control part, similarly from the stochastic Hamilton–Jacobi–Bellman equation, we define

$$V(\hat{x}(t),t) = E\left\{\hat{x}^T(t_f)S\hat{x}(t_f) + \int_t^{t_f}[\hat{x}(\tau)^TQ\hat{x}(\tau)+u^T(\tau)Ru(\tau)]\,d\tau \mid \hat{x}(t)\right\}. \quad (69)$$

The stochastic Hamilton–Jacobi–Bellman equation then implies that

$$\min_{u(t)}\left\{\frac{\partial V}{\partial t}+\hat{x}^TQ\hat{x}+u^TRu+\left(\frac{\partial V}{\partial \hat{x}}\right)^T \times\left[\sum_{i=1}^L h_i(z(t))(A_i\hat{x}(t)+B_iu(t))\right] + \frac{1}{2}\mathrm{tr}\left(\sum_{i=1}^L h_i(z(t)\sum_{j=1}^L h_j(z(\tau)) \times\left(L_iD_jD_j^TL_i^T\frac{\partial^2V}{\partial\hat{x}^2}\right)\right)=0 \quad (70)$$

with endpoint condition $V(\hat{x}(t_f),t_f)=\hat{x}^T(t_f)S\hat{x}(t_f)$.

By the same argument as that in noncooperative case, a solution of above equation is of the following form:

$$V(\hat{x}(t),t)=\hat{x}^T(t)P^*\hat{x}(t)+\eta(t). \quad (71)$$

By substituting (71) into (70), at steady state, we get

$$\sum_{i=1}^L h_i(z(t))\sum_{j=1}^L h_j(z(t))\left\{\hat{x}^T\left[Q+P^*A_i+A_i^TP^* - \frac{1}{2}\left(P^*B_iR^{-1}B_j^TP^*+P^*B_jR^{-1}B_i^TP^*\right)\right]\hat{x}\right\} + u^TRu+\hat{x}^TP^*\left(\sum_{i=1}^L h_i(z(t))(B_iu)\right) + \left(\sum_{i=1}^L h_i(z(t))(B_iu)\right)^T P^*\hat{x} + \frac{1}{2}\sum_{i=1}^L h_i(z(t))\sum_{j=1}^L h_j(z(t))\left[\hat{x}^T \times\left(P^*B_iR^{-1}B_j^TP^*+P^*B_jR^{-1}B_i^TP^*\right)\hat{x}\right]=0 \quad (72)$$

By the fact that $\sum_{i=1}^L h_i(z(t)) = \sum_{j=1}^L h_j(z(t))$, (72) can be rewritten as

$$\sum_{i=1}^L h_i(z(t))\sum_{j=1}^L h_j(z(t))\left\{\hat{x}^T\left[Q+P^*A_i+A_i^TP^* - \frac{1}{2}\left(P^*B_iR^{-1}B_j^TP^*+P^*B_jR^{-1}B_i^TP^*\right)\right]\hat{x}\right\} + \left(Ru+\sum_{i=1}^L h_i(z(t))(B_iP^*\hat{x})\right)^T \times R^{-1}\left(Ru+\sum_{j=1}^L h_j(z(t)(B_jP^*\hat{x})\right)=0. \quad (73)$$

Observe that if we let

$$u(t)=-\sum_{i=1}^L h_i(z(t))(K_i^*\hat{x}(t)) \quad (74)$$

be denoted as $u^*(t)$, where $K_i^* = R^{-1}B_iP^*$ for $i = 1,2,\ldots,L$, then by substituting (74) into (73) we obtain

$$\sum_{i=1}^L\sum_{j=1}^L h_i(z(t))h_j(z(t))\left\{\hat{x}^T\left[Q-\frac{1}{2} \times\left(P^*B_iR^{-1}B_j^TP^*+P^*B_jR^{-1}B_i^TP^*\right) + P^*A_i+A_i^TP^*\right]\hat{x}\right\}=0. \quad (75)$$

A sufficient condition for (75) implies that

$$P^*A_i+A_i^TP^*-\frac{1}{2}\left(P^*B_iR^{-1}B_j^TP^* + P^*B_jR^{-1}B_i^TP^*\right)+Q=0. \quad (76)$$

Therefore, the cooperative optimal performance for $J^{C1}(u^*)$ is obtained as

$$J^{C1}(u^*)=\lim_{t_f\to\infty}\frac{1}{t_f}\left\{\int_0^{t_f}\mathrm{tr}\left(\sum_{i=1}^L h_i(z(\tau))\sum_{j=1}^L h_j(z(\tau)) \times\left(L_iD_jD_j^TL_i^TP^*\right)\right)d\tau\right\}. \quad (77)$$

Furthermore, the cooperative optimal performance

$$J^C(u^*)=\mathrm{tr}(Q\tilde{P})+\lim_{t_f\to\infty}\frac{1}{t_f}\left\{\int_0^{t_f}\mathrm{tr}\left(\sum_{i=1}^L h_i(z(\tau)) \times\sum_{j=1}^L h_j(z(\tau))\left(L_iD_jD_j^TL_i^TP^*\right)\right)d\tau\right\}.$$

Similarly, it is difficult to solve P^* from the Riccati-like equations in (76). The following suboptimal solution is employed to deal with this problem. Recall that

$$J^{C1}(u^*)<\lim_{t_f\to\infty}\frac{1}{t_f}\left\{\int_0^{t_f}\mathrm{tr}\left(\sum_{i=1}^L h_i(z(\tau)) \times\sum_{j=1}^L h_j(z(\tau))\left(L_iD_jD_j^TL_i^TP\right)\right)d\tau\right\} \quad (78)$$

for any $P=P^T>0$ such that

$$PA_i+A_i^TP-\frac{1}{2}\left(PB_iR^{-1}B_j^TP + PB_jR^{-1}B_i^TP\right)+Q<0. \quad (79)$$

With $W=P^{-1}$, (79) is equivalent to

$$A_iW+WA_i^T-\frac{1}{2}\left(B_iR^{-1}B_j^T+B_jR^{-1}B_i^T\right)+WQW<0. \quad (80)$$

By the Schur complements [9], (80) is equivalent to the following LMIs:

$$\begin{bmatrix} \begin{pmatrix} A_iW + WA_i^T - \frac{1}{2} \\ \times \left(B_iR^{-1}B_j^T + B_jR^{-1}B_i^T\right) \end{pmatrix} & W \\ W & -Q^{-1} \end{bmatrix} < 0 \quad (81)$$

for $i, j = 1, 2, \ldots, L$. Therefore, the upper bound of $J^{C1}(u^*)$ can be found by solving the following EVP:

$$\begin{aligned} &\min_{W} \quad \mathrm{tr}(\Psi) \\ &\text{subject to} \quad W = W^T > 0 \\ &\begin{bmatrix} \Psi & D_j^T L_i^T \\ L_i D_j & W \end{bmatrix} > 0 \\ &\text{and (81)} \end{aligned} \quad (82)$$

for $i, j = 1, 2, \ldots, L$. Therefore, from (78), we get

$$J^{C1}(u^*) < \mathrm{tr}(\Psi). \quad (83)$$

Based on the analysis above, we obtain the following result.

Theorem 2: In the cooperative fuzzy differential game with the fuzzy observer of (19), if the observer parameters are chosen as

$$L_i^{\mathrm{sub}} = \hat{W}^{-1} C_i^T \quad (84)$$

where $\hat{W} = \hat{W}^T > 0$ is a common solution of the EVP in (53) and suppose the fuzzy control law

$$u^{\mathrm{sub}}(t) = -\sum_{i=1}^{L} h_i(z(t))[K_i^{\mathrm{sub}} \hat{x}(t)] \quad (85)$$

is employed with

$$K_i^{\mathrm{sub}} = R^{-1} B_i^T W^{-1} \quad (86)$$

for $i = 1, 2, \ldots, L$, where $R = R^T > 0$ is a weighting matrix and $W = W^T > 0$ can be obtained by solving the EVP in (82) then the fuzzy estimator (19) is suboptimal and $u^{\mathrm{sub}}(t)$ in (85) is the suboptimal fuzzy control for the cooperative control performance in (5).

Proof: Based on the analysis of suboptimal approach, the proof is immediately followed. ■

Based on the above analysis, the control design for the suboptimal noncooperative or cooperative game problems with fuzzy observer are summarized as the following design procedure.

Design Procedure:

Step 1) Select membership function and construct fuzzy model to approximate the nonlinear system.

Step 2) Select weighting matrices Q and R_k for the noncooperative game (or Q and R for the cooperative game) according to the design objective.

Step 3) Solve the EVP in (53) for the noncooperative suboptimal fuzzy observer (or solve the EVP in (53) for the cooperative suboptimal fuzzy observer) to obtain $\hat{W}$.

Step 4) Solve the minimization problem in (59) for noncooperative game to obtain W_k (or solve the EVP in (82) for cooperative game to obtain W).

Step 5) Obtain fuzzy observer parameters L_i^{sub} from (65) for noncooperative case (or from (84) for cooperative case) and then construct the fuzzy observer in (19).

Step 6) Obtain control parameters K_{ik}^{sub} from (67) for noncooperative case (or K_i^{sub} from (86) for cooperative case) and then obtain the fuzzy control rule of (20).

Remark 1: The fuzzy observer for the noncooperative and cooperative fuzzy game problems is the same. ■

IV. Simulation Examples

We consider a three-machine interconnected power system as follows [16]:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + w_1(t) \\ \dot{x}_2(t) &= f_2(t) + w_2(t) \\ \dot{x}_3(t) &= x_4(t) + w_3(t) \\ \dot{x}_4(t) &= f_4(t) + w_4(t) \\ \dot{x}_5(t) &= x_6(t) + w_5(t) \\ \dot{x}_6(t) &= f_6(t) + w_6(t) \\ y(t) &= \begin{bmatrix} x_1(t) \\ x_3(t) \\ x_5(t) \end{bmatrix} + \begin{bmatrix} v_1(t) \\ v_3(t) \\ v_5(t) \end{bmatrix} \end{aligned} \quad (87)$$

where

$$\begin{aligned} f_2(t) = &-\frac{D_1}{M_1}x_2(t) + \frac{1}{M_1}u_1(t) + \frac{E_1E_2Y_{12}}{M_1} \\ &\times [\cos(\delta_{12}^0 - \theta_{12}) - \cos(\theta_{12} - x_1(t) \\ &+ x_3(t))] + \frac{E_1E_3Y_{13}}{M_1} \\ &\times [\cos(\delta_{13}^0 - \theta_{13}) \\ &- \cos(\theta_{13} - x_1(t) + x_5(t))] \\ f_4(t) = &-\frac{D_2}{M_2}x_4(t) + \frac{1}{M_2}u_2(t) + \frac{E_2E_1Y_{21}}{M_2}[\cos(\delta_{21}^0 - \theta_{21}) \\ &- \cos(\theta_{21} - x_3(t) + x_1(t))] + \frac{E_2E_3Y_{23}}{M_2} \\ &\times [\cos(\delta_{23}^0 - \theta_{23}) - \cos(\theta_{23} - x_3(t) + x_5(t))] \\ f_6(t) = &-\frac{D_3}{M_3}x_6(t) + \frac{1}{M_3}u_3(t) + \frac{E_3E_1Y_{31}}{M_3}[\cos(\delta_{31}^0 - \theta_{31}) \\ &- \cos(\theta_{31} - x_5(t) + x_1(t))] + \frac{E_3E_2Y_{32}}{M_3} \\ &\times [\cos(\delta_{32}^0 - \theta_{32}) - \cos(\theta_{32} - x_5(t) + x_3(t))] \end{aligned}$$

where $x_1(t) = \delta_1(t), x_3(t) = \delta_2(t)$ and $x_5(t) = \delta_3(t)$ are the absolute rotor angle of the 1st, 2nd and 3rd machine, respectively, and assume that $x_i(t) \in [-\pi/2, \pi/2]$ for $i = 1, 3, 5; x_2(t) = \dot{\delta}_1(t), x_4(t) = \dot{\delta}_2(t)$ and $x_6(t) = \dot{\delta}_3(t)$ are the absolute angular velocity of the 1st, 2nd and 3rd machine, respectively; M_i is the inertia coefficient; D_i is the damping coefficient; E_i is the internal voltage; Y_{ij} is the modulus of the transfer admittance between the ith and jth machines; θ_{ij} is the phase angle of the transfer admittance between the ith and jth machines; for $i, j = 1, 2, 3$.

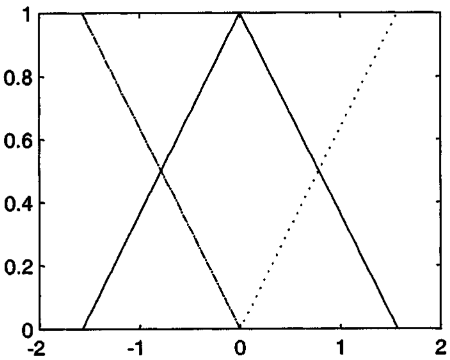

Fig. 1. Membership functions and fuzzy sets for $x_i \in [-\pi/2, \pi/2] (i = 1, 3, 5)$.

At the steady state of multimachine systems, the mechanical power delivered to the ith machine is equal to electrical power delivered to the network and the synchronization is achieved. In this situation, $u_i(t) = 0$. However, some initial conditions and disturbances due to short circuit and sudden increment of power load may occur in the interconnected power system. The control $u_i(t)$ must be employed to eliminate the transient phenomenon of multimachine system or the synchronization will be destroyed.

We assume the three-machine interconnected systems' parameters as follows [16]:

$$\begin{aligned} &E_1 = 1.017 \quad E_2 = 1.005 \quad \text{and} \quad E_3 = 1.033 \\ &M_1 = 1.03 \quad M_2 = 1.25 \quad \text{and} \quad M_3 = 1.4 \\ &D_1 = 0.8 \quad D_2 = 1.2 \quad \text{and} \quad D_3 = 1.1 \\ &Y_{12} = 1.98 \quad Y_{13} = 1.14 \quad \text{and} \quad Y_{23} = 1.06 \\ &\theta_{12} = 1.5 \quad \theta_{13} = 1.55 \quad \text{and} \quad \theta_{23} = 1.56 \\ &\delta_{12}^0 = 0.5 \quad \delta_{13}^0 = 0.18 \quad \text{and} \quad \delta_{23}^0 = -0.32 \end{aligned}$$

and $w(t) = [w_1(t), w_2(t), w_3(t), w_4(t), w_5(t), w_6(t)]^T$ and $v(t) = [v_1(t), v_3(t), v_5(t)]^T$ are external disturbance and measurement noise, respectively, with $E\{w(t), w(\tau)\} = I\delta(t - \tau)$ and $E\{v(t), v(\tau)\} = I\delta(t - \tau)$.

Example 1: In the above three-machine interconnected power system, in order to achieve synchronization, each machine designs a fuzzy controller to minimize its individual performance in (2) to eliminate the transient behavior due to short circuit and sudden changes of power load. This is a noncooperative differential game design problem. Now, following the **Design Procedure** in the previous section, the suboptimal control policy for the noncooperative game using suboptimal fuzzy observer is determined by the following steps: Step 1): To use the fuzzy control approach, we must have a fuzzy model which represents the dynamics of the nonlinear plant. In these examples, we specify three fuzzy sets for x_1, x_3 and x_5, respectively, to construct the fuzzy model. This makes twenty-seven $(3 \times 3 \times 3)$ fuzzy rules for the example where membership functions and fuzzy sets are shown in Fig. 1. Step 2): Select $Q = I$ and $R_k = 0.001$ for $k = 1, 2, 3$. Step 3): Solve the EVP in (53) for the suboptimal fuzzy observer to obtain $\hat{W}$. Step 4): Solve the iterative EVP in (61) for noncooperative game to obtain W_k (for $k = 1, 2, 3$). The updating process stops after four iterations with $\epsilon = 0.001$ (refer to Fig. 2). Step 5): Construct the suboptimal fuzzy observer. Step 6): Construct the noncooperative fuzzy control law.

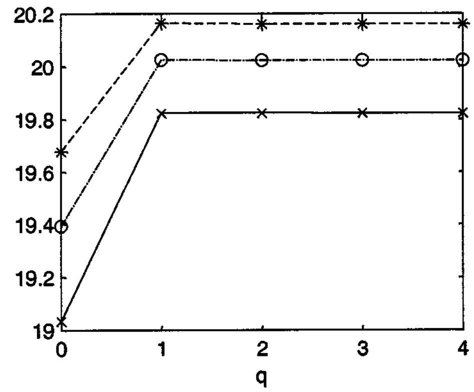

Fig. 2. Iterations of $\text{tr}(\Psi^{(q)}{}_1)$—denoted by "×," $\text{tr}(\Psi^{(q)}{}_2)$—denoted by "$o$," and $\text{tr}(\Psi^{(q)}{}_3)$—denoted by "∗."

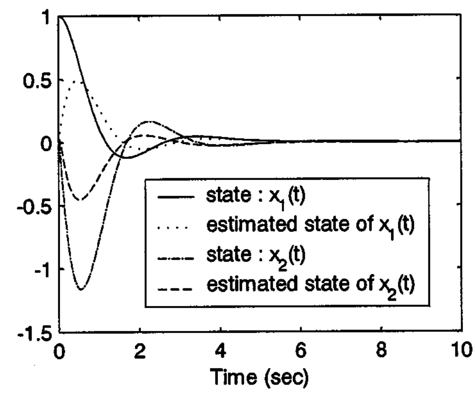

Fig. 3. The trajectories of the states x_1 and x_2 including estimated states $\hat{x}_1$ and $\hat{x}_2$ (noncooperative case).

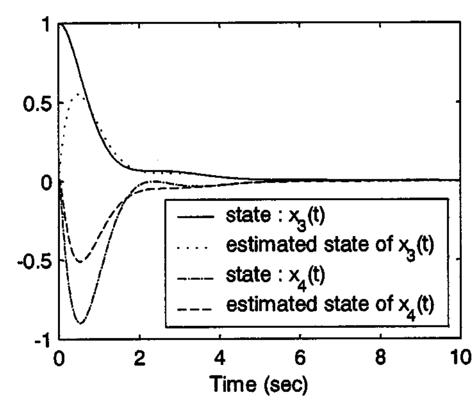

Fig. 4. The trajectories of the states x_3 and x_4 including estimated states $\hat{x}_3$ and $\hat{x}_4$ (noncooperative case).

Figs. 3–6 present the simulation results for the noncooperative fuzzy control. The initial condition is assumed to be $(x_1(0), x_2(0), x_3(0), x_4(0), x_5(0), x_6(0), \hat{x}_1(0), \hat{x}_2(0), \hat{x}_3(0), \hat{x}_4(0), \hat{x}_5(0), \hat{x}_6(0))^T = (1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0)^T$. The external disturbance $w(t)$ and measurement noise $v(t)$ are assumed to be white noise with identity power spectrum.

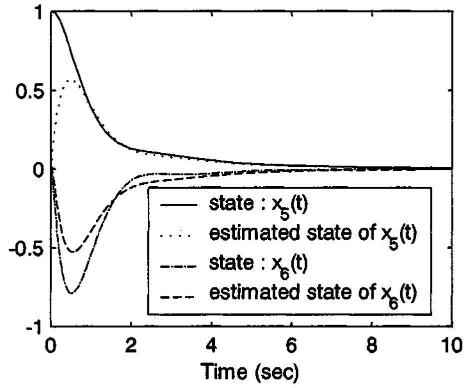

Fig. 5. The trajectories of the states x_5 and x_6 including estimated states $\hat{x}_5$ and $\hat{x}_6$ (noncooperative case).

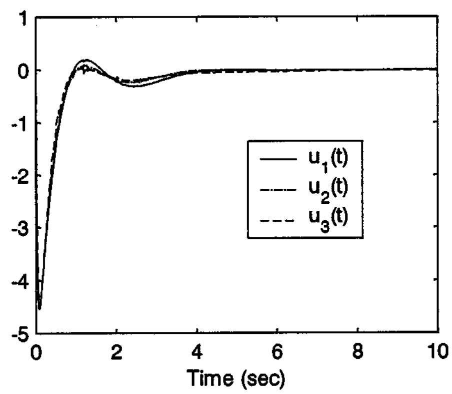

Fig. 6. The noncooperative control inputs.

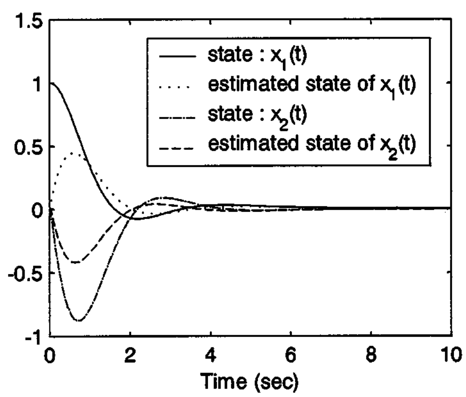

Fig. 7. The trajectories of the states x_1 and x_2 including estimated states $\hat{x}_1$ and $\hat{x}_2$ (cooperative case).

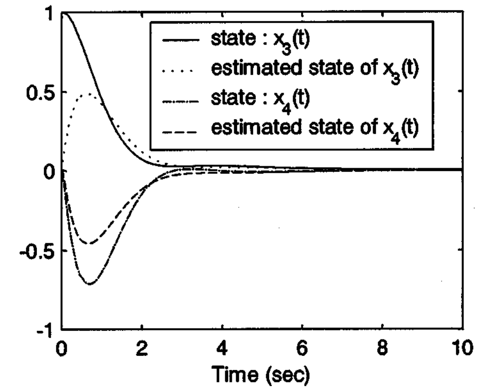

Fig. 8. The trajectories of the states x_3 and x_4 including estimated states $\hat{x}_3$ and $\hat{x}_4$ (cooperative case).

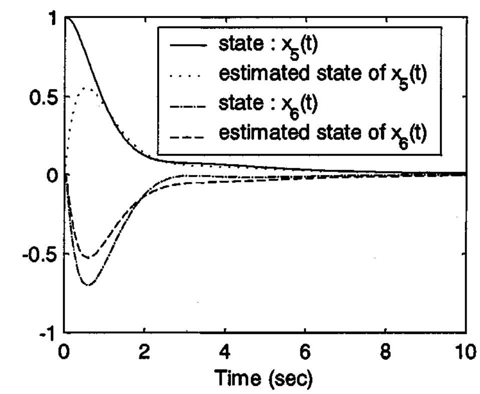

Fig. 9. The trajectories of the states x_5 and x_6 including estimated states $\hat{x}_5$ and $\hat{x}_6$ (cooperative case).

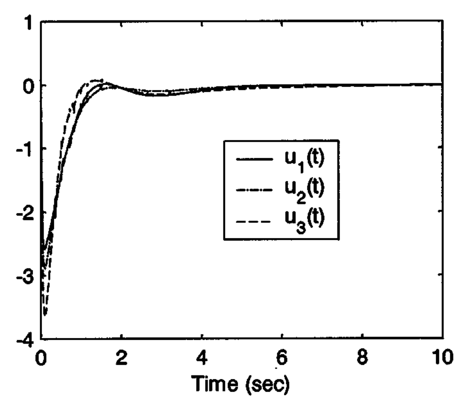

Fig. 10. The cooperative control inputs.

Fig. 3 shows the trajectories of the states $x_1(t)$, and $x_2(t)$ (including the estimated states $\hat{x}_1(t)$, and $\hat{x}_2(t)$). Fig. 4 shows the trajectories of the states $x_3(t)$, and $x_4(t)$ (including the estimated states $\hat{x}_3(t)$, and $\hat{x}_4(t)$). Fig. 5 shows the trajectories of the states $x_5(t)$, and $x_6(t)$ (including the estimated states $\hat{x}_5(t)$ and $\hat{x}_6(t)$). The control inputs are presented in Fig. 6.

Example 2: In the above three-machine interconnected power system, suppose all three machines cooperate to design their fuzzy controller to compensate its transient behavior to achieve synchronization by minimizing the common control performance (5). This is a cooperative differential game design problem. The suboptimal control policy for the cooperative game using suboptimal fuzzy observer can be determined by the same procedure as **Example 1** with $Q = I$ and $R = 0.001 \times I$.

Figs. 7–11 present the simulation results for the suboptimal fuzzy observer-based cooperative fuzzy control. Fig. 7 shows the trajectories of the states $x_1(t)$, and $x_2(t)$ (including the estimated states $\hat{x}_1(t)$, and $\hat{x}_2(t)$). Fig. 8 shows the trajectories of the states $x_3(t)$, and $x_4(t)$ (including the estimated states $\hat{x}_3(t)$, and $\hat{x}_4(t)$). Fig. 9 shows the trajectories of the states $x_5(t)$, and $x_6(t)$ (including the estimated states $\hat{x}_5(t)$, and $\hat{x}_6(t)$). The control inputs are presented in Fig. 10. The simulation results show

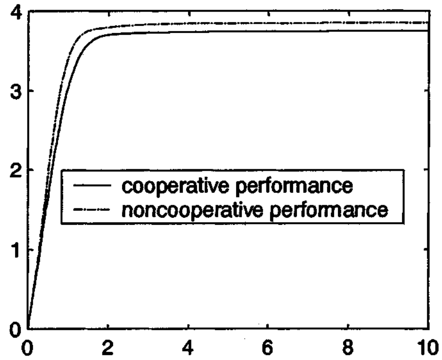

Fig. 11. The plots of $\int_0^{t_f}(x^TQx + u^TRu)dt$ for cooperative case and noncooperative case.

that the cooperative fuzzy controller yields better performance which is shown in Fig. 11 since it features the property that no other joint decision of the players can improve the performance of at least one of them, without degrading the performance of others.

V. Conclusion

In this paper, both noncooperative and cooperative fuzzy differential game problems of nonlinear stochastic systems are solved using suboptimal approach. Based on the fuzzy model and the suboptimal approach, the outcome of the noncooperative and cooperative fuzzy differential game problems is parameterized in terms of an EVP. A suboptimal fuzzy observer has also been introduced in the case that the state variables are unavailable. Based on the separation method, the solution of the observer-based fuzzy differential game problems is also parameterized in terms of an EVP. The proposed design methods are very simple and more efficient than other control methods to deal with the general n-person nonlinear differential game problems. Simulation examples indicate that the desired performance of noncooperative and cooperative game control designs for nonlinear interconnected power systems can be achieved using the proposed methods. Hence, the proposed methods are suitable for solving the practical differential game problems in real applications.

References

[1] M. Jamshidi, *Large-Scale Systems-Modeling and Control*. Amsterdam, The Netherlands: North-Holland, 1982.
[2] T. Basar and G. J. Olsder, *Dynamic Noncooperative Game Theory*. New York: Academic, 1982.
[3] E. Altman and T. Basar, "Multiuser rate-based flow control," *IEEE Trans. Commun.*, vol. 46, July 1998.
[4] B. D. O. Anderson and J. B. Moore, *Optimal Control: Linear Quadratic Methods*. Upper Saddle River, NJ: Prentice-Hall, 1990.
[5] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modeling and control," *IEEE Trans. System, Man, Cybern.*, vol. SMC-15, pp. 116–132, 1985.
[6] K. Tanaka, T. Ikeda, and H. O. Wang, "Fuzzy regulators and fuzzy observers: Relaxed stability conditions and LMI-based designs," *IEEE Trans. Fuzzy Syst.*, vol. 6, Apr. 1998.
[7] B. S. Chen, C. S. Tseng, and H. J. Uang, "Robustness design of nonlinear dynamic systems via fuzzy linear control," *IEEE Trans. Fuzzy Syst.*, vol. 7, pp. 571–585, Oct. 1999.
[8] A. P. Sage and J. L. Melsa, *Estimation Theory with Application to Communication and Control*. New York: McGraw-Hill, 1971.
[9] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM, 1994.
[10] C. Scherer and P. Gahinet, "Multiobjective output-feedback control via LMI optimization," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 896–911, July 1997.
[11] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali, *LMI Control Toolbox*. Natick, MA: The Math Works, 1995.
[12] G. C. Hwang and S. C. Lin, "A stability approach to fuzzy control design for nonlinear systems," *Fuzzy Sets Syst.*, vol. 48, pp. 279–287, 1992.
[13] J. J. Buckley, "Theory of fuzzy controller: An introduction," *Fuzzy Sets Syst.*, vol. 51, pp. 249–258, 1992.
[14] E. Tse, "On the optimal control of stochastic linear systems," *IEEE Trans. Automat. Contr.*, vol. AC-16, pp. 776–784, Dec. 1971.
[15] A. P. Sage and C. C. White III, *Optimal Systems Control*, 2nd ed. Upper Saddle River, NJ: Prentice-Hall, 1977.
[16] D. D. Siljak, *Large-Scale Dynamic Systems-Stability and Structure*. Amsterdam, The Netherlands: North-Holland, 1978.