Chapter 13
The Laplace Transform in Circuit Analysis

13.1	Circuit Elements in the s Domain
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13.6	The Transfer Function and the Convolution Integral
13.7	The Transfer Function and the Steady-State Sinusoidal Response
13.8	The Impulse Function in Circuit Analysis

Key points

- How to represent the initial energy of L, C in the s-domain?
- Why the functional forms of natural and steady-state responses are determined by the poles of transfer function $H(s)$ and excitation source $X(s)$, respectively?
- Why the output of an LTI circuit is the convolution of the input and impulse response? How to interpret the memory of a circuit by convolution?

Section 13.1
Circuit Elements in the s Domain

1. Equivalent elements of R, L, C

A resistor in the s domain

- iv-relation in the time domain

$$v(t) = R \cdot i(t).$$

- By operational Laplace transform:

$$L\{v(t)\} = L\{R \cdot i(t)\} = R \cdot L\{i(t)\} \ ,$$
$$\Rightarrow V(s) = R \cdot I(s).$$

- Physical units: $V(s)$ in volt-seconds, $I(s)$ in ampere-seconds.

An inductor in the s domain

- *iv*-relation in the time domain

$$v(t) = L \cdot \frac{d}{dt} i(t).$$

- By operational Laplace transform:

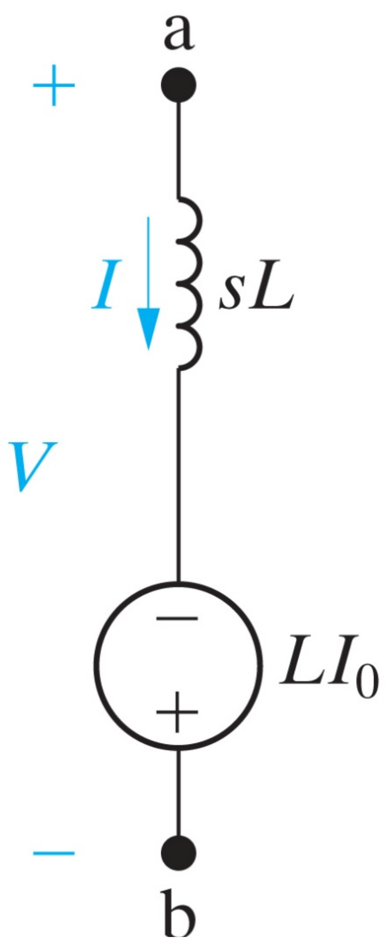
$$\mathcal{L}\{v(t)\} = \mathcal{L}\{L \cdot i'(t)\} = L \cdot \mathcal{L}\{i'(t)\} \ ,$$

$$\Rightarrow V(s) = L \cdot [sI(s) - I_0] = sL \cdot I(s) - LI_0.$$

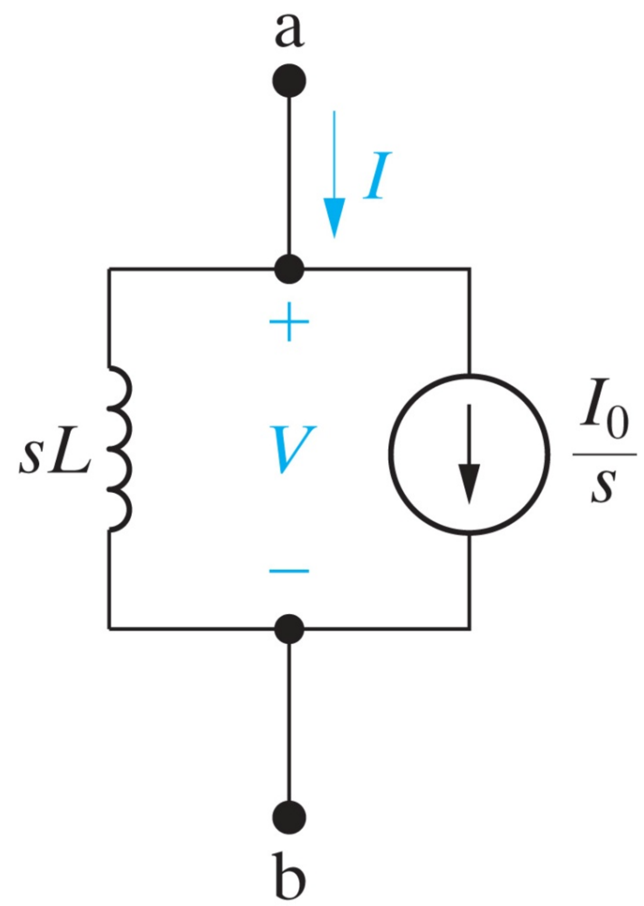
initial current

Equivalent circuit of an inductor

- Series equivalent:
- Parallel equivalent:



Thévenin → Norton



A capacitor in the s domain

- *iv*-relation in the time domain

$$i(t) = C \cdot \frac{d}{dt} v(t).$$

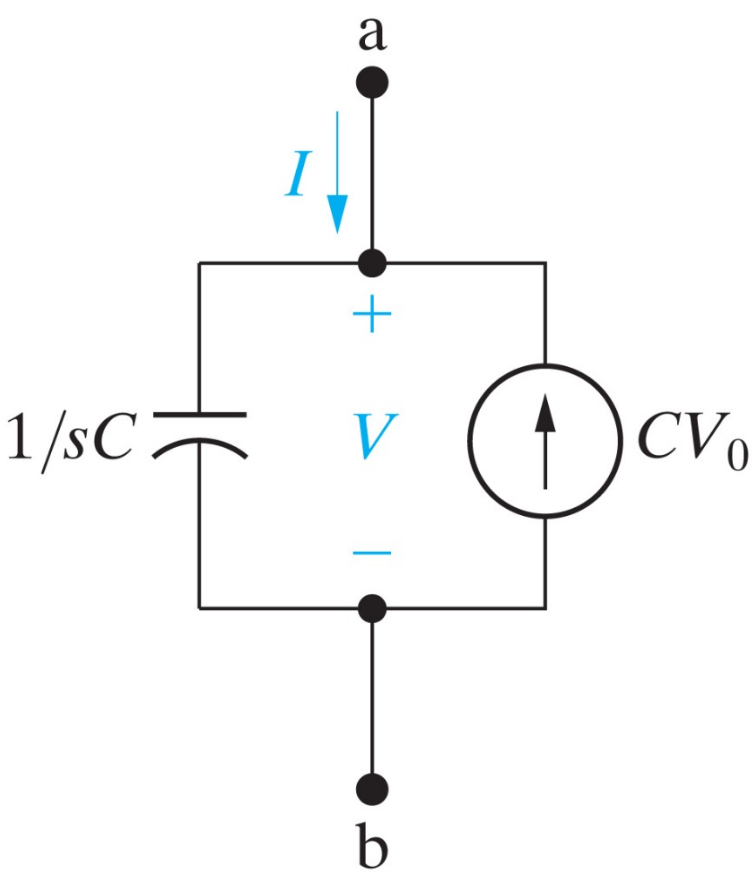
- By operational Laplace transform:

$$L\{i(t)\} = L\{C \cdot v'(t)\} = C \cdot L\{v'(t)\} \ ,$$
$$\Rightarrow I(s) = C \cdot [sV(s) - V_0] = sC \cdot V(s) - CV_0.$$

initial voltage

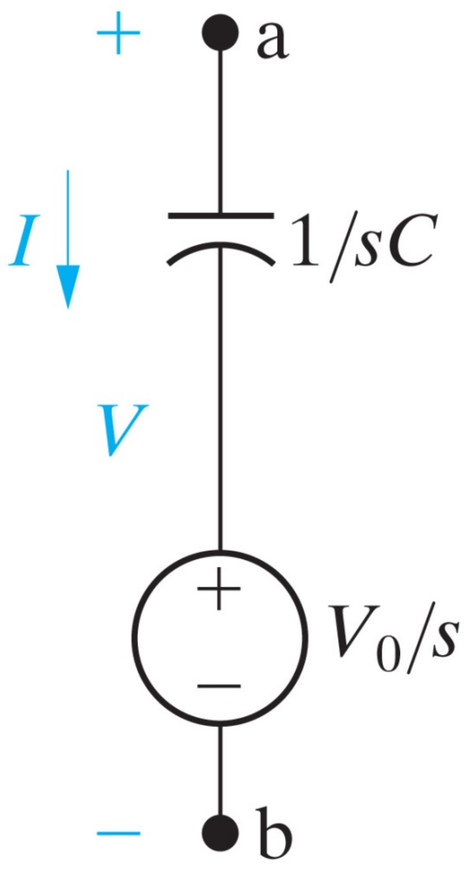
Equivalent circuit of a capacitor

- Parallel equivalent:



Norton → Thévenin

- Series equivalent:



Section 13.2, 13.3 Circuit Analysis in the s Domain

1. Procedures
2. Nature response of RC circuit
3. Step response of RLC circuit
4. Sinusoidal source
5. MCM
6. Superposition

How to analyze a circuit in the s-domain?

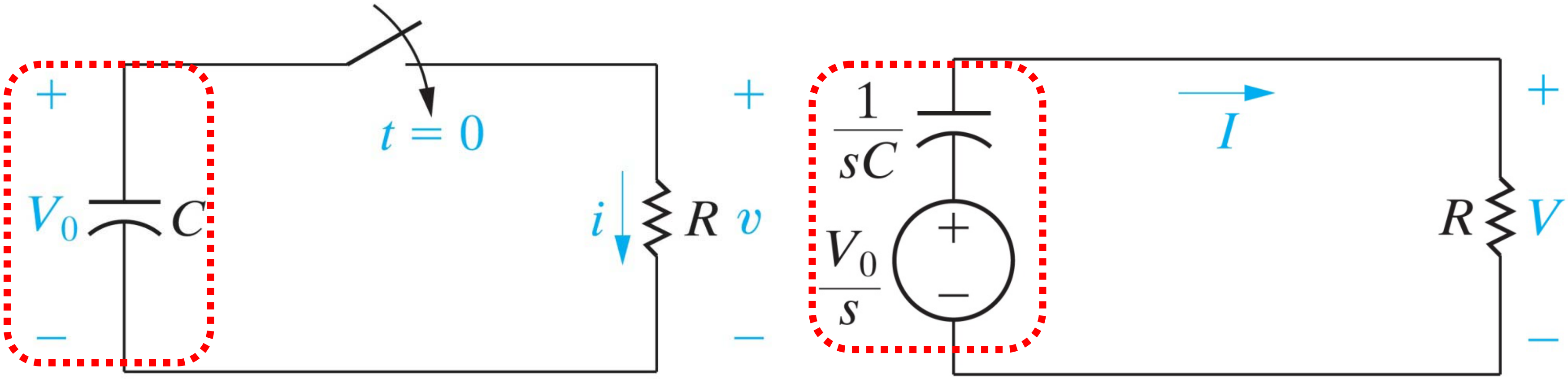
1. Replacing each circuit element with its s-domain equivalent. The initial energy in L or C is taken into account by adding independent source in series or parallel with the element impedance.
2. Writing & solving algebraic equations by the same circuit analysis techniques developed for resistive networks.
3. Obtaining the t-domain solutions by inverse Laplace transform.

Why to operate in the s-domain?

- It is convenient in solving transient responses of linear, lumped parameter circuits, for the initial conditions have been incorporated into the equivalent circuit.
- It is also useful for circuits with multiple essential nodes and meshes, for the simultaneous ODEs have been reduced to simultaneous algebraic equations.
- It can correctly predict the impulsive response, which is more difficult in the t-domain (Sec. 13.8).

Nature response of an RC circuit (1)

- Q: $i(t),\ v(t) = ?$



- Replacing the charged capacitor by a Thévenin equivalent circuit in the s-domain.
- KVL, $\Rightarrow$ algebraic equation & solution of $I(s)$:

$$\frac{V_0}{s} = \frac{I}{sC} + IR, \ \Rightarrow I(s) = \frac{CV_0}{1+RCs} = \frac{V_0/R}{s+(RC)^{-1}}.$$

Nature response of an RC circuit (2)

- The t-domain solution is obtained by inverse Laplace transform:

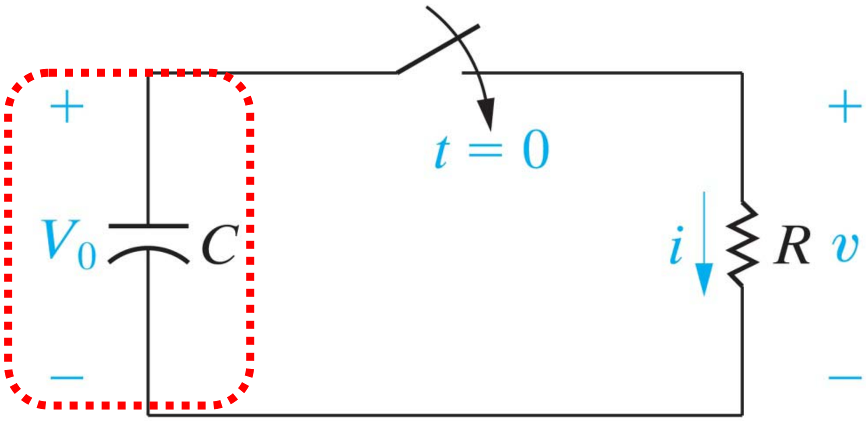
$$i(t) = L^{-1}\left\{\frac{V_0/R}{s+(RC)^{-1}}\right\} = \frac{V_0}{R}e^{-t/(RC)}L^{-1}\left\{\frac{1}{s}\right\}$$

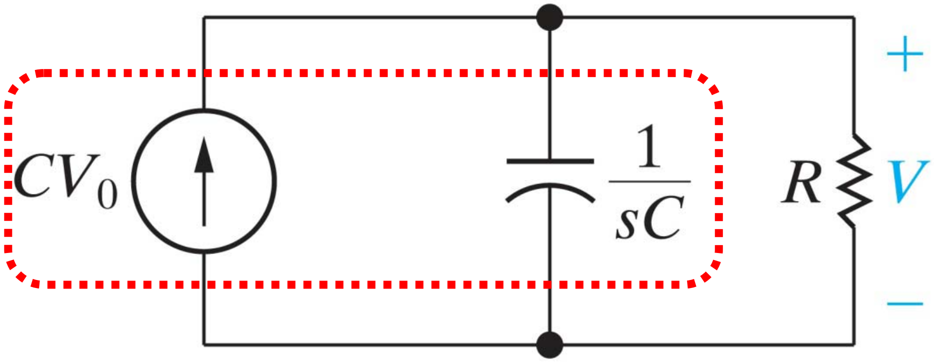
$$= \frac{V_0}{R}e^{-t/(RC)}u(t).$$

- $i(0^+) = V_0/R$, which is true for $v_C(0^+) = v_C(0^-) = V_0$.
- $i(\infty) = 0$, which is true for capacitor becomes open (no loop current) in steady state.

Nature response of an RC circuit (3)

- To directly solve $v(t)$, replacing the charged capacitor by a Norton equivalent in the s-domain.





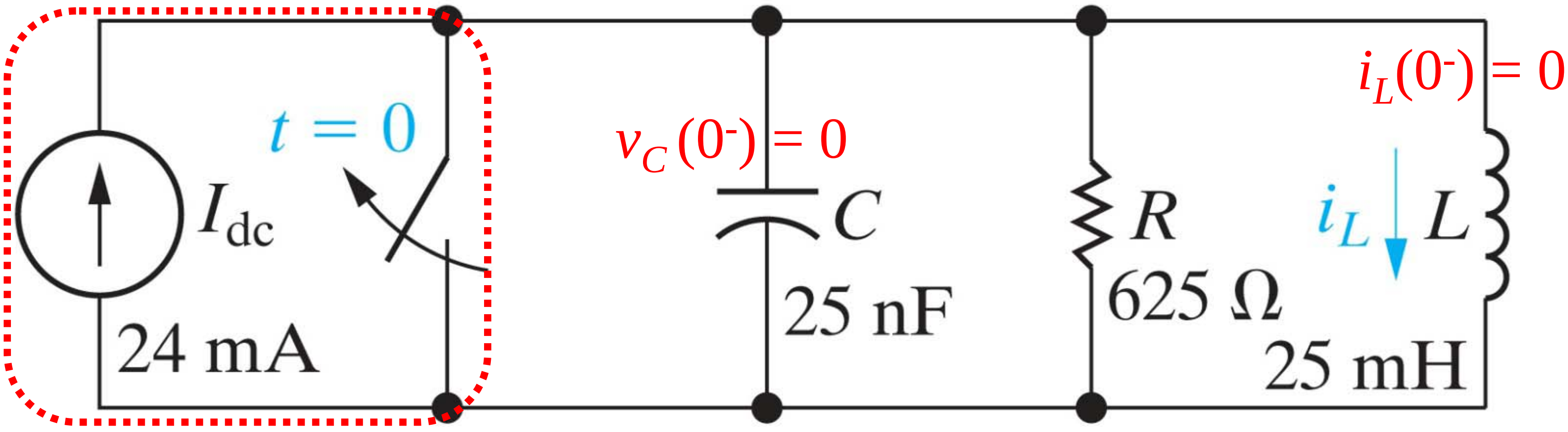
- Solve $V(s)$, perform inverse Laplace transform:

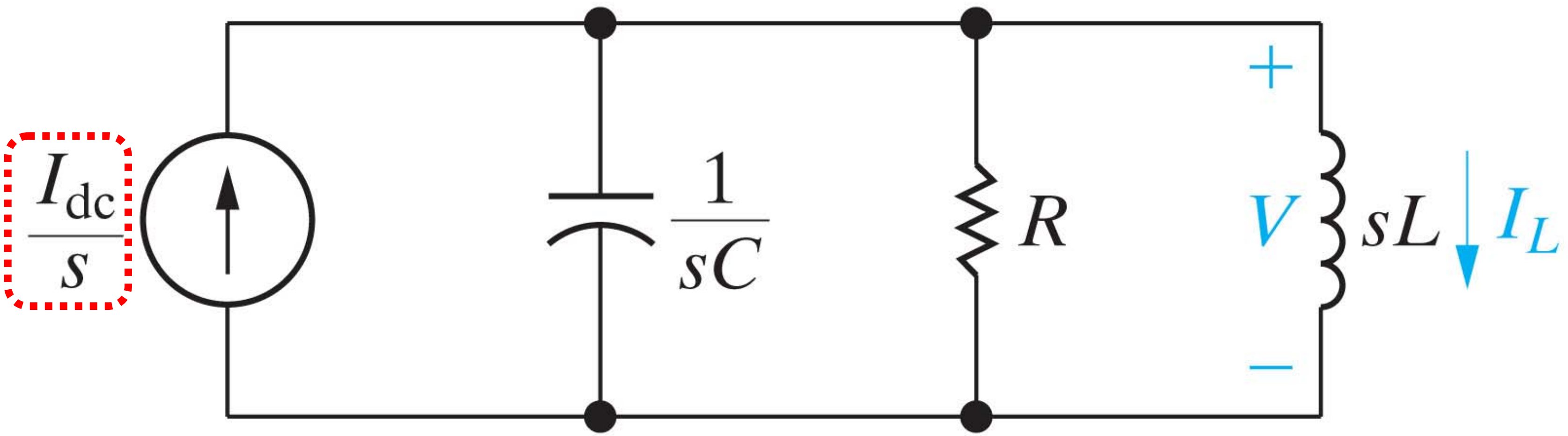
$$CV_0 = sCV + \frac{V}{R}, \Rightarrow V(s) = \frac{V_0}{s + (RC)^{-1}}.$$

$$\Rightarrow v(t) = L^{-1}\left\{V_0 \Big/ \left[s + (RC)^{-1}\right]\right\} = V_0 e^{-t/(RC)} u(t) = Ri(t).$$

Step response of a parallel RLC (1)

- Q: $i_L(t) = ?$





Step response of a parallel RLC (2)

- KCL, $\Rightarrow$ algebraic equation & solution of $V(s)$:

$$\frac{I_{dc}}{s} = sCV + \frac{V}{R} + \frac{V}{sL}, \ \Rightarrow V(s) = \frac{I_{dc}/C}{s^2 + (RC)^{-1}s + (LC)^{-1}}.$$

- Solve $I_L(s)$:

$$I_L(s) = \frac{V(s)}{sL} = \frac{I_{dc}(LC)^{-1}}{s\left[s^2 + (RC)^{-1}s + (LC)^{-1}\right]}$$

$$= \frac{3.84\times 10^7}{s\left[s^2 + (6.4\times 10^4)s + (1.6\times 10^9)\right]}.$$

Step response of a parallel RLC (3)

- Perform partial fraction expansion and inverse Laplace transform:

$$I_L(s) = \frac{24}{s} + \frac{20\angle 127^\circ}{s-(-32\text{k}+j24\text{k})} + \frac{20\angle -127^\circ}{s-(-32\text{k}-j24\text{k})}\,(\text{mA}\cdot\text{s}).$$

$$\begin{aligned} i_L(t) &= 24u(t) + \left[20e^{j127^\circ}e^{-(32\text{k})t}e^{j(24\text{k})t}u(t) + c.c.\right] \\ &= \left\{24 + 40e^{-(32\text{k})t}\cos\left[(24\text{k})t + 127^\circ\right]\right\}\ u(t)\,(\text{mA}) \\ &= \left\{24 - e^{-(32\text{k})t}\left[24\cos(24\text{k})t - 32\sin(24\text{k})t\right]\right\}\ u(t)\,(\text{mA}). \end{aligned}$$

Transient response due to a sinusoidal source (1)

- For a parallel RLC circuit, replace the current source by a sinusoidal one: $i_g(t) = I_m \cos\omega t \cdot u(t)$. The algebraic equation changes:

$$sCV + \frac{V}{R} + \frac{V}{sL} = \boxed{I_g = \frac{sI_m}{s^2+\omega^2}},$$

$$\Rightarrow V(s) = \frac{(I_m/C)s^2}{(s^2+\omega^2)\left[s^2+(RC)^{-1}s+(LC)^{-1}\right]},$$

$$\Rightarrow I_L(s) = \frac{V}{sL} = \frac{I_m(LC)^{-1}s}{(s^2+\omega^2)\left[s^2+(RC)^{-1}s+(LC)^{-1}\right]}.$$

Transient response due to a sinusoidal source (2)

- Perform partial fraction expansion and inverse Laplace transform:

$$I_L(s) = \frac{K_1}{s - j\omega} + \frac{K_1^*}{s + j\omega} + \frac{K_2}{s - (-\alpha + j\beta)} + \frac{K_2^*}{s - (-\alpha - j\beta)}.$$

Driving frequency

Neper frequency

Damped frequency

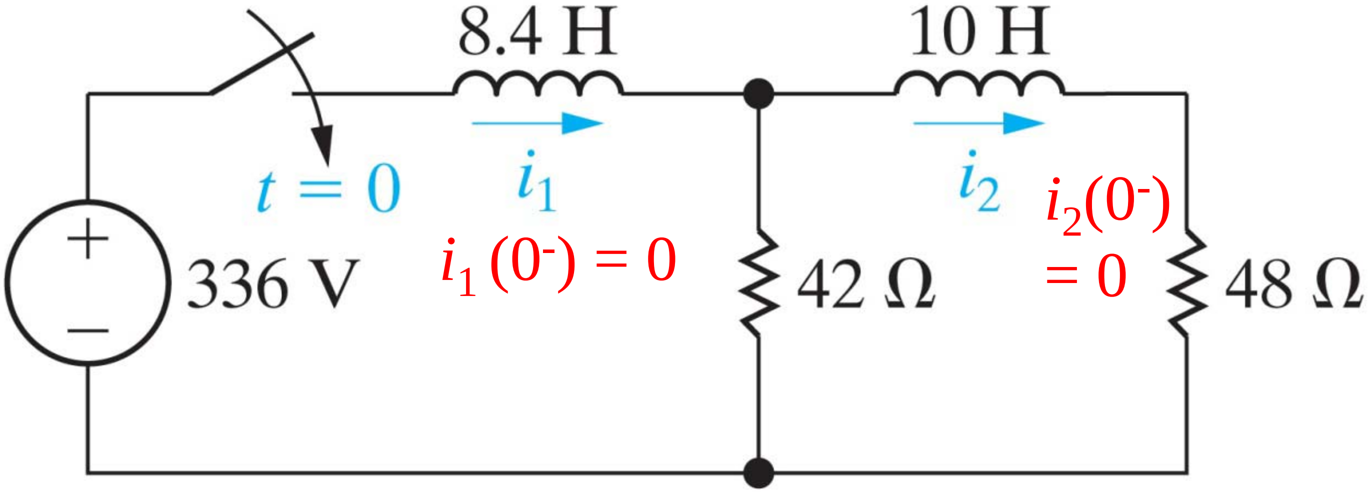
$$i_L(t) = \left\{ \underline{2|K_1|\cos(\omega t + \angle K_1)} + \underline{2|K_2|e^{-\alpha t}\cos(\beta t + \angle K_2)} \right\} u(t).$$

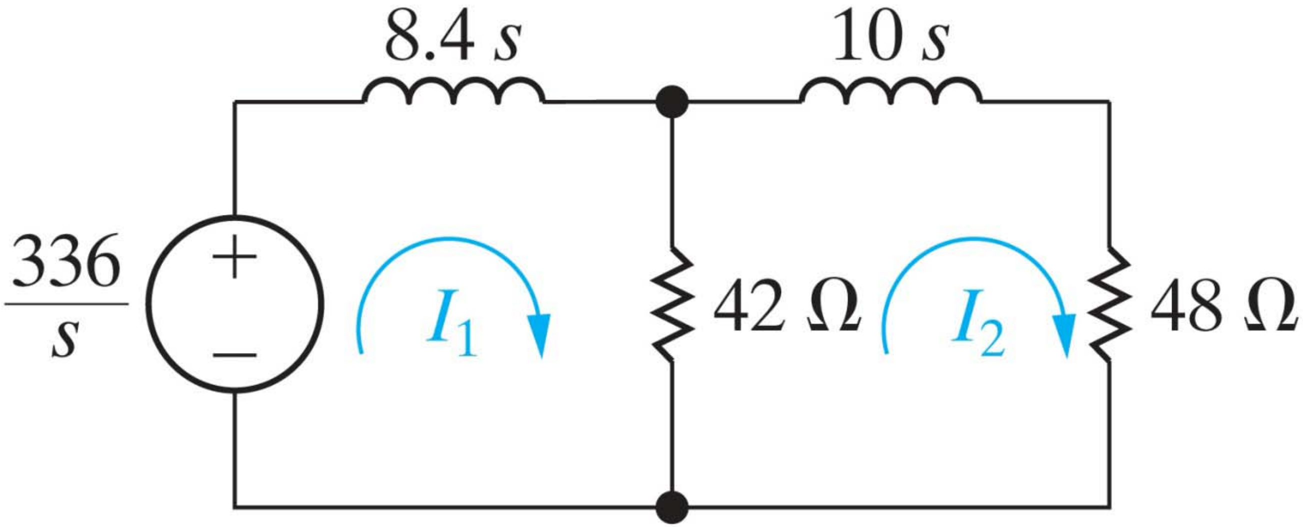
Steady-state response (source)

Natural response (RLC parameters)

Step response of a 2-mesh circuit (1)

- Q: $i_1(t),\ i_2(t) = ?$





Step response of a 2-mesh circuit (2)

- MCM, $\Rightarrow$ 2 algebraic equations & solutions:

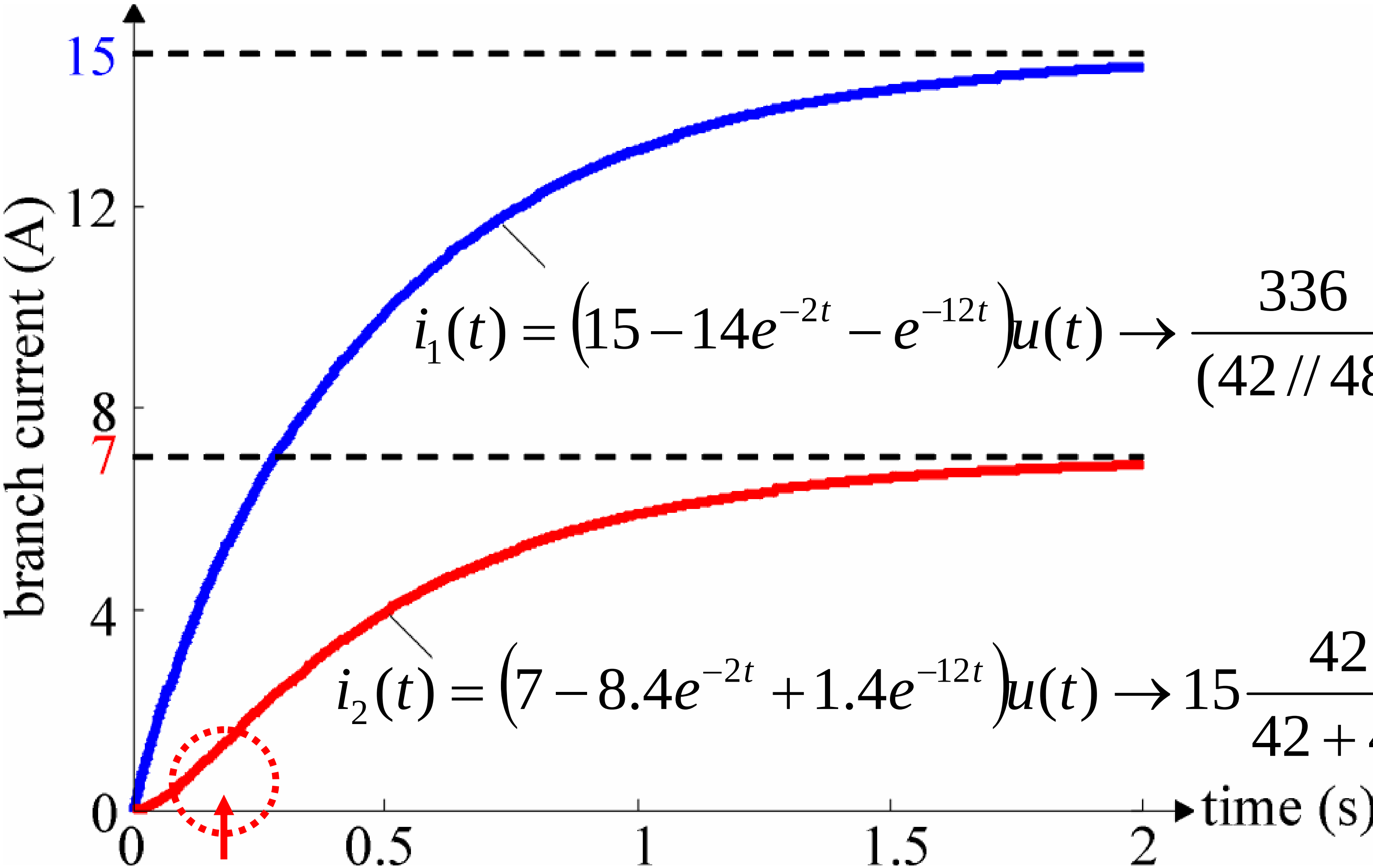
$$\begin{cases} 8.4sI_1 + 42(I_1 - I_2) = \dfrac{336}{s} \cdots (1) \\ 42(I_2 - I_1) + (10s + 48)I_2 = 0 \cdots (2) \end{cases}$$

$$\Rightarrow \begin{bmatrix} 42 + 8.4s & -42 \\ -42 & 90 + 10s \end{bmatrix} \times \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 336/s \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 42 + 8.4s & -42 \\ -42 & 90 + 10s \end{bmatrix}^{-1} \times \begin{bmatrix} 336/s \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{15}{s} - \dfrac{14}{s+2} - \dfrac{1}{s+12} \\ \dfrac{7}{s} - \dfrac{8.4}{s+2} + \dfrac{1.4}{s+12} \end{bmatrix}.$$

Step response of a 2-mesh circuit (3)

- Perform inverse Laplace transform:

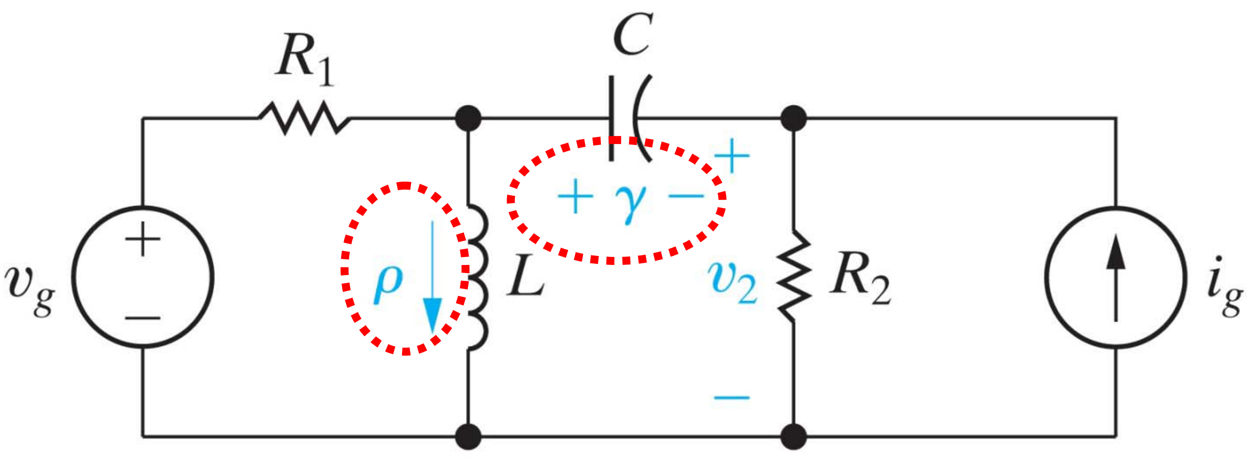


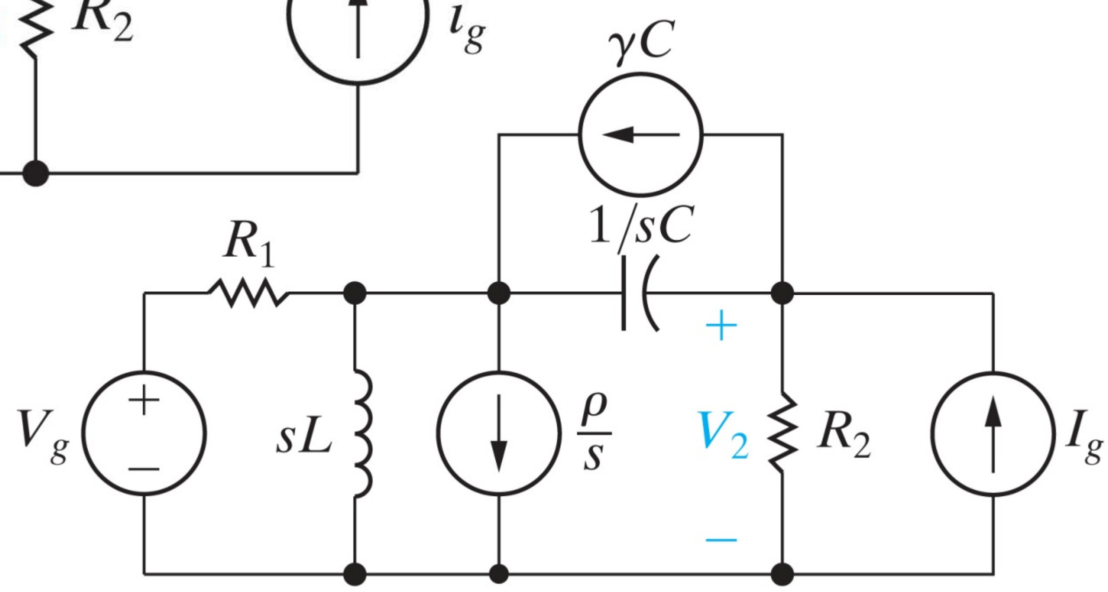
$$i_1(t) = \left(15 - 14e^{-2t} - e^{-12t}\right)u(t) \rightarrow \frac{336}{(42 // 48)} = 15\ \text{A}.$$

$$i_2(t) = \left(7 - 8.4e^{-2t} + 1.4e^{-12t}\right)u(t) \rightarrow 15\frac{42}{42+48} = 7\ \text{A}.$$

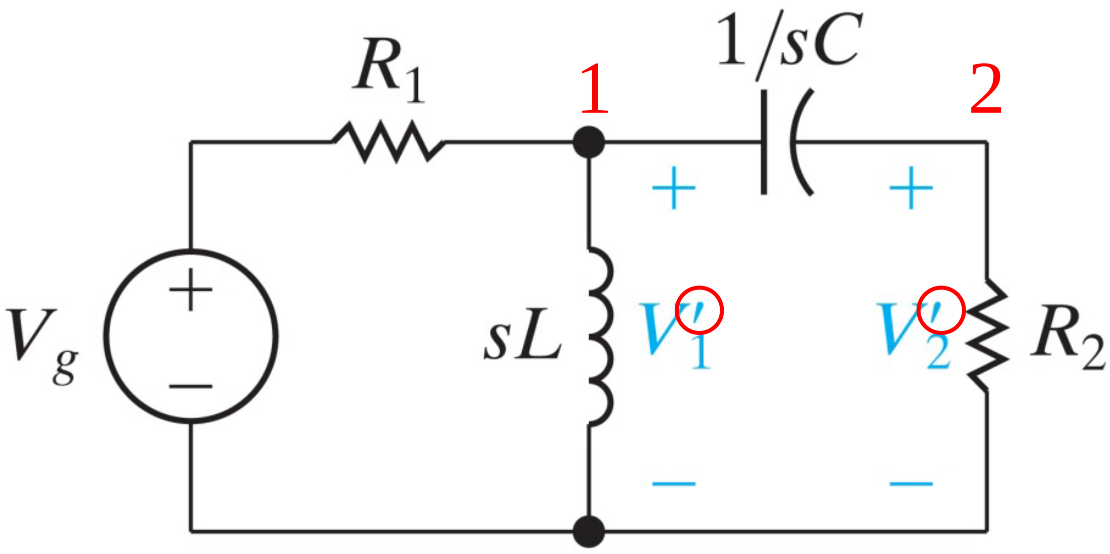
Use of superposition (1)

- Given 2 independent sources v_g, i_g and initially charged C, L, $\Rightarrow v_2(t) = ?$





Use of superposition: V_g acts alone (2)



$$\begin{cases} \dfrac{V_1'-V_g}{R_1}+\dfrac{V_1'}{sL}+\dfrac{V_1'-V_2'}{(sC)^{-1}}=0, \\ \dfrac{V_2'-V_1'}{(sC)^{-1}}+\dfrac{V_2'}{R_2}=0. \end{cases} \Rightarrow \begin{cases} \left(\dfrac{1}{R_1}+\dfrac{1}{sL}+sC\right)V_1'-sCV_2'=\dfrac{V_g}{R_1}, \\ -sCV_1'+\left(\dfrac{1}{R_2}+sC\right)V_2'=0. \end{cases}$$

Use of superposition (3)

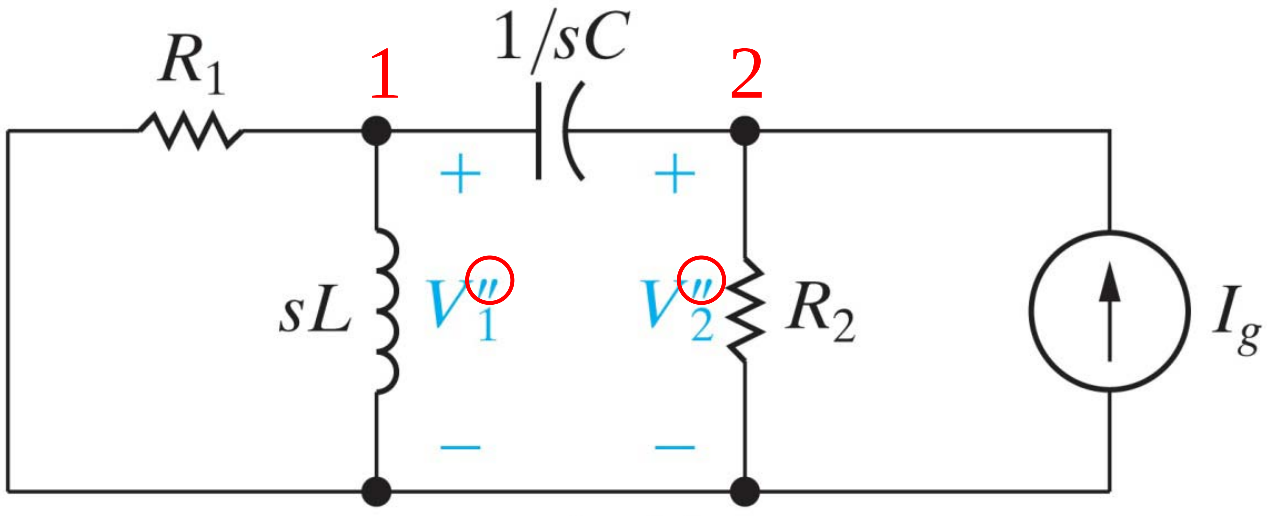
- For convenience, define admittance matrix:

$$\begin{bmatrix} \frac{1}{R_1}+\frac{1}{sL}+sC & -sC \\ -sC & \frac{1}{R_2}+sC \end{bmatrix} \times \begin{bmatrix} V_1' \\ V_2' \end{bmatrix}$$

$$= \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12} & Y_{22} \end{bmatrix} \times \begin{bmatrix} V_1' \\ V_2' \end{bmatrix} = \begin{bmatrix} V_g / R_1 \\ 0 \end{bmatrix}.$$

$$\Rightarrow V_2' = \frac{-Y_{12}/R_1}{Y_{11}Y_{22}-Y_{12}^2} V_g.$$

Use of superposition: I_g acts alone (4)

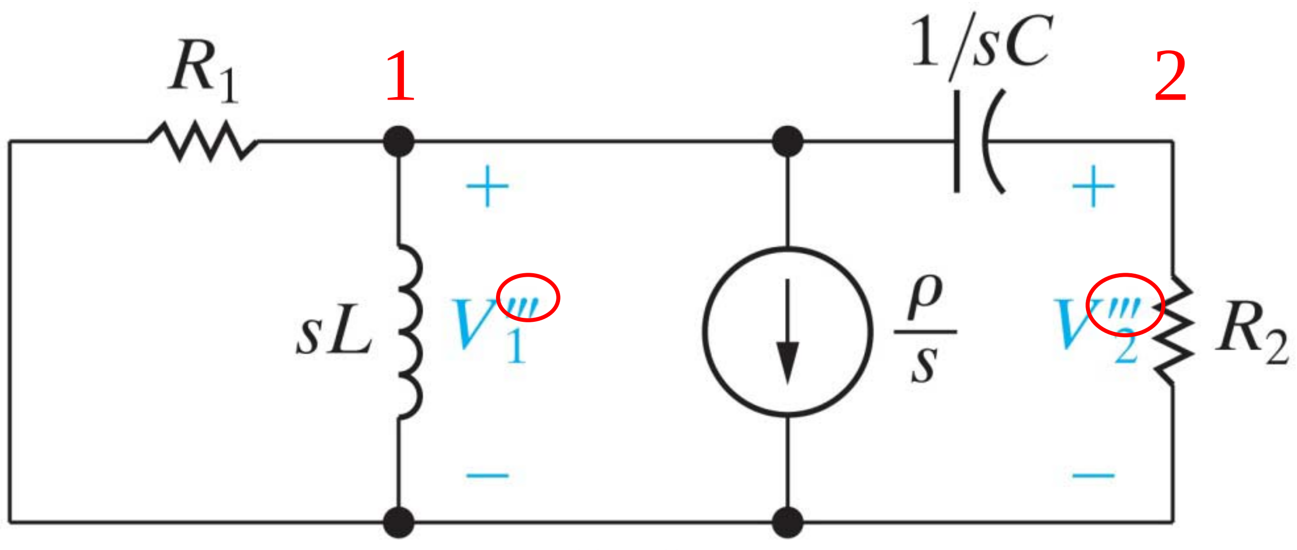


$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12} & Y_{22} \end{bmatrix} \times \begin{bmatrix} V_1'' \\ V_2'' \end{bmatrix} = \begin{bmatrix} 0 \\ I_g \end{bmatrix}, \Rightarrow V_2'' = \frac{Y_{11}}{Y_{11}Y_{22} - Y_{12}^2} I_g ;$$

Same matrix

Same denominator

Use of superposition: Energized L acts alone (5)

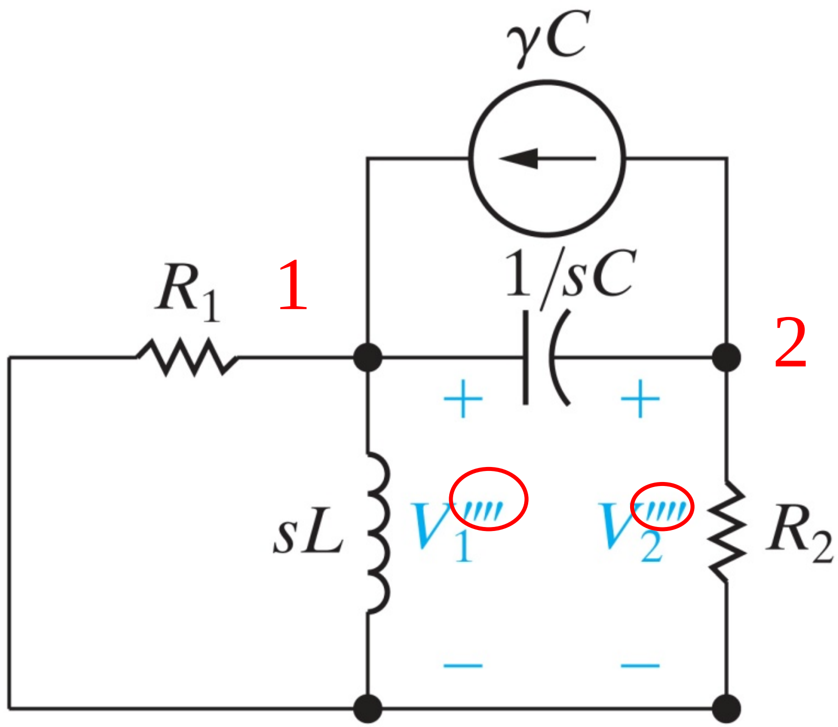


$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12} & Y_{22} \end{bmatrix} \times \begin{bmatrix} V_1''' \\ V_2''' \end{bmatrix} = \begin{bmatrix} -\rho/s \\ 0 \end{bmatrix}, \Rightarrow V_2''' = \frac{Y_{12}/s}{Y_{11}Y_{22} - Y_{12}^2}\rho,$$

Same matrix

Same denominator

Use of superposition: Energized C acts alone (6)



$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12} & Y_{22} \end{bmatrix} \times \begin{bmatrix} V_1'''' \\ V_2'''' \end{bmatrix} = \begin{bmatrix} \gamma C \\ -\gamma C \end{bmatrix},$$

$$\Rightarrow V_2'''' = \frac{-(Y_{11}+Y_{12})C}{Y_{11}Y_{22}-Y_{12}^2}\gamma.$$

- The total voltage is: $V_2 = V_2' + V_2'' + V_2''' + V_2''''$.

Section 13.4, 13.5
The Transfer Function and Natural Response

What is the transfer function of a circuit?

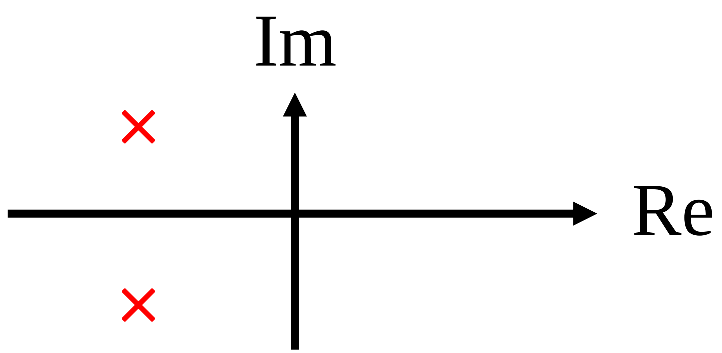
- The ratio of a circuit's output to its input in the s-domain:

$$H(s) = \frac{Y(s)}{X(s)}$$

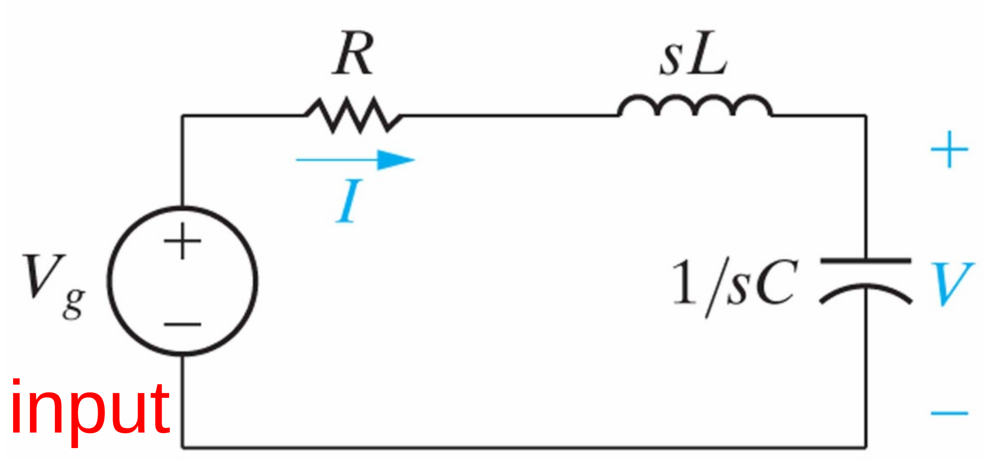
- A single circuit may have many transfer functions, each corresponds to some specific choices of input and output.

Poles and zeros of transfer function

- For linear and lumped-parameter circuits, $H(s)$ is always a rational function of s.
- Poles and zeros always appear in complex conjugate pairs.
- The poles must lie in the left half of the s-plane if bounded input leads to bounded output.



Example: Series RLC circuit



- If the output is the loop current *I*:

$$H(s) = \frac{I}{V_g} = \frac{1}{R + sL + (sC)^{-1}} = \frac{sC}{s^2LC + sRC + 1}.$$

- If the output is the capacitor voltage *V*:

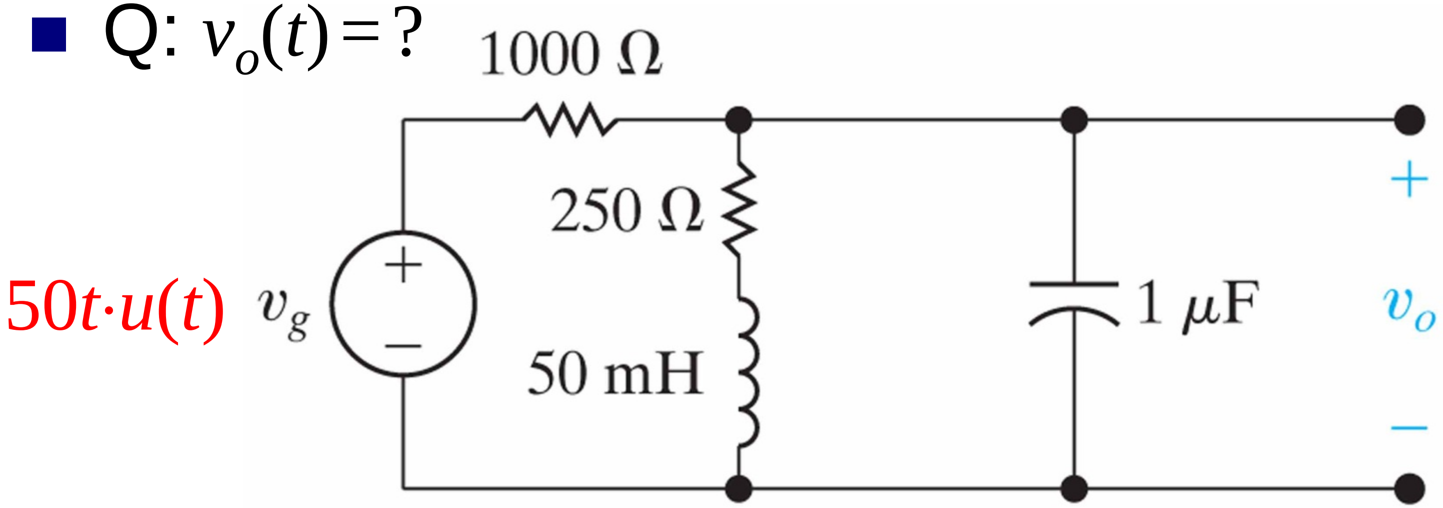
$$H(s) = \frac{V}{V_g} = \frac{(sC)^{-1}}{R + sL + (sC)^{-1}} = \frac{1}{s^2LC + sRC + 1}.$$

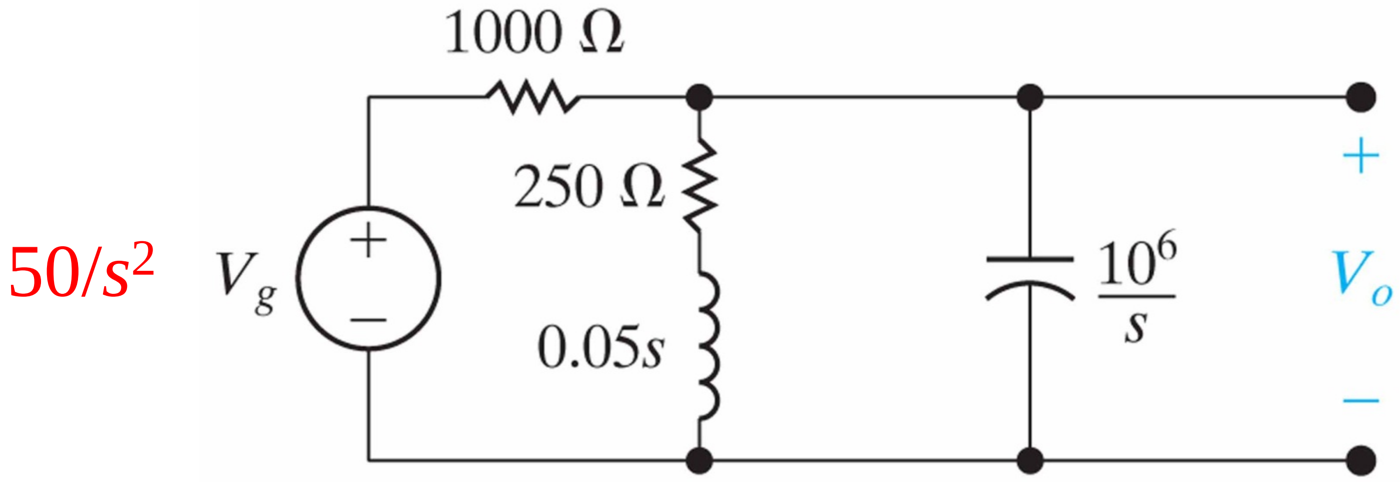
How do poles, zeros influence the solution?

- Since $Y(s)=H(s)X(s)$, $\Rightarrow$ the partial fraction expansion of the output $Y(s)$ yields a term $K/(s\text{-}a)$ for each pole $s=a$ of $H(s)$ or $X(s)$.
- The functional forms of the transient (natural) and steady-state responses $y_{tr}(t)$ and $y_{ss}(t)$ are determined by the poles of $H(s)$ and $X(s)$, respectively.
- The partial fraction coefficients of $Y_{tr}(s)$ and $Y_{ss}(s)$ are determined by both $H(s)$ and $X(s)$.

Example 13.2: Linear ramp excitation (1)

- Q: $v_o(t) = ?$





Example 13.2 (2)

- Only one essential node, $\Rightarrow$ use NVM:

$$\frac{V_o - V_g}{1000} + \frac{V_o}{250 + 0.05s} + \frac{V_o}{10^6/s} = 0,$$

$$\Rightarrow H(s) = \frac{V_o}{V_g} = \frac{1000(s + 5000)}{s^2 + 6000s + \left(2.5 \times 10^7\right)}.$$

- $H(s)$ has 2 complex conjugate poles:

$$s = -3000 \pm j4000.$$

- $V_g(s) = 50/s^2$ has 1 repeated real pole: $s = 0^{(2)}$.

Example 13.2 (3)

- The total response in the s-domain is:

$$V_o(s) = H(s)V_g(s) = \frac{5\times10^4(s+5000)}{s^2\left(s^2+6000s+2.5\times10^7\right)} = Y_{tr} + Y_{ss}$$

expansion coefficients depend on $H(s)$ & $V_g(s)$

$$= \boxed{\frac{5\sqrt{5}\times10^{-4}\angle 80^\circ}{s+3000-j4000} + \frac{5\sqrt{5}\times10^{-4}\angle -80^\circ}{s+3000+j4000}} + \boxed{\frac{10}{s^2} - \frac{4\times10^{-4}}{s}}.$$

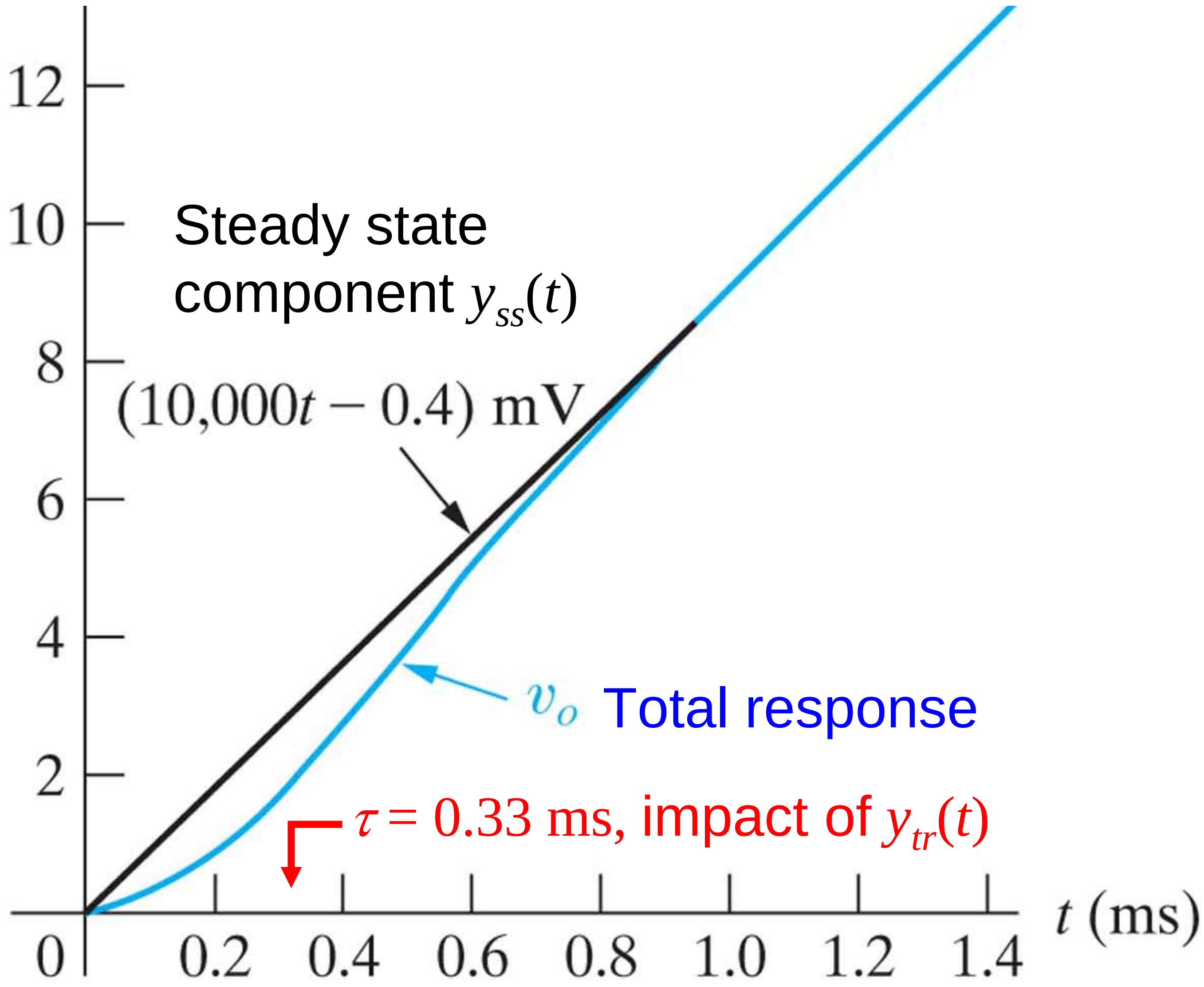
poles of $H(s)$: -3k $\pm j$4k

pole of $V_g(s)$: $0^{(2)}$

- The total response in the t-domain:

$$v_o(t) = y_{tr} + y_{ss} = \left[\sqrt{5}\times10^{-3}e^{-3{,}000t}\cos(4{,}000t+80^\circ)\right]u(t) + \left(10t - 4\times10^{-4}\right)u(t).$$

Example 13.2 (4)



Section 13.6
The Transfer Function and the Convolution Integral

1. Impulse response
2. Time invariant
3. Convolution integral
4. Memory of circuit

Impulse response

- If the input to a linear, lumped-parameter circuit is an impulse $\delta(t)$, the output function $h(t)$ is called impulse response, which happens to be the natural response of the circuit:

$$X(s) = L\{\delta(t)\} = 1,\; Y(s) = H(s) \times 1 = H(s),$$

$$y(t) = L^{-1}\{Y(s)\} = L^{-1}\{H(s)\} = h(t).$$

- The application of an impulse source is equivalent to suddenly storing energy in the circuit. The subsequent release of this energy gives rise to the natural response.

Time invariant

- For a linear, lumped-parameter circuit, delaying the input $x(t)$ by τ simply delays the response $y(t)$ by τ as well (time invariant):

$$X(s,\tau) = L\{x(t-\tau)u(t-\tau)\} = e^{-\tau s}X(s),$$

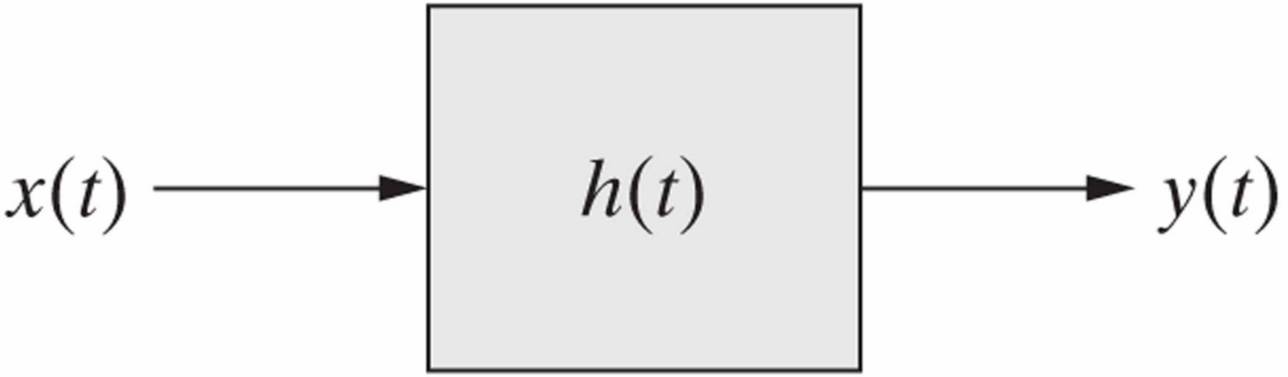
$$Y(s,\tau) = H(s)X(s,\tau) = e^{-\tau s}H(s)X(s) = e^{-\tau s}Y(s),$$

$$y(t,\tau) = L^{-1}\{Y(s,\tau)\} = L^{-1}\{Y(s)\}\Big|_{t\to t-\tau}$$

$$= y(t-\tau)u(t-\tau).$$

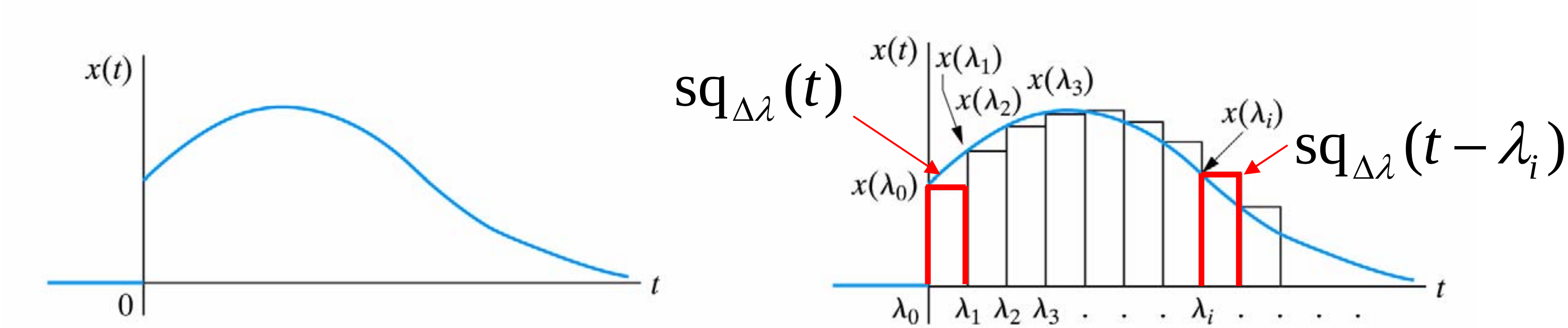
Motivation of working in the time domain

- The properties of impulse response and time-invariance allow one to calculate the output function $y(t)$ of a "linear and time invariant (LTI)" circuit in the t-domain only.
- This is beneficial when $x(t)$, $h(t)$ are known only through experimental data.

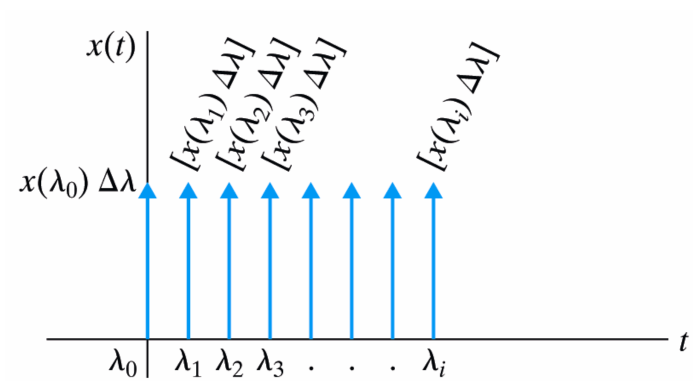


Decompose the input source $x(t)$

- We can approximate $x(t)$ by a series of rectangular pulses $\mathrm{rec}_{\Delta\lambda}(t-\lambda_i)$ of uniform width $\Delta\lambda$:



- By having $\Delta\lambda \to 0$, $\Rightarrow$ $\mathrm{rec}_{\Delta\lambda}(t-\lambda_i)/\Delta\lambda \to \delta(t-\lambda_i)$, $x(t)$ converges to a train of impulses:



$$x(t) = \sum_{i=0}^{\infty} x(\lambda_i) \times \lim_{\Delta\lambda \to 0} \mathrm{rec}_{\Delta\lambda}(t-\lambda_i)$$

$$= \sum_{i=0}^{\infty} x(\lambda_i) \times \Delta\lambda \times \delta(t-\lambda_i).$$

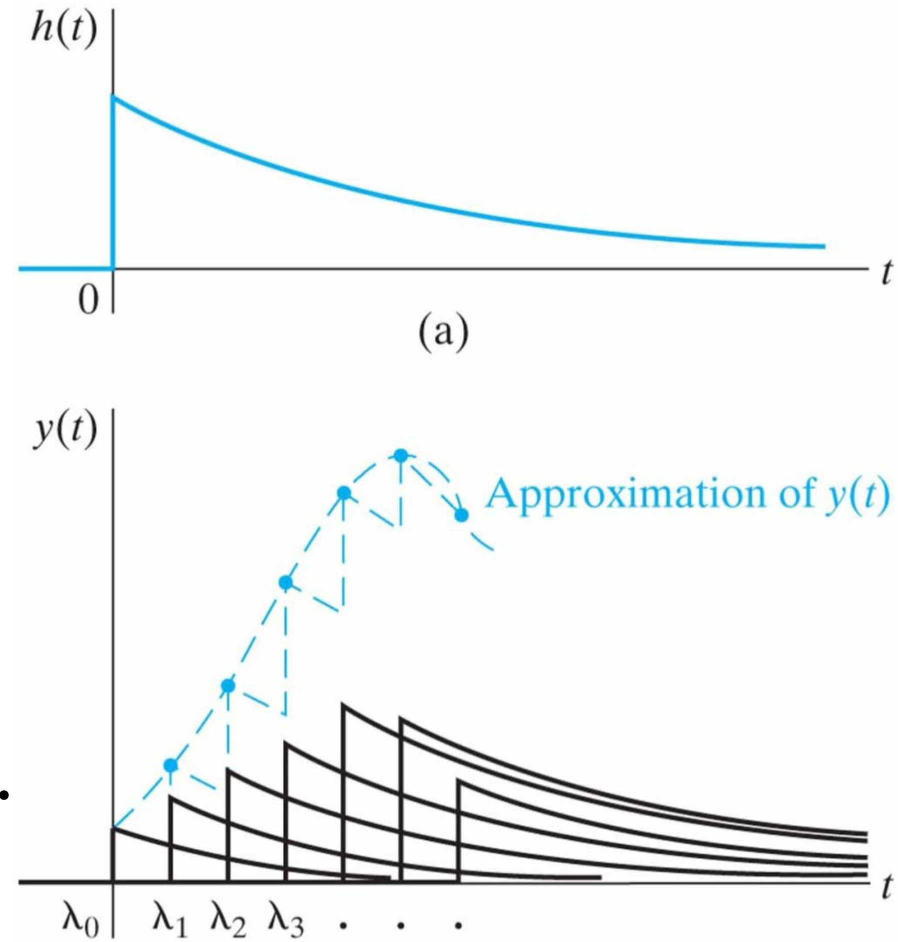
Synthesize the output $y(t)$ (1)

- Since the circuit is LTI:

$$\begin{cases} \delta(t) \to h(t), \\ \delta(t-\lambda_i) \to h(t-\lambda_i), \\ \sum a_i x_i(t) \to \sum a_i y_i(t); \end{cases}$$

$$\Rightarrow \sum_{i=0}^{\infty} x(\lambda_i)\Delta\lambda \cdot \delta(t-\lambda_i)$$

$$\to \sum_{i=0}^{\infty} x(\lambda_i)\Delta\lambda \cdot h(t-\lambda_i).$$



Synthesize the output $y(t)$ (2)

- As $\Delta\lambda \to 0$, summation $\to$ integration:

if $x(t)$ extends $(-\infty, \infty)$

$$y(t) = \int_0^\infty x(\lambda)h(t-\lambda)d\lambda \to \int_{-\infty}^\infty x(\lambda)h(t-\lambda)d\lambda.$$

- By change of variable $u = t - \lambda$, $\Rightarrow$

$$y(t) = \int_{-\infty}^\infty x(t-u)h(u)du.$$

- The output of an LTI circuit is the convolution of input and the impulse response of the circuit:

$$y(t) = x(t) * h(t)$$

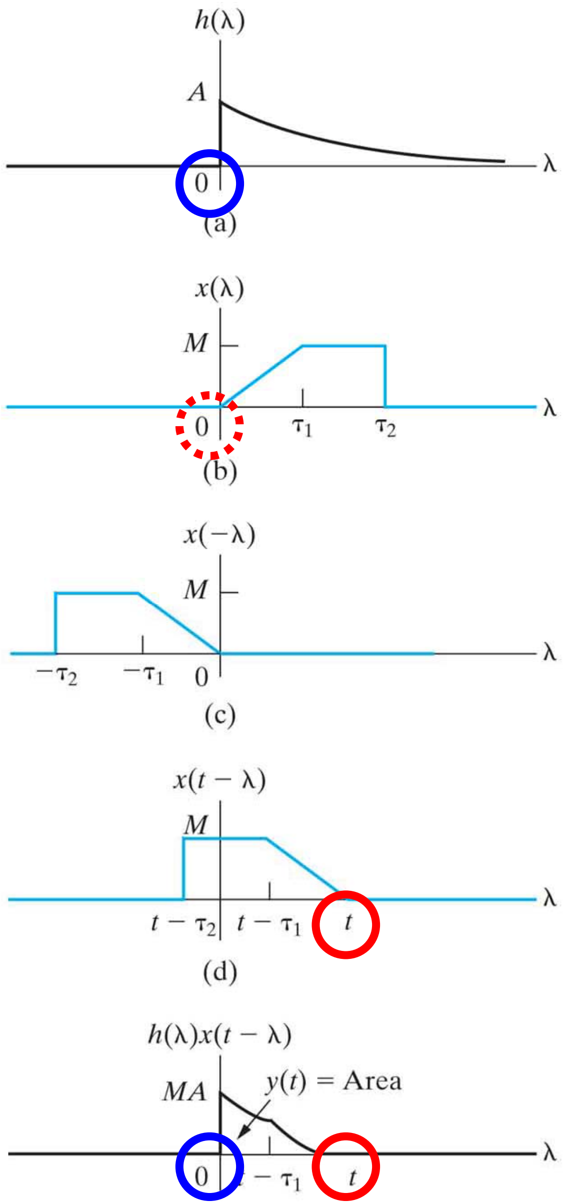
$$= \int_{-\infty}^\infty x(\lambda)h(t-\lambda)d\lambda = \int_{-\infty}^\infty x(t-\lambda)h(\lambda)d\lambda.$$

Convolution of a causal circuit

- For physically realizable circuit, no response can occur prior to the input excitation (causal), $\Rightarrow \{h(t)=0 \text{ for } t<0\}$.
- Excitation is turned on at $t=0$, $\Rightarrow \{x(t)=0 \text{ for } t<0\}$. $\Rightarrow$

$$y(t) = x(t) * h(t)$$

$$= \int_0^t x(t-\lambda)h(\lambda)d\lambda.$$

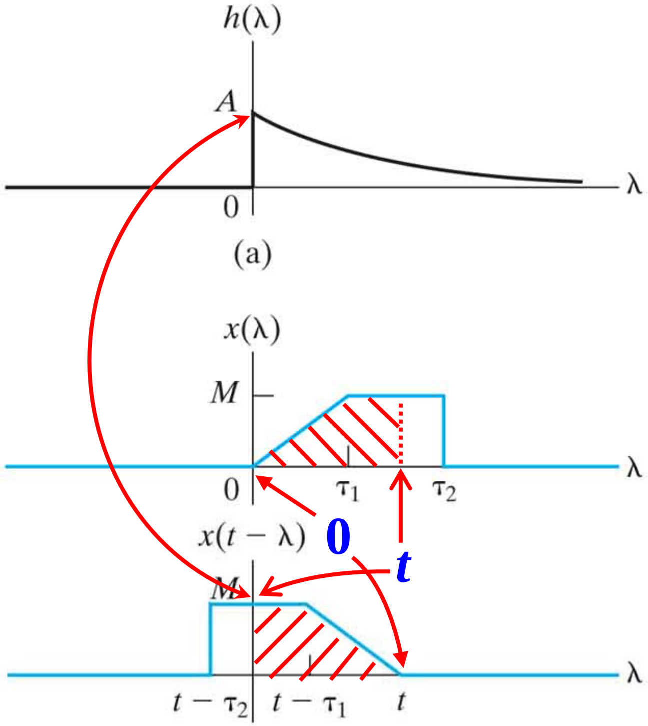


Effect of $x(t)$ is weighted by $h(t)$

- The convolution integral

$$y(t)=\int_0^t x(t-\lambda)h(\lambda)d\lambda$$

shows that the value of $y(t)$ is the weighted average of $x(t)$ from $t=0$ to $t=t$ [from $\lambda=t$ to $\lambda=0$ for $x(t\text{-}\lambda)$].



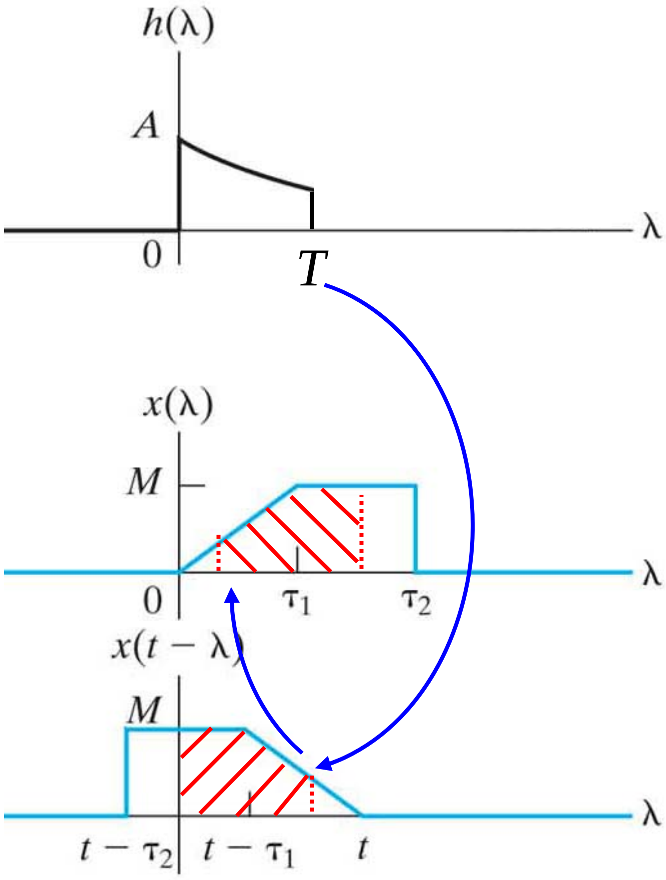
- If $h(t)$ is monotonically decreasing, the highest weight is given to the present $x(t)$.

Memory of the circuit

- If $h(t)$ only lasts from $t = 0$ to $t = T$, the convolution integral

$$y(t) = \int_0^t x(t-\lambda)h(\lambda)d\lambda.$$

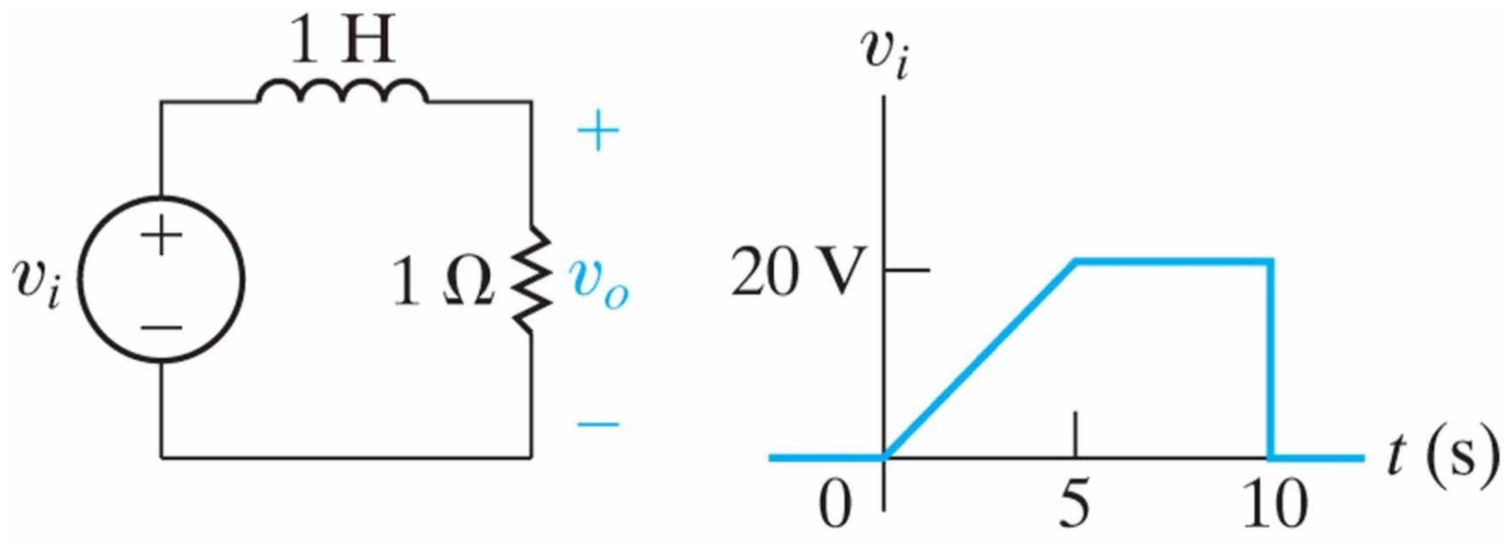
implies that the circuit has a memory over a finite interval $t = [t\text{-}T, t]$.



- If $h(t) = \delta(t)$, no memory, output at t only depends on $x(t)$, $\Rightarrow y(t) = x(t) * \delta(t) = x(t)$, no distortion.

Example 13.3: RL driven by a trapezoidal source (1)

- Q: $v_o(t) = ?$

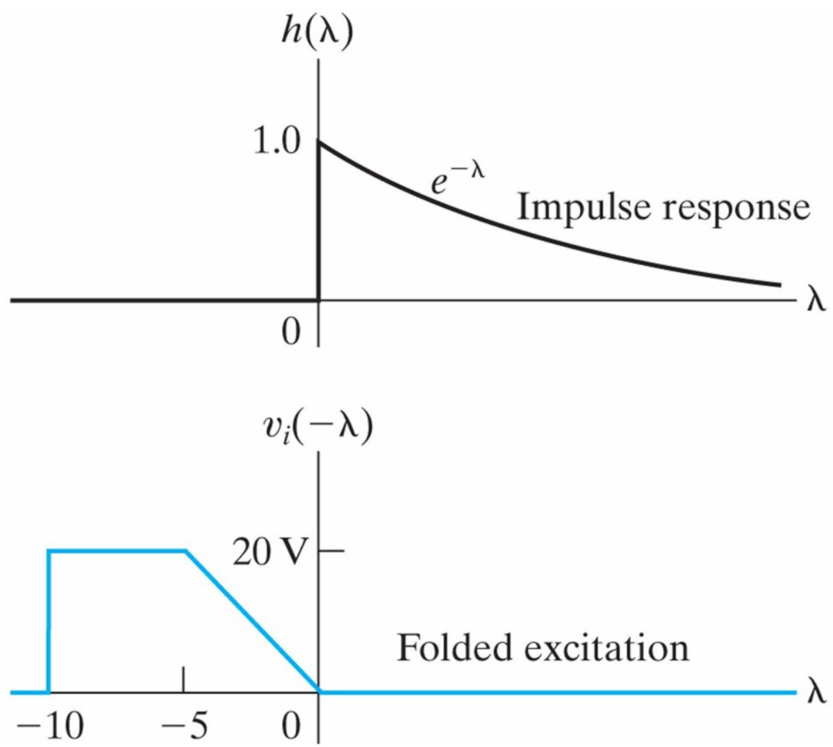


$$V_o = \frac{1}{s+1} V_i, \Rightarrow H(s) = \frac{V_o}{V_i} = \frac{1}{s+1}.$$

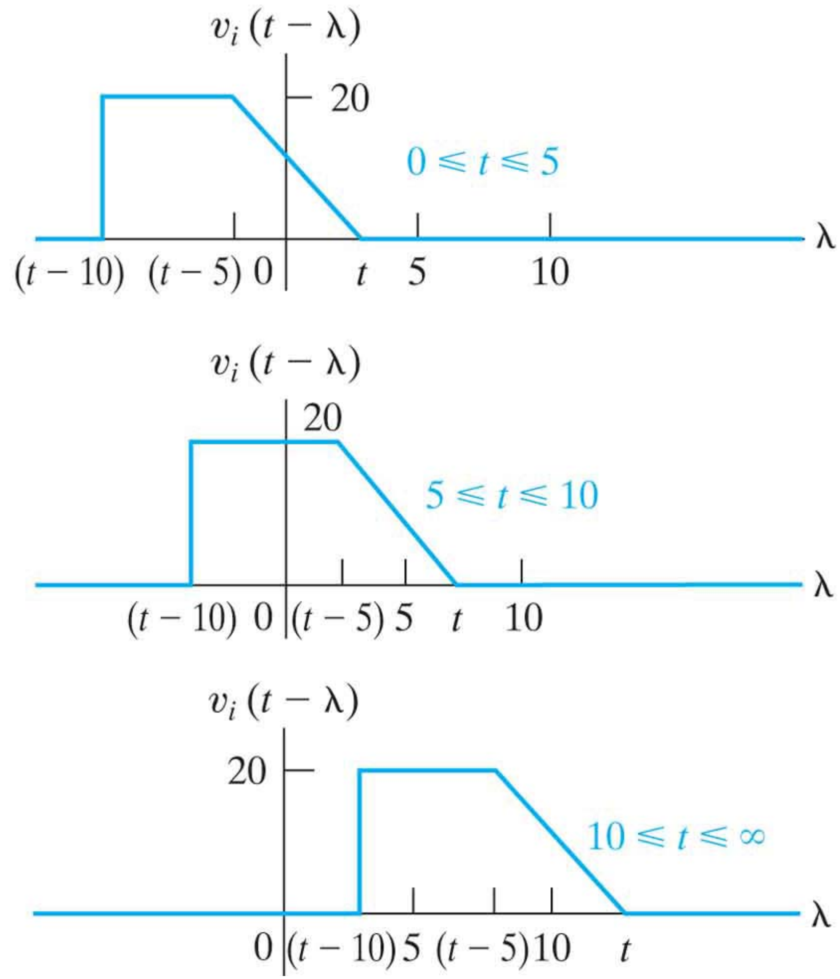
$$\Rightarrow h(t) = L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}u(t).$$

Example 13.3 (2)

$$v_o(t) = \int_0^t v_i(t-\lambda)h(\lambda)d\lambda.$$

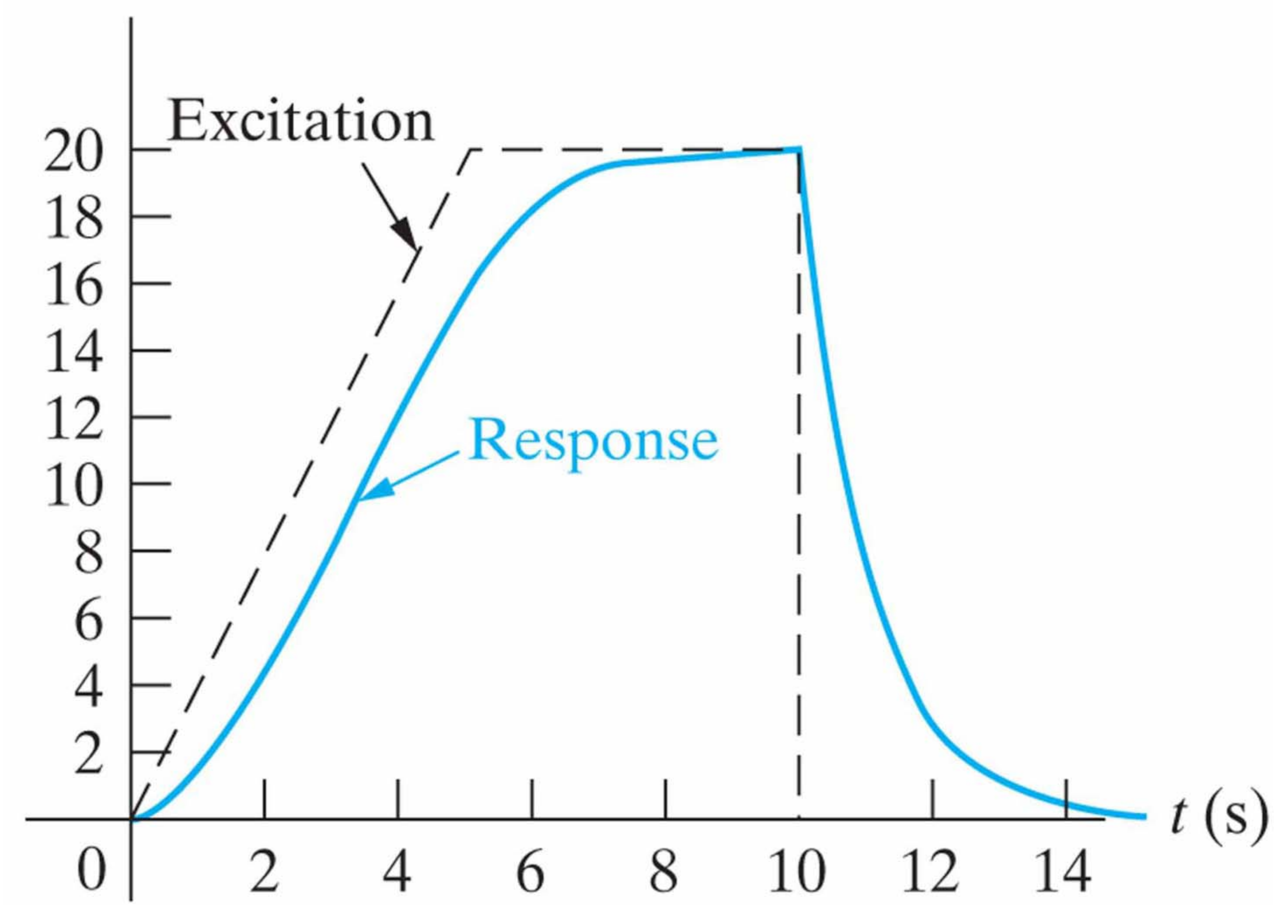


- Separate into 3 intervals:



Example 13.3 (3)

- Since the circuit has certain memory, $v_o(t)$ has some distortion with respect to $v_i(t)$.



Section 13.7
The Transfer Function and the Steady-State Sinusoidal Response

How to get sinusoidal steady-state response by $H(s)$?

- In Chapters 9-11, we used phasor analysis to get steady-state response $y_{ss}(t)$ due to a sinusoidal input $x(t) = A\cos(\omega t + \phi)$.
- If we know $H(s)$, $y_{ss}(t)$ must be:

$$y_{ss}(t) = |H(j\omega)| A\cos[\omega t + \phi + \theta(\omega)],$$

$$\text{where } H(j\omega) = H(s)\big|_{s=j\omega} = |H(j\omega)| e^{j\theta(\omega)}.$$

- The changes of amplitude and phase depend on the sampling of $H(s)$ along the imaginary axis.

Proof

$$x(t) = A\cos(\omega t + \phi) = A\cos\phi \underline{\cos\omega t} - A\sin\phi \underline{\sin\omega t}.$$

$$X(s) = A\cos\phi \boxed{\frac{s}{s^2+\omega^2}} - A\sin\phi \boxed{\frac{\omega}{s^2+\omega^2}} = A\frac{s\cos\phi - \omega\sin\phi}{s^2+\omega^2}.$$

$$Y(s) = H(s)X(s) = H(s)\cdot A\frac{s\cos\phi - \omega\sin\phi}{\boxed{s^2+\omega^2}} = Y_{tr}(s) + \boxed{Y_{ss}(s)},$$

where $Y_{ss}(s) = \dfrac{K_1}{s-j\omega} + \dfrac{K_1^*}{s+j\omega}$, $K_1 = Y(s)(s-j\omega)\big|_{s=j\omega}$

$$= H(s)A\frac{s\cos\phi - \omega\sin\phi}{s+j\omega}\bigg|_{s=j\omega} = H(j\omega)A\frac{j\omega\cos\phi - \omega\sin\phi}{2j\omega} = \frac{H(j\omega)Ae^{j\phi}}{2}.$$

$$y_{ss}(t) = L^{-1}\left\{\frac{|H(j\omega)|e^{j\theta(\omega)}Ae^{j\phi}}{2(s-j\omega)}\right\} + c.c. = A|H(j\omega)|\cos[\omega t + \phi + \theta(\omega)].$$

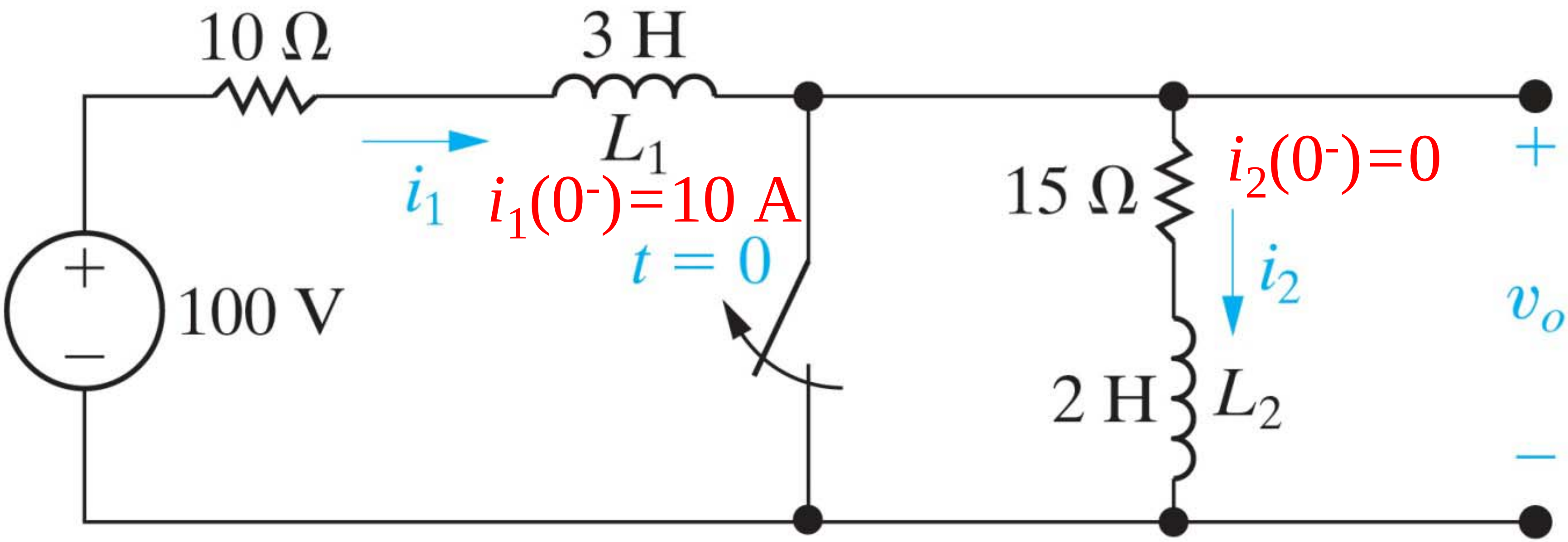
Obtain $H(s)$ from $H(j\omega)$

- We can reverse the process: determine $H(j\omega)$ experimentally, then construct $H(s)$ from the data (not always possible).
- Once we know $H(s)$, we can find the response to other excitation sources.

Section 13.8
The Impulse Function in Circuit Analysis

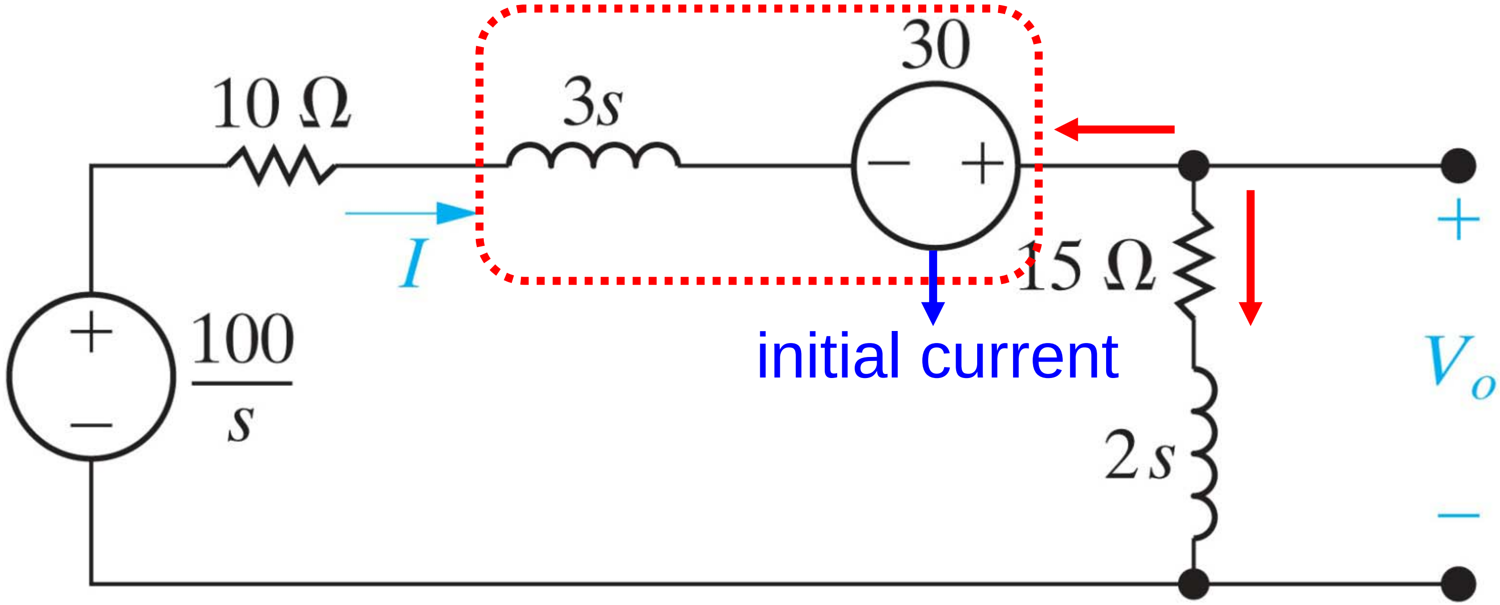
E.g. Impulsive inductor voltage (1)

- Q: $v_o(t) = ?$



- The opening of the switch forces the two inductor currents i_1, i_2 change immediately by inducing an impulsive inductor voltage $[v = L \cdot i'(t)]$.

E.g. Equivalent circuit & solution in the s-domain (2)



$$\frac{V_0 - (100/s + 30)}{10 + 3s} + \frac{V_0}{15 + 2s} = 0,$$

improper rational

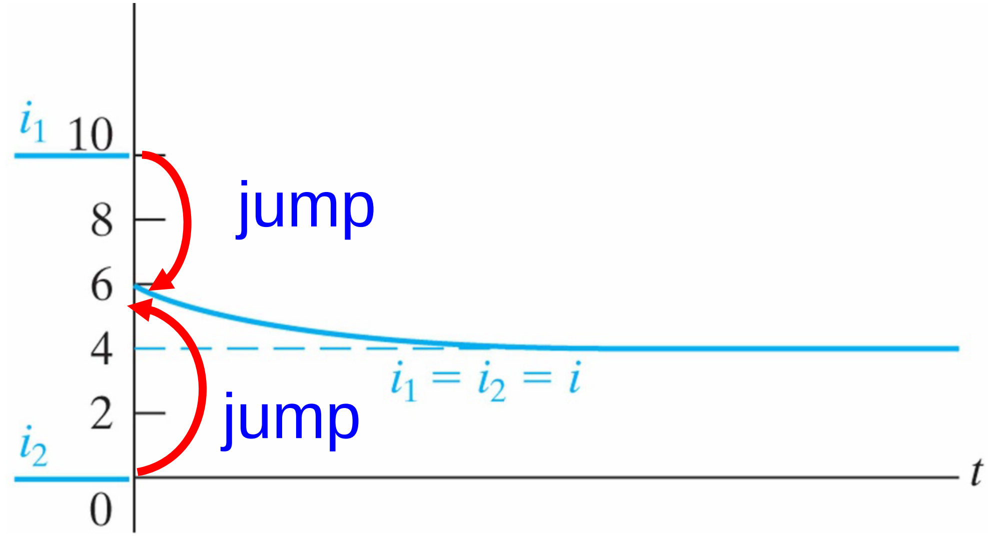
$$\Rightarrow V_0(s) = \boxed{\frac{2(6s^2 + 65s + 150)}{s(s+5)}} = 12 + \frac{60}{s} + \frac{10}{s+5}.$$

E.g. Solutions in the t-domain (3)

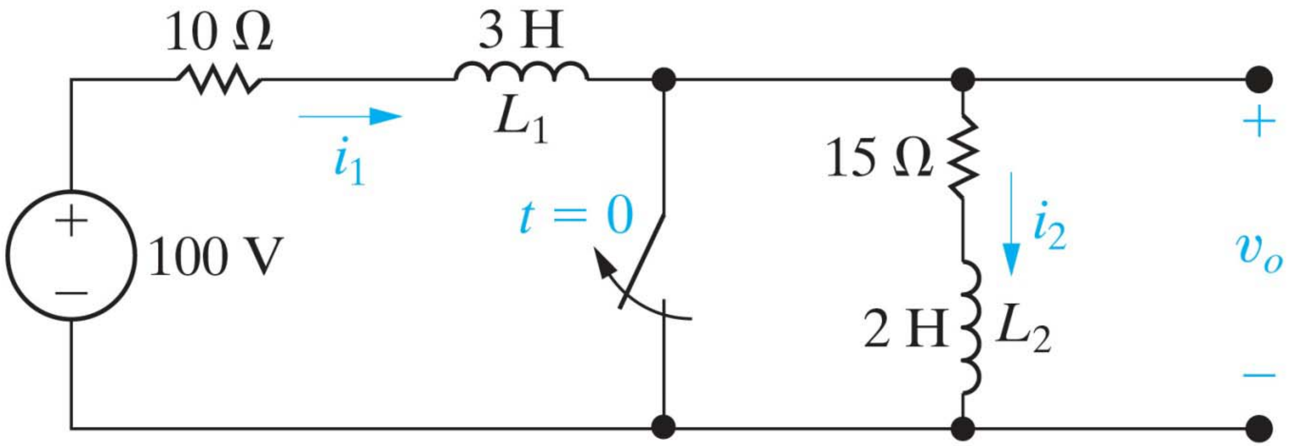
$$v_0(t) = L^{-1}\left\{12 + \frac{60}{s} + \frac{10}{s+5}\right\} = \underline{12\delta(t)} + (60 + 10e^{-5t})u(t).$$

- To verify whether this solution $v_o(t)$ is correct, we need to solve $i(t)$ as well.

$$I(s) = \frac{100/s + 30}{10 + 3s + 15 + 2s} = \frac{4}{s} + \frac{2}{s+5}, \Rightarrow i(t) = (4 + 2e^{-5t})u(t).$$



Impulsive inductor voltage (4)



- The jump of $i_2(t)$ from 0 to 6 A causes $i_2'(t) = 6\delta(t)$, contributing to a voltage impulse $L_2 i_2'(t) = 12\delta(t)$.
- After $t > 0^+$,

$$v_o(t) = (15\Omega)i_2(t) + (2\mathrm{H})i_2'(t)$$
$$= 15(4 + 2e^{-5t}) + 2(-10e^{-5t}) = 60 + 10e^{-5t},$$

consistent with that solved by Laplace transform.

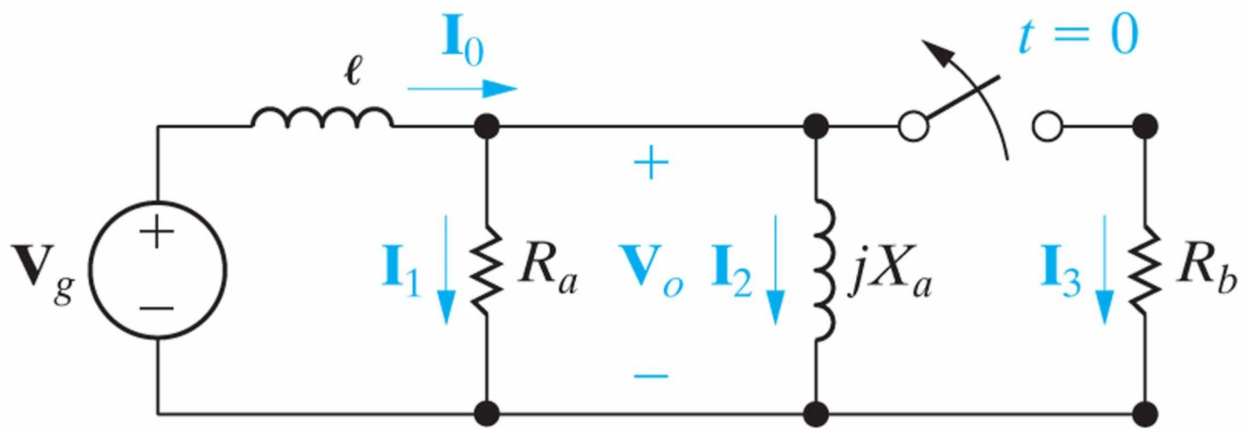
Key points

- How to represent the initial energy of L, C in the s-domain?
- Why the functional forms of natural and steady-state responses are determined by the poles of transfer function $H(s)$ and excitation source $X(s)$, respectively?
- Why the output of an LTI circuit is the convolution of the input and impulse response? How to interpret the memory of a circuit by convolution?

Practical Perspective
Voltage Surges

Why can a voltage surge occur?

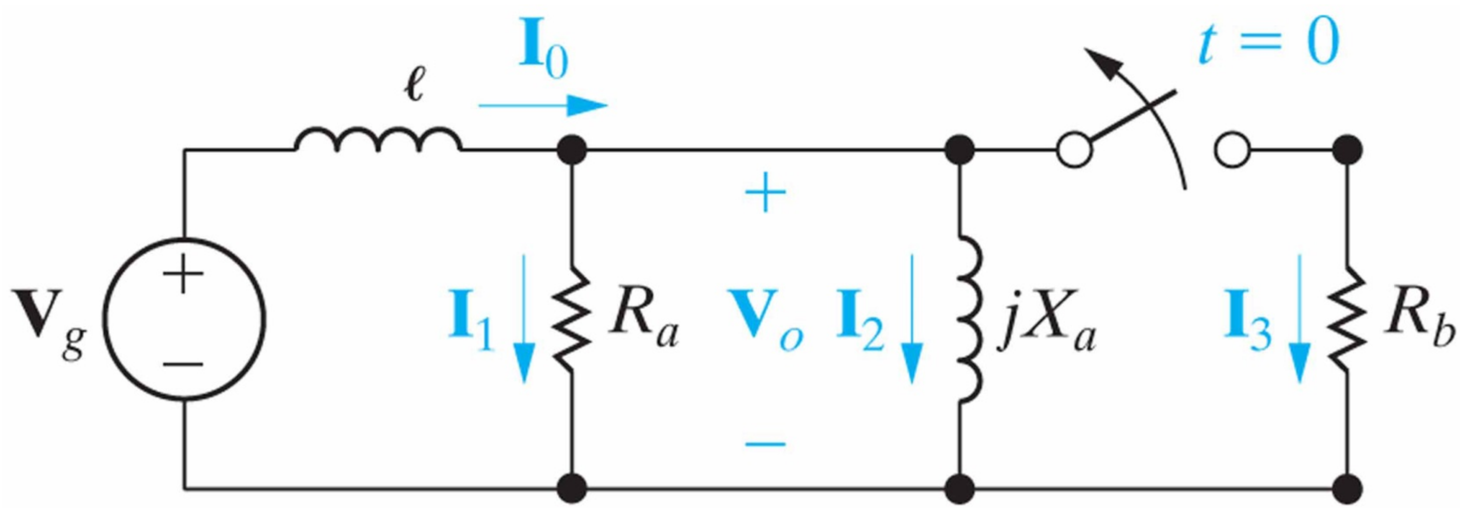
- Q: Why a voltage surge is created when a load is switched off?
- Model: A sinusoidal voltage source drives three loads, where R_b is switched off at $t = 0$.



- Since $i_2(t)$ cannot change abruptly, $i_1(t)$ will jump by the amount of $i_3(0^-)$, $\Rightarrow$ voltage surge occurs.

Example

- Let $\mathbf{V}_o = 120\angle 0°$ (rms), $f = 60$ Hz, $R_a = 12\ \Omega$, $R_b = 8\ \Omega$, $X_a = 41.1\ \Omega$ (i.e. $L_a = X_a/\omega = 109$ mH), $X_l = 1\ \Omega$ (i.e. $L_l = 2.65$ mH). Solve $v_o(t)$ for $t > 0^-$.



- To draw the s-domain circuit, we need to calculate the initial inductor currents $i_2(0^-)$, $i_0(0^-)$.

Steady-state before the switching

- The three branch currents (rms phasors) are:

$\mathbf{I_1} = \mathbf{V}_{\boldsymbol{o}}/R_a = (120\angle 0°)/(12\ \Omega) = 10\angle 0°\ \mathrm{A}$,

$\mathbf{I_2} = \mathbf{V}_{\boldsymbol{o}}/(jX_a) = (120\angle 0°)/(j41.1\ \Omega) = 2.92\angle\text{-}90°\ \mathrm{A}$,

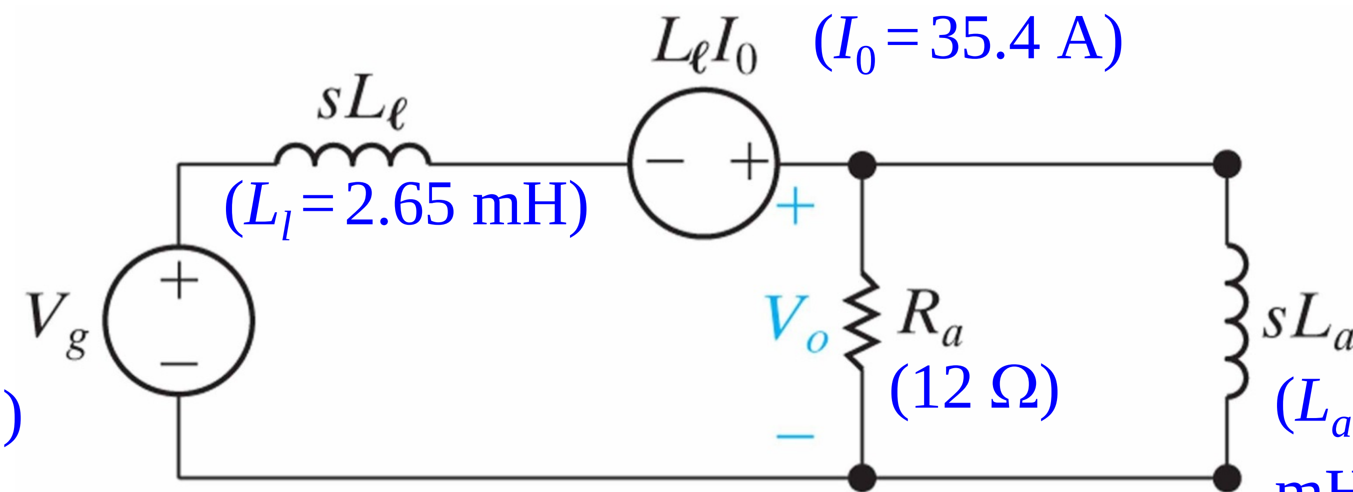
$\mathbf{I_3} = \mathbf{V}_{\boldsymbol{o}}/R_b = (120\angle 0°)/(8\ \Omega) = 15\angle 0°\ \mathrm{A}$,

- The line current is: $\mathbf{I_0} = \mathbf{I_1} + \mathbf{I_2} + \mathbf{I_3} = 25.2\angle\text{-}6.65°\ \mathrm{A}$.
- Source voltage: $\mathbf{V_g} = \mathbf{V}_{\boldsymbol{o}} + \mathbf{I_0}(jX_l) = 125\angle\text{-}11.5°\ \mathrm{V}$.
- The two initial inductor currents at $t = 0^-$ are:
- $i_2(t) = 2.92(\sqrt{2})\cos(120\pi t\text{-}90°)$, $\Rightarrow$ $i_2(0^-) = 0$;
- $i_0(t) = 25.2(\sqrt{2})\cos(120\pi t\text{-}6.65°)$, $\Rightarrow$ $i_0(0^-) = 35.4\ \mathrm{A}$.

S-domain analysis

- The s-domain circuit is:

$\mathbf{V_g}$ = 125∠-11.5° V (rms)



$(L_a = 109\text{ mH})$

- By NVM: $\dfrac{V_o - L_l I_0 - V_g}{sL_l} + \dfrac{V_o}{R_a} + \dfrac{V_o}{sL_a} = 0,$

$$V_o = \frac{(R_a / L_l)V_g(s) + I_0 R_a}{s + [R_a(L_a + L_l)]/(L_a L_l)} = \frac{253}{s + 1475\pi} + \frac{86\angle 6.85^\circ}{s - j120\pi} + \frac{86\angle -6.85^\circ}{s + j120\pi}$$

Inverse Laplace transform

- Given $V_o(s)=\dfrac{253}{s+1475\pi}+\dfrac{86\angle 6.85^\circ}{s-j120\pi}+\dfrac{86\angle -6.85^\circ}{s+j120\pi}$,

$$\Rightarrow v_o(t)=\left[253e^{-1475\pi t}+173\cos(120\pi t+6.85^\circ)\right]\cdot u(t).$$