Lesson 03 Transient Response of Transmission Lines

■ Introduction

Partial reflection and transmission occur whenever a wave meets with a “discontinuity”, i.e., an interface between different materials. Multiple discontinuities cause successive counter-propagating waves, and the superposition of all component waves will determine the exact signal along the line. In this lesson, we will quantitatively analyze transient response of a terminated transmission line or cascaded lines excited by a step-like voltage source, which is important in digital integrated electronics and computer communications.

■ Reflection at discontinuity

Example 3-1: A step voltage source of amplitude $V_0$ and internal resistance $R_s$ drives a lossless transmission line of characteristic impedance $Z_o$, length $l$, phase velocity $v_p$ (one-way signal traveling time $t_d = l/v_p$), and is terminated by a load resistance $R_L$ (Fig. 3-1).

As in Example 2-3, a voltage signal starts to propagate in the $+z$ direction with velocity $v_p$ at $t = 0$, so $v_1^+(z,t) = \begin{cases} V_1^+, & \text{if } z < v_p t \\ 0, & \text{otherwise} \end{cases}$ (valid for all $t > 0$, but is only interested during
$z \in [0,l)$, where $V_1^+ = \frac{Z_0}{Z_0 + R_S} V_0$. When the disturbance $v_1^+(z,t)$ arrives at the load ($z = l$) at $t = t_d$, a reflected voltage wave $v_1^-(z,t) = \begin{cases} V_1^- \text{, if } z > l - v_p(t - t_d) \\ 0, \text{otherwise} \end{cases}$ (valid for all $t > t_d$)

will be generated and propagate in the $-z$ direction. The total voltage at the load at $t = t_d^+$ is equal to their superposition:

$$v_L(t) = v_1^+(l,t) + v_1^-(l,t) \quad (3.1)$$

By eq. (2.9), the total current at $z = l$, $t = t_d^+$ is:

$$i_L(t) = \frac{v_1^+(l,t)}{Z_0} - \frac{v_1^-(l,t)}{Z_0} \quad (3.2)$$

Substituting eq’s (3.1), (3.2) into the boundary condition $v_L(t) = i_L(t) \cdot R_L(t)$ (imposed by the load resistance) gives:

$$v_1^+(l,t) + v_1^-(l,t) = \frac{R_L}{Z_0} \left[ v_1^+(l,t) - v_1^-(l,t) \right].$$

Dividing the above equation by $v_1^+(l,t)$ for both sides of equality, and defining the load voltage reflection coefficient $\Gamma_L$ as the ratio of the reflected voltage $v_1^-(z,t)$ to the incident voltage $v_1^+(z,t)$, we arrive at:

$$1 + \Gamma_L = \frac{R_L}{Z_0} \left( 1 - \Gamma_L \right), \quad \Rightarrow \quad \Gamma_L \equiv \frac{v_1^-(l,t)}{v_1^+(l,t)} = \frac{R_L - Z_0}{R_L + Z_0} \quad (3.3)$$

The reflected voltage wave $v_1^-(z,t)$ will arrive at the source ($z = 0$) at $t = 2t_d$, generating a reflected voltage $v_2^+(z,t) = \begin{cases} V_2^+, \text{if } z < v_p(t - 2t_d) \\ 0, \text{otherwise} \end{cases}$ (valid for all $t > 2t_d$) propagating in the $+z$ direction. This process can be viewed as a voltage disturbance propagating on a line of characteristic impedance $Z_0$ and being incident on a resistance of $R_S$. By eq. (3.3), the source voltage reflection coefficient $\Gamma_S$ is:
\[ \Gamma_S = \frac{v_1^-(l,t)}{v_1^-(l,t)} = \frac{R_S - Z_0}{R_S + Z_0} \]  

(3.4)

Note that \( v_i^+(z,t) \) is created at \( t = 0 \), and will continue to exist “forever”. The total voltage and current at the source at \( t = 2t_v^+ \) are formulated as:

\[ v_S(t) = v_1^+(0,t) + v_1^-(0,t) + v_2^+(0,t) = v_1^+(0,t)\left(1 + \Gamma_L + \Gamma_L \Gamma_S \right) \]

\[ i_S(t) = \frac{v_1^+(0,t)}{Z_0} - \frac{v_1^-(0,t)}{Z_0} + \frac{v_2^+(0,t)}{Z_0} = \frac{v_1^+(0,t)}{Z_0} \left(1 - \Gamma_L + \Gamma_L \Gamma_S \right) \]

This process will continue indefinitely. The total voltage at the source will converge to:

\[ \lim_{t \to \infty} v_S(t) = v_1^+(0,t \to \infty) \cdot \left(1 + \Gamma_L + \Gamma_L \Gamma_S + \Gamma_L^2 \Gamma_S + \Gamma_L^3 \Gamma_S^2 + \ldots \right) \]

\[ = V_i^+ \cdot \left[\left(1 + \Gamma_L\right) + \Gamma_L \Gamma_S \left(1 + \Gamma_L\right) + \Gamma_L^2 \Gamma_S^2 \left(1 + \Gamma_L\right) + \ldots \right] = V_i^+ \frac{1 + \Gamma_L}{1 - \Gamma_L \Gamma_S}. \]

By \( V_i^- = \frac{Z_0}{Z_0 + R_S} V_o \), and eq’s (3.3), (3.4), we have:

\[ \lim_{t \to \infty} v_S(t) = \frac{R_L}{R_S + R_L} V_o \]  

(3.5)

<Comment>

You will find that the total voltage at arbitrary position \( z \in [0,l] \) will also converge to eq. (3.5). In other words, the steady state appears as if the transmission line were absent!

■ Bounce diagram

Bounce diagram is a distance vs. time plot, illustrating successive reflections along a transmission line driven by a “step voltage source” (Fig. 3-2a). It can be used conveniently to determine:

1) The spatial voltage distribution at some instant \( v(z,t_0) \): Mark a point \( P_o(z_0,t_0) \) on the plot (Fig. 3-2b). The solution becomes:
where \( V_{\text{left}} \), \( V_{\text{right}} \) result from the superposition of proper numbers of bouncing voltage waves \( v_k^+(z,t) \), \( v_k^-(z,t) \) \( (k = 1, 2, \ldots) \), respectively. For example, the case of Fig. 3-2b leads to a solution of Fig. 3-2c, where

\[
V_{\text{left}} = v_1^+(z,t) + v_1^-(z,t) + v_2^+(z,t) = V_1^+ \left(1 + \Gamma_L + \Gamma_R \right),
\]

\[
V_{\text{right}} = v_1^+(z,t) + v_1^-(z,t) = V_1^+ \left(1 + \Gamma_L \right).
\]

Fig. 3-2. (a) A transmission line excited by a step voltage source and terminated by a resistive load. (b) The corresponding bounce diagram. (c) The spatial voltage distribution at the time \( t = t_0 \).

2) The temporal voltage distribution at some position \( v(z_a,t) \) : Draw a vertical line \( z = z_a \), intersecting with the lines of the plot at successive times \( t = t_k^+ \), \( t_k^- \) \( (k = 1, 2, \ldots) \), which are the instants when the voltage wave \( v_k^+(z,t) \), \( v_k^-(z,t) \) arrives at the point of interest \( z = z_a \). The solution becomes:
\[
V(z_a, t) = \begin{cases} 
0, & \text{if } 0 < t < t_1^* \\
V_1^+ (1 + \Gamma_L), & \text{if } t_1^* < t < t_1^+ \\
V_1^+ (1 + \Gamma_L + \Gamma_L \Gamma_S), & \text{if } t_1^+ < t < t_2^+ \\
\ldots, & \ldots 
\end{cases}
\] (3.6)

Single transmission line with resistive termination

**Example 3-2**: Consider a system shown in Fig. 3-1 where \( R_s = 0.25Z_0 \), \( R_L = \infty \) (open circuited load). Find the terminal voltages \( v_S(t) \), \( v_L(t) \).

Fig. 3-3. (a) The bounce diagram. (b) The temporal voltage distribution at the position \( z = z_a \).

Fig. 3-4. (a) The bounce diagram. (b-c) The normalized terminal voltages of Example 3-2.
Ans: \( V_1^+ = \frac{Z_0}{Z_0 + R_s} V_0 = 0.8V_0 \). By eqs (3.3), (3.4), \( \Gamma_L = 1 \), \( \Gamma_S = -0.6 \). Draw the corresponding bounce diagram (Fig. 3-4a).

(1) For \( z = z_a = 0 \), \( t_1^+ = 0 \), \( t_1^- = t_2^+ = 2t_d \) (degenerate), \( t_2^- = t_3^+ = 4t_d \), … By eq. (3.6),

\[
v_S(t) = v(0,t) = \begin{cases} V_1^+ = 0.8V_0, & \text{if } 0 < t < 2t_d \\ V_1^+(1 + \Gamma_L + \Gamma_L \Gamma_S) = 1.12V_0, & \text{if } 2t_d < t < 4t_d \end{cases}
\]

(See Fig. 3-4b).

(2) For \( z = z_a = l \), \( t_1^+ = t_1^- = t_d \), …, \( t_k^+ = t_k^- = (2k-1)t_d \) (degenerate). By eq. (3.6),

\[
v_L(t) = v(l,t) = \begin{cases} 0, & \text{if } 0 < t < t_d \\ V_1^+(1 + \Gamma_L) = 1.6V_0, & \text{if } t_d < t < 3t_d \end{cases}
\]

(See Fig. 3-4c).

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1) Both \( v_S(t) \) and \( v_L(t) \) converge to \( V_0 \) as predicted by eq. (3.5).

2) Overshooting and ringing effects during the transient state could be harmful for circuits.

Example 3-3: Consider a system shown in Fig. 3-1 where \( R_s = 4Z_0 \), \( R_L = Z_0 \) (matched load). Find the terminal voltages \( v_S(t) \), \( v_L(t) \).

Fig. 3-5. (a) The bounce diagram. (b-c) The normalized terminal voltages of Example 3-3.
Ans: \( V_1^+ = \frac{Z_0}{Z_0 + R_s} V_0 = 0.2V_0 \). By eq’s (3.3), (3.4), \( \Gamma_L = 0 \), \( \Gamma_s = 0.6 \). Draw the corresponding bounce diagram (Fig. 3-5a).

(1) For \( z = z_a = 0 \), \( \Rightarrow t_1^+ = 0 \), \( t_1^- = t_2^+ = 2t_d \) (degenerate), \( t_2^- = t_3^+ = 4t_d \), … By eq. (3.6),
\[
v_s(t) = v(0, t) = 0.8V_0, \text{if } t > 0 \quad \text{(See Fig. 3-5b).}
\]

(2) For \( z = z_a = l \), \( \Rightarrow t_1^+ = t_1^- = t_d \), …, \( t_{k-1}^+ = t_k^- = (2k-1)t_d \) (degenerate). By eq. (3.6),
\[
v_L(t) = v(l, t) = \begin{cases} 0, & \text{if } 0 < t < t_d \\ V_1^+(1 + \Gamma_L) = 0.2V_0, & \text{if } t > 3t_d \end{cases} \quad \text{(See Fig. 3-5c).}
\]

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1) Both \( v_s(t) \) and \( v_L(t) \) converge to \( V_0 \) as predicted by eq. (3.5).

2) No overshooting and ringing for the matched load prevents successive reflections.
Example 3-4: Consider a system shown in Fig. 3-6a. Find the terminal voltages \( v_s(t) \), \( v_L(t) \).

Fig. 3-6  (a) System configuration. (b) The bounce diagram. (c-d) The normalized terminal voltages of Example 3-4.

Ans: When a voltage disturbance \( v_A^r(z,t) \) arrives at the junction \( z = l_j \) between lines \( A \) and \( B \) at time \( t = t_{d1} \), a reflected wave \( v_A^r(z,t) \) and a transmitted wave \( v_B^r(z,t) \) are generated simultaneously. Boundary condition requires that the total voltages on both sides of the junction must be equal:

\[
v_A^r(l_j,t_{d1}) + v_A^r(l_j,t_{d1}) = v_B^r(l_j,t_{d1}).
\]
Dividing $V_{iA}^+ (l_j, t_{d1})$ for both sides of the equality leads to: $1 + \frac{V_{iA}^+ (l_j, t_{d1})}{V_{iA}^+ (l_j, t_{d1})} = \frac{V_{iB}^+ (l_j, t_{d1})}{V_{iA}^+ (l_j, t_{d1})}$. By eq. (3.3), we have:

$$T_{AB} = \frac{V_{iB}^+ (l_j, t_{d1})}{V_{iA}^+ (l_j, t_{d1})} = 1 + \Gamma_{AB},$$

(3.7)

where $T_{AB}$ is the transmission coefficient.

$$V_{iA}^+ = \frac{Z_0}{Z_0 + R_S} V_0 = 0.5V_0 = 0.75V.$$ By eq’s (3.3), (3.4), $\Gamma_S = 0$, $\Gamma_{AB} = -\frac{1}{3}$, $T_{AB} = \frac{2}{3}$,

$$\Gamma_{BA} = \frac{1}{3}, \quad T_{BA} = \frac{4}{3}, \quad \Gamma_L = 0.6.$$ Draw the corresponding bounce diagram (Fig. 3-6b).

(1) $v_s(t) = v(0, t) = \begin{cases} V_{iA}^+ = 0.75V, & \text{if } 0 < t < 2t_{d1} = 1\text{ ns}; \\ V_{iA}^+ + V_{iA}^- = 0.5V, & \text{if } 1\text{ ns} < t < 2t_{d1} + 2t_{d2} = 1.4\text{ ns}; \\ V_{iA}^+ + V_{iA}^- + V_{2A}^- = 0.9V, & \text{if } 1.4\text{ ns} < t < 2t_{d1} + 4t_{d2} = 1.8\text{ ns}; \ldots \end{cases}$, (Fig. 3-6c).

(2) $v_L(t) = v(l, t) = \begin{cases} 0, & \text{if } 0 < t < t_{d1} + t_{d2} = 0.7\text{ ns}; \\ V_{iB}^+ + V_{iB}^- = 0.8V, & \text{if } 0.7\text{ ns} < t < t_{d1} + 3t_{d2} = 1.1\text{ ns}; \\ V_{iB}^+ + V_{iB}^- + V_{2B}^- = 0.96V, & \text{if } 1.1\text{ ns} < t < t_{d1} + 5t_{d2} = 1.5\text{ ns}; \ldots \end{cases}$ (Fig. 3-6d).

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