Lecture 6

- Electrostatic Actuation
  - Derived from an energy perspective
  - Parallel-plate actuator
  - Comb-drive actuator
- Mechanics of Material
  - Beam Deflection
    - Integration method
    - Energy method
- Bending of Plates
- Flexure Spring Constants
- Residual Stress and Stress Gradient

Energy Perspective: Charge Control

- Due to the conservation of the energy of the system:
  \[ dW_e = \int q' dq' \]

- By partial differentiation:
  \[ F_e = -\frac{\partial W_e}{\partial z} \mid_{q \text{ const}} \]

- Calculate stored energy by integration along the path:
  \[ W_e = \int F_e dz \]

Cont’d

- The amount of charge on conductor is:
  
  \[ q' = CV \]

- By substitution, the energy \( W_e \) is:
  
  \[ W_e(z,q) = \int q' \frac{q'}{C(z)} dq' = \frac{q^2}{2C(z)} \]

- The electrostatic force in the z direction is:
  
  \[ F_e = -\frac{\partial W_e}{\partial z} \bigg|_{q \text{ const}} = \frac{q^2}{2C(z)^2} \frac{dC(z)}{dz} \]

- Is the charge control easy to implement?

---

Energy Perspective: Voltage Control

- Introduce the co-energy expression:

- By partial differentiation:
  
  \[ F_e = -\frac{\partial W_e'}{\partial z} \bigg|_{v \text{ const}} \]

- Integrate along the path as shown:
  
  \[ W_e' = \]

- The electrostatic force in the z direction is:
  
  \[ F_e = -\frac{\partial W_e'}{\partial z} \bigg|_{v \text{ const}} = \frac{1}{2} \frac{dC(z)}{dz} v^2 \]
**Parallel-Plate Actuators**

- The parallel-plate electrostatic force is:
  \[
  F_z = \frac{1}{2} \frac{d}{dz} V^2 = \frac{1}{2} \frac{d}{dz} \left( \frac{\varepsilon_o A}{(g-z)} \right) V^2 = \frac{1}{2} \frac{\varepsilon_o A}{(g-z)^2} V^2
  \]

- The force is nonlinear with respect to the applied voltage and the displacement.

- Pull-in would happen due to a positive-feedback mechanism.

**Electrostatic Pull-in: the Transfer-Function Perspective**

- The dynamic equation is:
  \[
  m \ddot{z} + b \dot{z} + k z = F_z = \frac{1}{2} \frac{\varepsilon_o A}{(g-z)^2} V^2
  \]

- At any equilibrium point \((Z_o, V_o)\):
  \[
  k Z_o = \frac{\varepsilon_o AV_o^2}{2(g-Z_o)^2}
  \]
  \[
  \Rightarrow V_o^2 = \frac{2k Z_o (g-Z_o)^2}{\varepsilon_o A} \quad (1)
  \]
  \[
  or \ (g-Z_o)^2 = \frac{\varepsilon_o AV_o^2}{2k Z_o} \quad (2)
  \]

- Consider a small variation of \(\Delta z\) and \(\Delta v\) around the operating point \((Z_o, V_o)\):
  \[
  m \Delta \ddot{z} + b \Delta \dot{z} + k (Z_o + \Delta z) = \left. F_z \right|_{Z_o, V_o} + \Delta F_e
  \]
  \[
  \therefore \ m \Delta \ddot{z} + b \Delta \dot{z} + k \Delta z = \Delta F_e
  \]
Cont’d

- Use Taylor-series expansion and substitute (1) and (2):
  \[
  \frac{\Delta F}{\Delta V} = \left. \frac{\partial F}{\partial z} \right|_{z = z_0} \Delta z + \left. \frac{\partial F}{\partial V} \right|_{z = z_0} \Delta V = \frac{2kZ_o}{g - Z_o} \Delta z + \frac{\varepsilon_o A V_o}{(g - Z_o)^2} \Delta V
  \]
  \[
  = \frac{2kZ_o}{g - Z_o} \Delta z + \frac{2kZ_o}{V_o} \Delta V
  \]

- Therefore,
  \[
  m\Delta z + b\Delta z + k\Delta z = \Delta F
  \]
  \[
  m\Delta z + b\Delta z + \Delta z \left( k - \frac{2kZ_o}{g - Z_o} \right) = \frac{2kZ_o}{V_o} \Delta V
  \]

- Define the “negative” electrical spring \( k_e \):
  - At \( Z_o = g/3 \), the electrical spring completely negates the mechanical spring (the actuator is marginally stable)
  \[
  k_e = -\frac{2Z_o}{g - Z_o} \Rightarrow \frac{2(Z_o/g)}{1 - (Z_o/g)} k
  \]

Cont’d

- The small-signal transfer function is:
  \[
  \Delta z = \frac{2kZ_o}{V_o} \frac{\Delta V}{m s^2 + b s + (k + k_e)}
  \]

- For displacements larger than \( g/3 \), the sum of \( k + k_e \) (will have right-half-plane pole) is negative \( \Rightarrow \) the system is unstable

- Substitute \( Z_o = g/3 \) into (1), we obtain the corresponding pull-in voltage:
  \[
  V_{pi} = \sqrt{\frac{8kg^3}{27\varepsilon_o A}}
  \]
Electrostatic Pull-in: Force-Balancing Perspective

- Solve $F_e = F_m$ (3rd-order algebraic equation); there are normally two intersections except at the pull-in
- No intersection (equilibrium state) after $V > V_{pi}$

\[
\frac{z}{g} \begin{cases} 
< 1/3: \text{stable equilibrium} \\
> 1/3: \text{unstable equilibrium} 
\end{cases}
\]

Lateral Comb-Drive Actuator

- Use the fringing field lines to drive, eliminating the 1/3 gap problem
- Force is linear with respect to the displacement $\Delta x$
- However the attainable force is smaller than parallel plates
- The capacitance and force are:

\[
C(x) = 2 \cdot \frac{\varepsilon_0 h (L + x)}{g}
\]

\[
F_e = \frac{\varepsilon_0 h V^2}{g}
\]
Lateral Comb Drive with a Ground Plane

- Three conductors: rotor (r), stator (s), and ground plane (p)
- The differential co-energy of the comb-drive is:
  \[ dW_e' = q_s = q_r = \]
- By integration of \( dW_e' \) along the 3-dimensional path:
  \[ W_e' = \]

Cont’d

- The lateral force is:
  \[
  F_{e,x} = \frac{1}{2} \frac{dC_{sp}(x)}{dx} V_s^2 + \frac{1}{2} \frac{dC_{sp}(x)}{dx} V_s^2 + \frac{1}{2} \frac{dC_{rs}(x)}{dx} (V_s - V_r)^2
  \]
  for \( V_r = 0 \):
  \[
  F_{e,x} = \frac{1}{2} \frac{d}{dx} (C_{sp} + C_{rs}) V_s^2
  \]
Comb-Drive Levitation Force

- With a bottom ground plane, the levitation force can be introduced due to the asymmetrical electric-field line distribution on top and bottom of the fingers.
- The expression is analogous to the lateral force expression:

\[
F_{r,z} = \frac{1}{2} \frac{dC_{m}(x)}{dz} V_s^2 + \frac{1}{2} \frac{dC_{p}(x)}{dz} V_r^2 + \frac{1}{2} \frac{dC_{s}(x)}{dz} (V_s - V_r)^2
\]
**Axial Stress and Strain**

- Axial stress is normal to the applied surface
  - $\sigma = \frac{F}{A}$ (typically in N/m$^2$, or Pa)
  - $F$ is positive if it is a tensile force, and negative if it is a compressive stress
- Strain $\epsilon = \frac{\Delta L}{L_o}$ (dimensionless)

![Axial Stress Diagram](attachment:axial_stress_diagram.png)

**Shear Stress and Strain**

- Shear stress is parallel to the applied surface: $\tau = \frac{F}{A}$
- Shear strain $\gamma$ (in radian) is the angular deformation with respect to original structural shape

![Shear Stress Diagram](attachment:shear_stress_diagram.png)
Shear Stress and Torsion

\[ T = \int \rho dF \]
\[ = \int \rho (\tau_{\text{max}} \rho) dA \]
\[ = \tau_{\text{max}} J \]
\[ \Rightarrow \tau_{\text{max}} = \frac{Tc}{J} \]
\[ \therefore \tau = \frac{T \rho}{J} \]

Take any cross-section

Assume that the shearing stress varies linearly with the distance \( \rho \)

Material Properties

- Modulus of Elasticity (Young's Modulus): \( E = \) stress / strain, \( \sigma / \varepsilon \)
- Modulus of Rigidity: \( G = \) shear stress / shear strain, \( \tau / \gamma \)
- Poisson's Ratio: \( v = \) lateral strain/axial strain, \((\Delta w/W_0)/(\Delta L/L_0)\)
- Relation between \( E, G, \) and \( v \): \( G = E / [2(1 + v)] \)
### MEMS Material Properties

<table>
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<tr>
<th>Material</th>
<th>Young's Modulus (GPa)</th>
<th>Density (kg/m³)</th>
<th>Thermal Conductivity (W/m-K)</th>
<th>Thermal Expansion Coefficient (K⁻¹)</th>
<th>Yield Strength (GPa)</th>
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<td>Diamond*</td>
<td>1035</td>
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</tbody>
</table>

* Asterisks mean for single-crystal material
* From K. Peterson, Proc. IEEE, May, 1982

**Material property database**
- MEMS Clearinghouse, http://www.memsnet.org/material

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### (Second) Moment of Inertia of an Area

#### Moment of Inertia w.r.t. the x axis

\[
dI_x = y^2 \, dA
\]

\[
I_x = \int y^2 \, dA
\]

#### Moment of Inertia w.r.t. the y axis

\[
dI_y = x^2 \, dA
\]

\[
I_y = \int x^2 \, dA
\]

#### Polar moment of Inertia

\[
J_o = \int r^2 \, dA = I_x + I_y
\]
Moment of Inertia Examples

<table>
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<tr>
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<th>1/12 bh³</th>
<th>1/12 b²h</th>
<th>1/3 bh³</th>
<th>1/3 b²h</th>
<th>1/12 bh(b²+h²)</th>
</tr>
</thead>
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Pure Bending

- Our goal is to understand how the bending moment M is related to the radius of curvature (and thus the deflection) of structures

\[
\frac{d^2y}{dx^2} = \frac{M(x)}{EI}
\]

- A member subjected to equal and opposite couples acting in the same longitudinal plane is said to be in pure bending
- For example, you can imagine M is the applied moment and M' is the reaction moment with the same magnitude in the reversed direction
A Beam Cross Section in Pure Bending

- Based on the free-body diagram, a counter-balancing moment exists in any cross section; the moment is a combinational effect of the normal stress and shear stress acting on the plane.

\[
\tau_{xy} \, dA + \tau_{xz} \, dA + \sigma_x \, dA = M
\]

How Do the Stresses Relate to the Moment?

- The total stresses contribute to a single moment \( M \):
  
  - **x components:**
  
    - Moments about y axis:
    
    - Moments about z axis:
How Does Longitudinal Strain Relate to the Radius of Curvature?

- The plane corresponding to the neutral axis AB has no deformation ($\varepsilon_x = 0$)

Longitudinal strain of CD:

$$\varepsilon_x = \frac{L_{CD} - L_{AB}}{L_{AB}} = \frac{(\rho - y)\theta - \rho\theta}{\rho\theta} = \frac{-y}{\rho}$$

The max. absolute value of strain: $\varepsilon_m = \frac{c}{\rho}$

A linear relationship:

$$\varepsilon_x = -\frac{y}{c} \varepsilon_m$$ (note the minus sign due to the direction of M)

Cont’d

- Remember previously we have:

$$\int (-y\sigma_x dA) = M$$

$$\int (-y\sigma_x dA) = \int (-y)(E\varepsilon_x) dA = \int (-y)[E(-\frac{y}{c}\varepsilon_m)]dA$$

$$= \frac{\sigma_m}{c} \int y^2 dA = M$$

$$\Rightarrow \sigma_m = \frac{Mc}{l}$$ (note that it is positive since $\varepsilon_m$ is positive)

$$\therefore \sigma_x = -\frac{My}{l}$$
Cont’d

Finally, bending moment is related to the radius of curvature by:

$$\sigma_x = \frac{M}{I} \cdot \frac{E}{I} = \frac{My}{E} = -\frac{1}{\rho} = M$$

Equation of the elastic curve (from your basic Calculus):

$$\frac{1}{\rho} = \frac{d^2 y}{dx^2} \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2}$$

Assuming a small beam deflection for an elastic beam:

$$\frac{d^2 y}{dx^2} = \frac{M(x)}{EI}$$

Boundary Conditions for Supported Beam

1. Cantilever beam ($y_A = 0$, $(dy/dx)_A = 0$)

2. Beam with a guided-end ($y_A = 0$, $(dy/dx)_B = 0$)

3. Simply supported beam ($y_A = 0$, $y_B = 0$)
Deflection of Beams by Integration

Task 1: Specify bending moment $M(x)$
- Use of free-body diagram: $\sum \text{Force} = 0$, and $\sum \text{Moment} = 0$ at equilibrium
- Example of a cantilever beam subjected to a pointed load $P$

\[ \sum \text{Force} = 0 \implies V = -P \]
\[ \sum \text{Moment} = 0 \implies M = -Px \]

Deflection of Beams by Integration

Task 2: Integration of the 2nd-order differential equation and insert boundary conditions:

First integration:
\[ EI \frac{d^2y}{dx^2} = M = -Px \]
\[ EI \frac{dy}{dx} = -\frac{1}{2}Px^2 + C_i \]

Boundary condition:
\[ \frac{dy}{dx} \bigg|_{x=L} = 0 \implies C_i = \frac{1}{2}PL^2 \]

Second integration:
\[ Ely = -\frac{1}{6}Px^3 + \frac{1}{2}PL^2x + C_2 \]

Boundary condition:
\[ y \bigg|_{x=L} = 0 \implies C_2 = -\frac{1}{3}PL^3 \]

Final beam curve:
\[ y = \frac{P}{6EI}(-x^3 + 3L^2x - 2L^3) \]

Tip deflection = $-\frac{PL^3}{3EI}$. So spring constant = $3EI / L^3$
Deflections of Beams

\[ y = -\frac{PL^3}{3EI} \text{ at } x = L \]
\[ y = -\frac{qL^4}{8EI} \text{ at } x = L \]
\[ y = -\frac{PL^3}{48EI} \text{ at } x = L/2 \]
\[ y = -\frac{qL^4}{384EI} \text{ at } x = L/2 \]

Energy Method – Strain Energy

- The strain energy stored in a structural body is the work done by the applied load \( P \) with a deformation \( x_1 \):
  \[ U = \int_0^{x_1} Pdx = \frac{1}{2}kx_1^2 \]

- Strain-energy density:
  \[ u = \frac{U}{V} = \]

- So the strain energy is:
  \[ U = \int \frac{\sigma^2}{2E} \, dV \]
**Strain Energy under Axial Loading**

\[ U = \int \frac{P^2}{2EA^2} dV \]

\[ = \int \frac{P^2}{2EA^2} (A dx) \]

\[ = \frac{P^2 L}{2EA} \]

**Strain Energy in Bending**

\[ U = \int \frac{\sigma_x^2}{2E} dV \]

\[ = \int \frac{M^2 y^2}{2EI^2} dV \]

\[ = \int_0^L \frac{M^2}{2EI^2} \left( \int y^2 dA \right) dx \]

\[ = \int_0^L \frac{M^2}{2EI} dx \]

Example:

\[ U = \int_0^L \frac{(-Px)^2}{2EI} dx = \frac{P^2 L^3}{6EI} \]
Elastic Strain Energy for Shearing Stresses

- The strain-energy density is expressed by:

\[ u = \int_0^{\pi/2} \tau_{xy} \gamma_{xy} \, d\gamma_{xy} \]
\[ = \int_0^{\pi/2} G \gamma_{xy} \gamma_{xy} \, d\gamma_{xy} \]
\[ = \frac{1}{2} G \gamma_{xy}^2 \]
\[ = \frac{\tau_{xy}^2}{2G} \]

- The total strain energy

\[ U = \int \frac{\tau_{xy}^2}{2G} \, dV \]

Strain Energy in Torsion

- The total strain energy:

\[ U = \int \frac{\tau_{xy}^2}{2G} \, dV \]
\[ = \int \frac{(T \rho / J)^2}{2G} \, dV \]
\[ = \int_0^L \frac{T^2}{2GJ^2} \left( \int \rho^2 \, dA \right) \, dx \]
\[ = \int_0^L \frac{T^2}{2GJ} \, dx \]
\[ = \frac{T^2 L}{2GJ} \]
**Deflections by the Castigliano’s Theorem**

- The most effective method to compute structural deflections (or angles of rotation) for complex structures (e.g. meandering springs) or structures under various loads
- By theorem, the deflection $x_i$ of a structure at the point of application of a load $P_i$ is determined by:
  - Step 1: calculate the total strain energy $U$
  - Step 2: do the partial differentiation so $x_i = \frac{\partial U}{\partial P_i}$
- For a beam:

**Example: Cantilever Beam under a Distributed Load**

- Goal: find the deflection and slope at $A$
- Deflection at $A$: add a dummy force $Q_A$ at $A$

Moment along the $x$ axis:

$$ M = \int_0^L \frac{dM}{dx} dx $$

Deflection at $A$:

$$ y_A = \frac{\partial U}{\partial Q_A} = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial Q_A} dx $$

Let $Q_A = 0$ in $M$:

$$ y_A = \frac{1}{EI} \int_0^L \frac{1}{2} w x^2 (-x) dx = \frac{wL^4}{8EI} $$

Diagram of the cantilever beam under a distributed load.
**Example: Cantilever Beam under a Distributed Load**

- Find the slope at A: add a dummy moment \( M_A \)

Moment along the x axis:

\[
M = -M_A - \frac{1}{2}wx^2
\]

Angle of rotation at A:

\[
\theta_A = \frac{\partial U}{\partial M_A} = \frac{\partial}{\partial M_A} \int_0^L \frac{M^2}{2EI} dx = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial M_A} dx
\]

Let \( M_A = 0 \) in \( M \):

\[
\theta_A = \frac{1}{EI} \int_0^L \left( -\frac{1}{2}wx^2 \right) (-1) dx = \frac{wL^3}{6EI}
\]

---

**Bending of Plates**

- The analysis is very useful for design of pressure sensors
- The small-amplitude deflection \( w(x,y) \) of a membrane under a two-dimensional pressure \( p \) is:

\[
D \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) = p(x,y)
\]

\( D \) is the flexural rigidity defined as:

\[
D = \frac{Eh^3}{12(1-\nu^2)} \quad h: \text{thickness}
\]

- Based on solved \( w(x,y) \), the bending moments can be calculated as:

\[
M_x = -D \left( \frac{1}{E} \rho_x + \frac{v}{E} \rho_y \right) = -D \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)
\]

\[
M_y = -D \left( \frac{v}{E} \rho_x + \frac{1}{E} \rho_y \right) = -D \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)
\]

Cont’d

- Assume the stress profile in the z direction at any point is triangular (zero at neutral plane and changes linearly), the stress at each position (x, y) can be calculated
- A square membrane has been solved by S.P. Timoshenko
  - Maximum bending moments appear at the center of the sides of the membrane, and decrease towards the corners and towards the center of the membrane
  - Accordingly, the maximum surface stress in the middle of the sides is:

\[
(\sigma_z)_{\text{max}} = 0.31 \cdot \frac{pa^2}{h^2}
\]

Flexure Spring Constants

**Procedure**

1. Reduce problem using symmetry
2. Determine boundary conditions (BC’s), giving set of simultaneous equations
3. Draw free-body diagram
4. Solve force, moment and tension equilibrium for each beam segment
5. Calculate moment along each beam segment
6. Solve BC equations from step 2 using energy methods
7. Solve for spring constant, \( k_x = \frac{F}{\delta} \)

---

**Clamped-Clamped Flexure**

- Modeled as four guided-end beams
- Apply a lateral force \( F_x \) to find the displacement \( \delta_x \), and thus the spring constant \( k_x \)
- An external bending moment, \( M_o \), constrains the angle in the analysis
- The beam bending moment is:
  \[
  M = M_o - F_x \delta_x
  \]
- The strain energy of the beam is:
  \[
  U = \frac{1}{2} \int \frac{M^2}{EI} dz
  \]
  \[
  I_z = \frac{w^3 t}{12}
  \]
Cont’d

- Apply Castigliano’s theorem, and the constraint \( \theta_o = 0 \):

\[
\theta_o = \frac{\partial U}{\partial M_o} = \int_0^L \frac{M}{EI_z} \frac{\partial M}{\partial M_o} d\xi = \frac{1}{EI_z} \int_0^L (M_o - F_x \xi) d\xi = 0
\]

so \( M_o = F_x L/2 \), and \( M = F_x (L/2 - \xi) \)

- Once more, apply the theorem:

\[
\delta_x = \frac{\partial U}{\partial F_x} = \int_0^L \frac{M}{EI_z} \frac{\partial M}{\partial F_x} d\xi = \frac{F_x}{EI_z} \left( \frac{L}{2} - \xi \right)^2 d\xi = \frac{F_x L^3}{12EI_z}
\]

\[
k_{x,\text{beam}} = \frac{F_x}{\delta_x} = 12EI_z / L^3
\]

\[
k_x = 4 \cdot k_{x,\text{beam}} = 48EI_z / L^3
\]

also \( k_x = 4 \cdot k_{x,\text{beam}} = 48EI_z / L^3 \)

Crab-leg Flexure: \( k_x \)

- Apply \( F_x, F_y, \) and \( M_o \) at the end of the thigh

- Bending moments of the thigh \( (M_a) \) and shin \( (M_b) \) are:

\[
M_a = M_o - F_y \xi
\]

\[
M_b = M_o - F_y (M_o - F_y L_a - F_x \xi)
\]

- Apply the boundary conditions:

\[
\theta_o = \frac{\partial U}{\partial M_o} = \int_0^L \frac{M_a}{EI_z} \frac{\partial M_a}{\partial M_o} d\xi + \int_0^{L'} \frac{M_b}{EI_z} \frac{\partial M_b}{\partial M_o} d\xi = 0 \quad (1)
\]

\[
\delta_x = \frac{\partial U}{\partial F_x} = \int_0^L \frac{M_a}{EI_z} \frac{\partial M_a}{\partial F_x} d\xi + \int_0^{L'} \frac{M_b}{EI_z} \frac{\partial M_b}{\partial F_x} d\xi = 0 \quad (2)
\]

\[
\delta_x = 0
\]

\[
\theta_o = 0
\]
Cont’d

- \( M_o \) is solved from (1), and \( F_y \) is solved from (2) and the solved \( M_o \):

\[
M_o = \frac{L_b^2 F_x + 2L_o L_b F_y + (w_b / w_o)^2 L_b^2 F_y}{2[L_o + (w_b / w_o)^2 L_o]}
\]

\[
F_y = -\frac{3L_b^2 F_x}{L_o (4L_o + (w_b / w_o)^2 L_o)}
\]

- \( M_o \) and \( F_y \) eventually are functions of \( F_x \)

- Finally,

\[
\delta_x = \frac{\partial U}{\partial F_x} = \int_0^1 M_o \frac{\partial M_o}{\partial F_x} d\xi + \int_0^1 M_b \frac{\partial M_b}{\partial F_x} d\xi = \frac{L_b^2}{3El_z,b} \left( \frac{w_b}{w_o} \right)^2 L_o F_x
\]

\[
k_z = 4 \cdot \frac{F_x}{\delta_x}
\]

Crab-leg Flexure: \( k_z \)

- Apply \( F_z, M_o \), and \( T_o \) at the end of the thigh
- Strain energy from torsion must be included in the \( z \) spring constant calculation

\[
U = \frac{1}{2} M_o \frac{\partial M_o}{\partial F_x} d\xi + \int_0^1 M_b \frac{\partial M_b}{\partial F_x} d\xi = \frac{L_b^2}{3El_z,b} \left( \frac{w_b}{w_o} \right)^2 L_o F_x
\]

\[
s = \frac{1}{2} \left( 1 - \frac{1}{1 + \frac{J}{tw}} \right)
\]

\[
J = \frac{wt}{3} \left( 1 - \frac{192}{\pi^2} \frac{t}{W} \sum_{i=0}^n \tan \left( \frac{tw_i}{2t} \right) \right)
\]
Cont’d

From the free-body diagram, we get:

\[ M_z = M_z - F_z \xi \]
\[ T_s = T_s \]
\[ M_b = T_b - F_z \xi \]
\[ T_b = M_t = M_z - F_z L_z \]

By solving

\[ \theta_s = \int_0^L \left( \frac{M_z}{E I_{x,a}} \frac{\partial M_z}{\partial M_o} + \frac{T_s}{G J_s} \frac{\partial T_s}{\partial M_o} \right) d\xi + \int_0^L \left( \frac{M_z}{E I_{x,b}} \frac{\partial M_z}{\partial M_o} + \frac{T_s}{G J_s} \frac{\partial T_s}{\partial M_o} \right) d\xi = 0 \]
\[ \phi_s = \int_0^L \left( \frac{M_z}{E I_{x,a}} \frac{\partial M_z}{\partial T_s} + \frac{T_s}{G J_s} \frac{\partial T_s}{\partial T_s} \right) d\xi + \int_0^L \left( \frac{M_z}{E I_{x,b}} \frac{\partial M_z}{\partial T_s} + \frac{T_s}{G J_s} \frac{\partial T_s}{\partial T_s} \right) d\xi = 0 \]
\[ \delta_s = \int_0^L \left( \frac{M_z}{E I_{x,a}} \frac{\partial M_z}{\partial F_z} + \frac{T_s}{G J_s} \frac{\partial T_s}{\partial F_z} \right) d\xi + \int_0^L \left( \frac{M_z}{E I_{x,b}} \frac{\partial M_z}{\partial F_z} + \frac{T_s}{G J_s} \frac{\partial T_s}{\partial F_z} \right) d\xi \]

Cont’d

The z-direction spring constant is:

\[ k_z = 4 \frac{F_z}{\delta_z} = \frac{48 S_{e_b} S_{e_b} L_z + S_{e_b} L_b + S_{e_b} L_b}{S_{e_b} S_{e_b} + 4 S_{e_b} S_{e_b} L_z L_b + S_{e_b} S_{e_b} S_{e_b} L_z L_b + 4 S_{e_b} S_{e_b} S_{e_b} L_z L_b + S_{e_b} S_{e_b} S_{e_b} L_z L_b + S_{e_b} S_{e_b} S_{e_b} L_z L_b} \]

where \( S_{e_b} = E I_{x,a} S_{e_b} + S_{e_b} S_{e_b} S_{e_b} + S_{e_b} S_{e_b} S_{e_b} + S_{e_b} S_{e_b} S_{e_b} \) and \( S_{e_b} = G J_s + S_{e_b} S_{e_b} \)
**Serpentine Flexure**

The moment of each beam segment is deduced from the free-body diagram:

\[
M_{x,i} = M_0 + F_x [\xi + (i-1)a] - \left(\frac{1}{2} - 1\right) F_y b ; \quad i = 1 \text{ to } n
\]

\[
M_{y,j} = M_0 + F_x a + F_y \left[(-1)\xi - \left(\frac{1}{2} - 1\right) b\right] ; \quad j = 1 \text{ to } n - 1
\]

The total strain energy in the serpentine spring is:

\[
U = \sum_{i=1}^{n} \int_{0}^{a} \frac{M_{x,i}^2}{2EI_{x,y}} \, d\xi + \sum_{j=1}^{n-1} \int_{0}^{b} \frac{M_{y,j}^2}{2EI_{y,b}} \, d\xi
\]

Simultaneously solve the three equations:

\[
\delta_y = \frac{\partial U}{\partial F_y} = 0
\]

\[
\theta_\phi = \frac{\partial U}{\partial M_0} = 0
\]

\[
\delta_\phi = \frac{\partial U}{\partial F_\phi}
\]
Stress

- Three main kinds of stress in materials
  - Externally applied stress
  - Thermal stress
  - Intrinsic stress
- Thermal stress arise from different temperature coefficients of expansion (TCE), \( \alpha \)
  - e.g., cool a thin-film material on substrate from \( T_d \) to \( T_r \)
  - \( \epsilon_{th} = (\alpha_2 - \alpha_1)(T_d - T_r) \)
- Intrinsic stress arises from interstitial atoms, mechanical annealing, microvoids, gas entrapment, grains
- After deposition of thin film, thermal + intrinsic stress = residue stress
Stress in Thin Films

- Single crystal silicon has almost no residual stress
- Polysilicon exhibits both compressive and tensile stresses, depending on deposition conditions
- Silicon nitride is highly tensile
  - Low-tensile-stress silicon nitride is silicon-rich, excellent for membranes
- Silicon oxides are always relatively highly compressive
- Most metal are tensile, but dependent on deposition conditions

Residual Stress and Stress Gradient

- In doubly-supported beams, residual stresses modify the bending stiffness, and possible buckling phenomenon if the stress is compressive
- Non-uniform residual stresses in cantilevers, due to either a gradient through the cantilever thickness, or to the deposition of a different material onto a structure, can cause the cantilever to curl
Stress Gradient and Radius of Curvature

- Assume that the axial stress in the beam is:
  \[ \sigma = \sigma_0 - \frac{\sigma_1}{(h/2)} y \]

- Therefore the induced bending moment is:
  \[ M = \int_{-h/2}^{h/2} y\sigma(y\cdot dy) = \frac{1}{6}wh^2\sigma \]

- We know that:
  \[ \frac{1}{\rho} = \frac{M}{Ei} \Rightarrow \rho = \frac{Ewh^3}{12M} = \frac{Eh}{2\sigma_1} \]

- The stress is released in the form of curl