Lecture 7: Noise

- Basics of noise analysis
- Thermomechanical noise
  - Air damping
- Electrical noise
  - Interference noise
    - Power supply noise (60-Hz hum)
    - Electromagnetic interference
  - Electronics noise
    - Thermal noise
    - Shot noise
    - Flicker (1/f) noise
- Calculation of total circuit noise


Why Do We Care?

- Noise affects the minimum detectable signal of a sensor
- Noise reduction:
  - The frequency-domain perspective: filtering
    - How much do you see within the measuring bandwidth
  - The time-domain perspective: probability and averaging
- Bandwidth Clarification
  - -3 dB frequency
  - Resolution bandwidth
Time-Domain Analysis

- **White-noise** signal appears randomly in the time domain with an average value of zero.
- Often use root-mean-square voltage (current), also the normalized noise power with respect to a 1-Ω resistor, as defined by:

\[
V_{n(\text{rms})} = \left[ \frac{1}{T} \int_0^T V_n^2(t) dt \right]^{1/2}
\]

\[
I_{n(\text{rms})} = \left[ \frac{1}{T} \int_0^T I_n^2(t) dt \right]^{1/2}
\]

- Signal-to-noise ratio (SNR) = 10 \cdot \log (\text{signal power}/\text{noise power})
- Noise summation

\[
V_n(t) = V_{n1}(t) + V_{n2}(t)
\]

0, for uncorrelated signals

\[
V_{n(\text{rms})}^2 = \frac{1}{T} \int_0^T [V_{n1}(t) + V_{n2}(t)]^2 dt = V_{n1(\text{rms})}^2 + V_{n2(\text{rms})}^2 + \frac{2}{T} \int_0^T V_{n1} V_{n2} dt
\]

Frequency-Domain Analysis

- A random signal/noise has its power spread out over the frequency spectrum.
- Noise spectral density is the average normalized noise power over a 1-Hz bandwidth (unit = volts-squared/Hz or amps-square/Hz).
- It is common to use root spectral density (e.g., V/√Hz).
Filtered Noise

- Root-mean-squared voltage can also be obtained in the frequency domain using
  \[ V_{n(rms)}^2 = \int_0^\infty V_n^2(f) df \]

- Total mean-squared value from a filter output is:
  \[ V_{no(rms)}^2 = \int_0^\infty |A(j2\pi f)|^2 V_n^2(f) df \]

- You should avoid designing circuits with a larger bandwidth than your signal actually requires

\[ V_n^2(f) \quad A(s) \quad V_{no}^2(f) = |A(j2\pi f)|^2 V_n^2(f) \]

Example: Filtered Noise

\[ V_n^2(f) \quad |A(2\pi f)|, \text{dB} \quad A(s) = 1/(1 + s/2\pi f_o) \]

\[ V_{no(rms)}^2 = \]

\[ f_o = 10^3 \]
**Noise Bandwidth**

\[ |A(2\pi f)| = \frac{1}{1 + s/2\pi f_o} \]

- Noise bandwidth \( f_x = \frac{\pi f_o}{2} \)

**Minimum Detectable Signal**

- Usually expressed in VOLTS
- Min. detectable signal = root spectral density \( \times \sqrt{BW} \) / sensitivity

For example: \( g \) \( \sqrt{V/Hz} \) \( \sqrt{Hz} \) \( V/g \)
Thermomechanical (Brownian) Noise

- For mechanical sensors, thermomechanical noise due to air damping is the ultimate performance limit.
- A dimensionless parameter, namely, the quality factor, is commonly used to describe the damping and also the Brownian noise.

First: Understand 2nd-order Lumped-Parameter System

- Transfer function \( x(s)/F(s) = 1/(ms^2 + bs + k) = 1/[m(s^2 + 2\xi\omega_n s + \omega_n^2)] \)
- Damping ratio \( \xi = b / (2m\omega_n) \)
  - Underdamped (\( \xi < 1 \), overshoot), critically damped (\( \xi = 1 \)), and overdamped (\( \xi > 1 \))
- Natural frequency \( \omega_n = (k/m)^{1/2} \)
- Damped natural frequency \( \omega_d = (1 - \xi^2)^{1/2}\omega_n \)

\[ \text{Force } F(s) \quad \frac{1}{ms^2 + bs + k} \quad \text{Displacement } x(s) \]
Cont’d: Transient-Response Characteristics

- Reduce rise time $t_r \Rightarrow$ have to increase $\omega_n$
- Reduce maximum overshoot $M_p \Rightarrow$ have to increase $\xi$
- Settling time $t_s \propto 1/(\xi \omega_n)$

Second: Definition of Quality Factor (Q)

- The most fundamental definition of Q is that it is proportional to the ratio of energy stored to the energy lost, per unit time:
  \[ Q = \frac{\text{(energy stored)}}{\text{(average energy dissipated per radian)}}; \]  
  dimensionless
- Q is related to the damping ratio: $Q = 1 / (2\xi)$
- Q can be measured using the half-power points on frequency spectrum:
  \[ Q = \frac{\omega_n}{(\omega_{h2} - \omega_{h1})} \]
- Note that the amplitude is amplified by Q times at resonance
**Derivation: \( Q = 1/2\xi \)**

- Take the 2nd-order mechanical system for example:
  \[
  m\ddot{x} + b\dot{x} + kx = f(t)
  \]

- The rate of change of energy with time: (the minus sign indicates energy dissipation)
  \[
  \frac{dW}{dt} = \text{force} \times \text{velocity} =
  \]

- Assume a simple harmonic motion of \( x(t) = X\sin(\omega t) \), the dissipated energy per cycle is:
  \[
  \Delta W = \int_{t=0}^{2\pi/\omega_d} F\dot{x}dt = \pi b\omega_d X^2
  \]
  \[
  \omega_d = \omega_n\sqrt{1 - \xi^2}
  \]

**Cont’d**

- The stored total energy of the system:
  \[
  W = \frac{1}{2}kX^2 = \frac{1}{2}m\nu_{\text{max}}^2 = \frac{1}{2}mX^2\omega_d^2
  \]

- The quality factor:
  \[
  Q =
  \]
Verify Q and Half-Power Bandwidth

- The amplitude at resonant is amplified by Q times of the amplitude ($X_o$) at d.c.

\[ Q \approx \frac{\omega_n}{\omega_{h2} - \omega_{h1}} \]

![Graph showing amplitude vs frequency with Q and Q/\sqrt{2} marked.]

Derivations

- Standard 2nd-order system:

\[ G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + j2\xi\frac{\omega}{\omega_n}} \]

- At $\omega = \omega_n$, the resonant amplitude is Q; at $\omega_h = r\omega_n$, the amplitude is Q/\sqrt{2}. Therefore:

\[ |G(s)|_{s=j\omega_h} = \frac{Q}{\sqrt{2}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \]

- Also, we already know that:

\[ Q = \frac{1}{2\xi} \]
Cont’d

Therefore we get:

\[ r^4 - r^2(2 - 4\xi^2) + (1 - 8\xi^2) = 0 \]
\[ r_1^2 = 1 - 2\xi^2 - 2\xi\sqrt{1 + \xi^2} \approx 1 - 2\xi \]
\[ r_2^2 = 1 - 2\xi^2 + 2\xi\sqrt{1 + \xi^2} \approx 1 + 2\xi \]
\[ \omega_{h2}^2 - \omega_{h1}^2 = r_2^2\omega_n^2 - r_1^2\omega_n^2 = 4\xi\omega_n^2 \]
\[ \therefore \omega_{h2} + \omega_{h1} = 2\omega_n \]
\[ \therefore \omega_{h2} - \omega_{h1} = 2\xi\omega_n, \quad \frac{\omega_n}{\omega_{h2} - \omega_{h1}} = \frac{1}{2\xi} = Q \]

Example: A Parallel RLC Tank Circuit

The impedance is zero at DC and at infinitely high frequency:

\[ Z = \]

At the resonant frequency \( \omega_r = 1/(LC)^{0.5} \), the reactive terms are cancelled; \( Z = R \)

Peak energy stored in the network at resonance is:

\[ E_{tot} = \]
Cont’d

- The average power dissipated in the resistor at resonance:
  \[ P_{\text{ave}} = \frac{1}{2} I_{pk}^2 R \]

- The quality factor Q is:
  \[ Q = \frac{\omega_n E_{\text{tot}}}{P_{\text{ave}}} = \frac{R}{\sqrt{L/C}} \]
  \[ Q = \omega_n RC \]  
  Large R is good in parallel network

- How does the bandwidth (here it means \( \Delta \omega \) at the half-power bandwidth) relate to Q? Let \( \omega = \omega_n + \Delta \omega \), the -3 dB point, then
  \[
  Z_{-3dB} = \frac{1}{\frac{1}{R} + \frac{j}{\omega_n L} [2\Delta \omega \omega_n + \Delta \omega^2] L C} = \frac{1}{\frac{1}{R} + j2C\Delta \omega} = R/\sqrt{2}
  \]

Cont’d

- Therefore we get \( \Delta \omega = 1 / (2RC) \)
- The bandwidth \( \omega_{h2} - \omega_{h1} = 2\cdot\Delta \omega = 1 / RC \)
- We can verify that:
  \[
  \frac{\omega_n}{\omega_{h2} - \omega_{h1}} = \frac{1}{\sqrt{LC}} = \frac{1}{1/RC} = \frac{R}{\sqrt{L/C}} = Q
  \]
Air-Damping Analysis

Couette Damping

- Motion-induced shear stress \( \tau \) sets up a velocity gradient in the fluid between two plates.
- Viscosity is the proportionality constant that relates \( \tau = \eta \frac{\partial v}{\partial x} \); \( (\eta_{\text{air}} = 77 \mu\text{N}\cdot\text{s/m}^2 \ @ \ 25 \ ^\circ\text{C}) \)
- Incompressible laminar flow; density change is negligible
- Linear velocity profile is valid for small gaps

Shear stress acting on the plate:

\[
\tau = \eta \frac{\partial v_y}{\partial y} = \eta \frac{V_g}{g}
\]

Total force acting on the plate:

\[
F_y = \int_0^s \frac{\eta A V_g}{g} \, dy
\]
Squeeze-Film Damping

- Damping caused by squeezing air in and out of plates
- The distribution of gas pressure between plates is governed by the compressible gas-film Reynolds equation; assuming a small plate motion, the linearized Reynolds equation is:

\[ \frac{\partial^2 P}{\partial z^2} + \frac{1}{\nu} \frac{\partial P}{\partial t} = \frac{1}{\rho} \frac{\partial^2 P}{\partial z^2} \]

Atmospheric pressure: \( P_0 \)
Pressure change: \( \Delta P \)
Effective gas viscosity: \( \eta_{\text{eff}} = \frac{\eta}{1 + 9.638 \lambda \sqrt{\frac{k}{M}} } \) (\( \eta = 22.6 \mu \text{N s/m}^2 @ 1 \text{atm and } 25 ^\circ \text{C} \))
Knudsen number: \( K_n = \frac{\lambda}{g} \) - Mean free path of air (70 nm @ 1 atm and 25 °C)

Cont’d

- The equation can be solved by the eigenfunction method, assuming that \( \Delta p(x,t) = \Delta p(x)e^{-\alpha t} \) in an one-dimensional problem, followed by separation of variables
- Total force \( F(t) = P_{\text{ref}} \int \Delta p(x,t) dx \)
- Squeeze-film damping exhibits damper (low-frequency) and spring-like (high-frequency) behaviors
- The solution can be simulated by an equivalent circuit model
  - \( L_{mn} \) and \( R_{mn} \) depend on the plate size (w and l), gap, and effective viscosity (m,n are odd numbers)

\[ L_{mn} = \frac{(mn)^2 \pi^4 (g-z)^3}{64lwP_0} \]
\[ R_{mn} = \frac{(mn)^2 (m^2 l^2 + n^2 w^2) \pi^4 (g-z)^3}{768lw} \eta_{\text{eff}} \]

Derivation of Brownian Noise

Brownian Motion

- The zigzag motion performed by suspended particles in a liquid or a gas; named after the English botanist Robert Brown
- Fluctuations from thermodynamic equilibrium
- Derived by Einstein and Smoluchowski (using different methods from kinetic theory) in 1906
  - Experimentally verified by Perrin, Svedberg, Seddig, and others
- A simpler derivation can follow Langevin’s method

Zigzag particle motion
Thermomechanical Noise

- Brownian noise force:

\[
\Delta F_n^2 = \\
\]

\( k_b \): Boltzmann's constant \((1.38 \times 10^{-23} \text{ J/K})\)
\( T \): temperature in Kelvin
\( b \): damping coefficient \((\text{unit} = \text{N/(m/s)})\)
\( \Delta f \): measuring bandwidth \((\text{unit} = \text{Hz})\)

Here is the Derivation

A particle of mass \( M \) with arbitrary acceleration:

\[
m\ddot{r} = -b\dot{r} + F_{\text{total}}
\]
(1)

Friction force
Other influences combined

Multiply by \( r \):

Equation (2) becomes:

\[
\frac{1}{2} m \frac{d^2 r^2}{dt^2} + \frac{1}{2} b \frac{dr^2}{dt} = r \cdot F_{\text{total}} + m v^2
\]
Cont’d

Using the Equipartition theorem:

\[
\frac{1}{2}mv_x^2 = \frac{1}{2}mv_y^2 = \frac{1}{2}k_b T
\]

\[k_b = 1.3803 \times 10^{-23} \text{ J} / K (\text{Boltzmann constant})\]

\[
\frac{1}{2}mv^2 = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 = k_b T
\]

Therefore:

Cont’d

By integration from \( t = 0 \) to \( t = \Delta t \) to obtain the traveled distance:

\[
\Delta r = r_f - r_i = \frac{4k_b T}{b} \Delta t = \frac{4k_b T}{b} \Delta f
\]

\[
\therefore \Delta r = \sqrt{\frac{4k_b T}{b} \Delta f}
\]

To get the Brownian force, we first get the noise velocity:

\[
\Delta v = \frac{\sqrt{\Delta r^2}}{\Delta t} = \frac{\sqrt{4k_b T / b}}{\Delta t}
\]

\[
\text{then } \Delta F_n = b \cdot \Delta v = \sqrt{4k_b T b \Delta f}
\]
**Noise Models for Circuit Elements**

- Thermal noise is due to thermal excitation of charge carriers in a conductor; occurs in all resistors above absolute zero temperature
  - Was first observed by J.B. Johnson (Johnson noise) and analyzed by H. Nyquist in 1928
  - A white noise
- Shot noise occurs in pn junctions because the dc bias current is not continuous and smooth due to pulsed current caused by individual flow of carriers
  - Was first studied by W. Schottky in 1918
  - Depend on bias current; also a white noise
- Flicker noise (1/f noise) arises due to release of trapped charges in semiconductors
  - A significant noise source in MOS transistors

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**Equivalent Resistor Model with Added Noise**

- Spectral density functions:
  
  \[
  V_R^2(f) = \\
  I_R^2(f) = \\
  \]

- \[ R \text{ (noisy)} = \] \[ R \text{ (noiseless)} \]
  \[ V_R^2(f) = \]
  \[ I_R^2(f) = \]
Forward-Biased Diodes

- Spectral density functions:
  \[ I_d^2(f) = \]
  \[ V_d^2(f) = \]

- Small-signal resistance of the diode \( r_d \) does not contribute any thermal noise:
  \[ I_d = I_s e^{V_d/(kT/q)} \Rightarrow r_d = \frac{k_b T}{q I_d} \]

\[ \begin{array}{c}
\text{diode (noisy)} = \quad \text{V}_d^2(f) = \quad r_d \text{ (noiseless)} = \quad \text{I}_d^2(f)
\end{array} \]

MOS Transistors in the Active Region

- Spectral density function is the sum of the flicker (1/f) noise and thermal noise
  - 1/f noise voltage source and thermal noise voltage source
    \[ V_f^2(f) = \]
    \[ V_t^2(f) = \]

- The equivalent voltage source of 1/f and thermal noise sources
  \[ V_i^2(f) = V_f^2(f) + V_t^2(f) \]

\[ \begin{array}{c}
\text{noisy} = \quad \text{V}_i^2(f)
\end{array} \]
Operational Amplifiers (Op-amps)

- Noise is modeled using three uncorrelated input-referred noise sources: $V_n^2(f)$, $I_{n+}^2(f)$, and $I_{n-}^2(f)$
- Op-amps with MOSFET input transistors can ignore current noises

![Noiseless op-amp diagram]

Capacitors and Inductors

- Capacitors and inductors do not generate any noises; however, they accumulate noise generated by other noise sources
- The rms noise voltage $V_{no}$ across a capacitor is equal to $(k_B T/C)^{1/2}$, regardless of the resistance seen across it (why?)

$$V_{no}^2 = 4k_BTR$$

![Capacitor and inductor diagram]
Cont’d

- The rms noise current $I_{no}$ across inductor is equal to $(k_B T/L)^{1/2}$, regardless of the resistance seen across it.

![Diagram of an inductor with noise](image)

Sampled Signal Noise

- The sample-and-hold circuit for A/D or D/A conversion, when turned off, the noise as well as the desired signal is held on the capacitor $C$.
- Over-sampling of the signal helps to reduce the noise level.
- Signal values add linearly, whereas their noise values add as the root of the sum of squares (i.e., $n_{eq} = (n_1^2 + n_2^2 + n_3^2 + ...)^{0.5}$).

![Sound-level meter diagram](image)
**Op-amp Example**

Low-pass filter

\[ V_{m1}^2(f) = [I_{n1}^2(f) + I_{p1}^2(f) + I_{n3}^2(f)] R_i \left[ \frac{R_f}{1 + j2\pi f R_i C_f} \right]^2 \]

\[ V_{m2}^2(f) = [I_{n2}^2(f) + I_{p2}^2(f) + I_{n4}^2(f)] R_i \left[ \frac{R_f}{1 + j2\pi f R_i C_f} \right]^2 \]

\[ V_{m3}^2(f) = V_{m01}^2(f) + V_{m02}^2(f) \]

**Example: CMOS Differential Input Pair**

- **Goal:** to obtain an equivalent input-referred voltage noise by summing up all the contributed noises at the output, then dividing by the gain of the circuit

\[ V_{m0}^2(f) = \left( \frac{16}{3} \frac{k_b T}{g_{m1}} + \frac{16}{3} k_b T \left( \frac{g_{m3}}{g_{m1}} \right)^2 \left( \frac{1}{g_{m3}} \right) \right) \]

Noise output due to various sources:

\[ \frac{V_{m0,1}}{V_{n1}} = \frac{V_{m0,2}}{V_{n2}} = g_{m1} R_v \]

\[ \frac{V_{m0,3}}{V_{n3}} = \frac{V_{m0,4}}{V_{n4}} = g_{m3} R_v \]
Cont’d

Due to symmetry, the drain of Q2 will track that of Q1:

\[ \frac{V_{m,5}}{V_{n,3}} = \frac{g_{m5}}{2g_{m3}} \]  
(can be neglected)

Output noise value:

\[ V_{n}^2(f) = 2(g_{m1}R_o)^2V_{n1}^2(f) + 2(g_{m3}R_o)^2V_{n3}^2(f) \]

Output value can be related back to an equivalent input noise value by dividing it by the gain, \( g_{m1}R_o \):

\[ V_{neq}^2(f) = 2V_{n1}^2(f) + 2V_{n3}^2(f) \left( g_{m1}R_o \right)^2 \]

Cont’d

Thermal noise (1/f noise neglected) of each transistor can be substituted:

\[ V_{n}^2(f) = 4k_BT \left( \frac{2}{3} \right) \left( \frac{1}{g_{mi}} \right) \]

\[ g_{mi} = \sqrt{\frac{2\mu C_{ox}}{L}} I_{Dn} \]

Can increase \( g_{m1} \) (more current) to minimize thermal noise contribution. Remember the price you pay for noise reduction is POWER!!