Lecture 9

- Equivalent circuit representation
  - One-port and two-port elements
  - Electromechanical transducers
- Thermal transduction
  - Thermal actuation
  - Equivalent circuit of a self-heating resistor

Reference: Stephen D. Senturia, Microsystem Design

Goal

- Circuit simulator (e.g., HSpice) is proved capable of performing large-scale circuit simulation
- Is there better solution, other than the circuit simulator, for mixed-domain (e.g., electrical, mechanical, thermal etc) simulations?
One-Port Elements

- **Ports:** A port is a pair of terminals on a circuit element that must carry an equal current entering and leaving the element.
  - Effort (Across) variable $e$: voltage ($elec.$), force ($mech.$), etc
  - Flow (Through) variable $f$: current ($elec.$), velocity or displacement ($mech.$), etc
- There is another convention which adopts force as the through variable and velocity (displacement) as the across variable.
- By the arrow direction, the product of $e$ and $f$ is the power entering the element.

\[ e \times f \]

Lumped element

One-Port Elements

- **One-port source (active) elements:**
  - Effort source
  
  \[ e \times f \]
  
  Lumped element

- **One-port circuit elements:**
  - Resistor
  
  \[ e \times f \]
  
  Lumped element

  - Capacitor
  
  \[ e \times f \]
  
  Lumped element

  - Inductor
  
  \[ e \times f \]
  
  Lumped element
**Circuit Representation of the Mass-Spring-Damper Systems**

From Fig. 1, Free-body diagram: \[ F(s) = ms^2 + bs + k \cdot x(s) = ms + b + \frac{k}{s} \cdot i(s) \]

From Fig. 2, Kirchhoff’s voltage Law: \[ V(s) = (Ls + R + \frac{1}{sC}) \cdot i(s) \]

- Analogies exist between electrical and mechanical systems:
  - \( F(s) \) can be equivalent to \( V(s) \), and \( i(s) \) to \( i(s) \)
  - The mechanical components \( m, b, \) and \( k \) can be substituted by \( L, R, \) and \( 1/C \), respectively

![Figure 1](image1.png)

**Electromechanical Two-Port Transducer**

- Four variables to consider: two efforts and two flows
- A total of six ways to pick two independent variables from a set of four

<table>
<thead>
<tr>
<th>Electrical</th>
<th>Mechanical</th>
</tr>
</thead>
<tbody>
<tr>
<td>effort</td>
<td>through</td>
</tr>
<tr>
<td>effort</td>
<td>through</td>
</tr>
<tr>
<td>( V )</td>
<td>( I )</td>
</tr>
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<td>( V )</td>
<td>( F )</td>
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<tr>
<td>( I )</td>
<td>( V )</td>
</tr>
<tr>
<td>( I )</td>
<td>( F )</td>
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</tbody>
</table>

![Figure 2](image2.png)
Transformers and Gyrators

- They are important elements used to aid in translating variables from one energy domain to another
- They neither store or dissipate energy

For a lossless two-port, the total energy entering the element through both ports must be Zero:

\[
\begin{bmatrix}
  e_2 \\
  f_2
\end{bmatrix}
= \begin{bmatrix}
  e_1 \\
  f_1
\end{bmatrix}
\]

- The unit of \( n \) depends on the two energy domains
Standard Matrix Representation

\[
\begin{bmatrix}
V \\
F
\end{bmatrix} =
\begin{bmatrix}
Z_{EB} & T_{EM} \\
T_{ME} & Z_{MO}
\end{bmatrix}
\begin{bmatrix}
I \\
U
\end{bmatrix}
\]

- \( Z_{EB} = V / I \mid_{U=0} \) Blocked electrical impedance
- \( Z_{MO} = F / U \mid_{I=0} \) Open-circuit mechanical impedance
- \( T_{EM} = V / U \mid_{I=0} \) Open-circuit electromechanical transduction impedance
- \( T_{ME} = F / I \mid_{U=0} \) Blocked mechanical-electro transduction impedance

\( T_{EM} \) (e.g., Volt-sec/meter) and \( T_{ME} \) (e.g., Newton/Ampere) capture the transduction, usually they are equal.

Circuit Representation (Your Homework!)

- Let \( T_{EM} = T_{ME} = k_{EB} \):

\[
\begin{bmatrix}
V \\
F
\end{bmatrix} =
\begin{bmatrix}
Z_{EB} & k_{EB}Z_{EB} \\
k_{EB}Z_{EB} & Z_{MO}
\end{bmatrix}
\begin{bmatrix}
I \\
U
\end{bmatrix}
\]

- It can be represented by the equivalent circuit:

With the short-circuit mechanical impedance \( Z_{MS} \) and the electromechanical coupling constant \( k_e \):

\[
Z_{MS} = \quad \phi Z_{EB}
\]
Representation of the Parallel-Plate Electrostatic Actuator

- Goal: to find the circuit representation of the parallel-plate actuator; namely, the impedance matrix

\[
\begin{align*}
W'(V,x) & = C \\
\delta F & = k \delta x \\
\delta F_{out} & = b \delta x \\
\delta V_{in} & = R \delta I \\
\delta I_{out} & = 1/k \delta x
\end{align*}
\]

Note: \( x = -U/s \)
Representation of the Parallel-Plate Electrostatic Actuator

By derivation using the co-energy \( W' \), the voltage-controlled electrostatic force is:

\[
F = \quad \text{(equation)}
\]

The force \( F_{\text{out}} \) is:

\[
F_{\text{out}} = \quad \text{(equation)}
\]

Cont’d

We can represent the voltage \( V \) and the force \( F_{\text{out}} \) as the function of current \( I \) and velocity \( U \), or use charge \( Q \) and displacement \( x \).

For linearization at \( Q = Q_o \) and \( x = x_o \), the Jacobian matrix \( J \):

\[
\begin{bmatrix}
\delta V \\
\delta F_{\text{out}}
\end{bmatrix} =
\begin{bmatrix}
\delta Q \\
\delta x
\end{bmatrix}
\]

\( V \) can be expressed as:

\[
V = \quad \text{(equation)}
\]

\( F_{\text{out}} \) can be expressed as:

\[
F_{\text{out}} = \quad \text{(equation)}
\]
Cont’d

- At the operating point \((Q_o, x_o)\), note that \(\delta x = -\delta U/s\):

\[
\begin{bmatrix}
\delta V \\
\delta F_{out}
\end{bmatrix} = 
\begin{bmatrix}
\frac{\partial V}{\partial Q} & \frac{\partial V}{\partial x} \\
\frac{\partial F_{out}}{\partial Q} & \frac{\partial F_{out}}{\partial x}
\end{bmatrix} 
\begin{bmatrix}
\delta Q \\
\delta x
\end{bmatrix} = 
\begin{bmatrix}
\delta I \\
\delta U
\end{bmatrix}
\]

\[
= 
\begin{bmatrix}
Z_{EB} & Z_{EM} \\
Z_{ME} & Z_{MO}
\end{bmatrix} 
\begin{bmatrix}
\delta I \\
\delta U
\end{bmatrix}
\]

\[
Z_{EB} = \frac{g_o - x_o}{s\varepsilon_o A}, \quad Z_{MO} = \frac{k}{s}, \quad T_{EM} = T_{ME} = \frac{Q_o}{s\varepsilon_o A}
\]

Cont’d

- To compute \(\phi\) and \(Z_{MS}\):

\[
\phi = \frac{T_{EM}}{Z_{EB}} = \frac{Q_o}{g_o - x_o} = \frac{\varepsilon_o A V_o}{(g_o - x_o)^2}
\]

\[
k^2 = \frac{T_{EM}^2}{Z_{EB}Z_{MO}} = \frac{Q_o^2}{\varepsilon_o A k(g_o - x_o)}
\]

\[
Z_{MS} = \frac{k}{s}(1 - \frac{Q_o^2}{\varepsilon_o A k(g_o - x_o)})
\]

- \(Z_{MS}\) illustrates the spring-softering effect of the electrostatic actuator
The electrostatic force with $n$ fingers is:

$$F = \frac{1}{L} \frac{\partial F}{\partial l} \frac{\partial V}{\partial x}$$

The voltage $V$ and $F_{\text{out}}$ are:

$$V = \frac{1}{S} \frac{\partial F}{\partial x}$$

$$F_{\text{out}} = \frac{1}{S} \frac{\partial F}{\partial x}$$

**Representation of the Comb Drives**

Following the same derivation, we obtain:

$$\begin{bmatrix} \delta V \\ \delta F_{\text{out}} \end{bmatrix} = \begin{bmatrix} \frac{\partial V}{\partial Q} & \frac{\partial V}{\partial F_{\text{out}}} \\ \frac{\partial F_{\text{out}}}{\partial Q} & \frac{\partial F_{\text{out}}}{\partial F_{\text{out}}} \end{bmatrix} \begin{bmatrix} \delta Q \\ \delta F \end{bmatrix} = \begin{bmatrix} \frac{1}{S} \frac{\partial F}{\partial x} & -\frac{1}{S} \frac{\partial F}{\partial x} \\ \frac{1}{S} \frac{\partial F}{\partial x} & \frac{1}{S} \frac{\partial F}{\partial x} \end{bmatrix} \begin{bmatrix} \delta l \\ \delta U \end{bmatrix}$$

$$Z_{\text{EM}} = \begin{bmatrix} Z_{\text{EB}} \\ Z_{\text{EM}} \\ Z_{\text{MS}} \end{bmatrix} \begin{bmatrix} \delta l \\ \delta U \end{bmatrix}$$

$$Z_{\text{EB}} = \frac{1}{s \cdot 2\pi \varepsilon_0 (1 + x_c) h / g_s} = \frac{1}{s C_s}$$

$$T_{\text{EM}} = \frac{Q_s}{s C_s (1 + x_c)}$$

$$\phi = T_{\text{EM}} / Z_{\text{EB}} = Q_s / (1 + x_c) \approx 2n \cdot \varepsilon_0 h V_s / g_s \quad \text{(no displacement dependency)}$$

$$Z_{\text{MS}} = k$$

As expected, there is no electromechanical coupling to affect the spring constant, as shown by $Z_{\text{MS}}$.
Example: A Comb-Driven Resonator

- The resonator is driven by the left-side comb drive, and the output current is sensed on the right.
- High transduction gain is desired to achieve large output signal.

Example: A Micromechanical Band-pass Filter

- The coupling spring experiences the velocity difference.
- The two resonant frequencies are \((k/m)^{1/2}\) and \([(k+2k_c)/m]^{1/2}\).
- The use of many resonators in series, along with suitable damping, can result in a flat passband.
Thermal Transduction

Thermal Actuation

- The basic working principle used the bimorph structure
  - Materials have different thermal expansion coefficients, $\alpha$, Young’s modulus, $E$, and moment of inertia, $I$
  - Electro-thermal-mechanical transduction
- Advantages:
  - Ease of implementation
  - Larger motion than electrostatic actuation
- Drawbacks:
  - Power consumption
  - Fluctuation due to temperature drift
  - Response time?
**Bi-morph Beam**

The bi-morph beam made of dissimilar materials extends differently on the top and the bottom; widely used for thermal actuation.

- The beam must be in equilibrium (force and moment):
- The strain of the two materials at the interface must be the same:
- Use the above equations to eliminate $P_1$ and $P_2$:

\[
\frac{4}{bh^3\rho} \left( E_1 I_1 + E_2 I_2 \right) \left( \frac{1}{E_1} + \frac{1}{E_2} \right) = (\alpha_2 - \alpha_1) T - \frac{h}{2\rho}
\]

Given $T$, the radius of curvature $\rho$ (and displacement) can be solved.
Heat Transfer

- Answers the question how fast the thermal actuation can be
- The first law of thermodynamics:

\[ Q = W + \frac{dU}{dt} \]

- Heat transfer rate
- Work done by the system
- Change of internal thermal energy

Heat Conduction

- Fourier's law
- Describes the steady-state heat conduction behavior
- The heat flux, \( q \) (W/m²), resulting from thermal conduction is proportional to the magnitude of the temperature gradient, and opposing to it in sign

\[ q = -k \frac{dT}{dx} \] (One-dimensional)
1-D Heat Diffusion Equation

- Combine the 1st law of thermodynamics and the Fourier's law for the derivation
- The solution can help understand the transient response of heat conduction

\[ Q = -kA \frac{\partial T}{\partial x} \]

\[ Q_{net} = \]

Cont’d

- 1st law of thermodynamics tells us, if no internal produced energy (e.g., chemical),

\[ Q_{net} = \]

- Combining (1) and (2):

\[ \frac{\partial^2 T}{\partial x^2} = \frac{\rho c}{k} \frac{\partial T}{\partial t} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \]

- Thermal diffusivity \( \alpha \) is a measure of how fast the heat can be carried away

\[ \alpha = \frac{k}{\rho c} \quad \text{unit: } \left( \frac{J}{m \cdot s \cdot ^oC} \right) = \left( m^2 / s \right) \]
**Circuit Representation for Steady-State Analysis**

- Taking heat conduction as the example, let’s re-visit the Fourier law:

\[ Q = \frac{L}{kA} \cdot \Delta T \]

- The term \( L/kA \) can be represented as the thermal resistance, and the temperature difference is the applying potential

**Example: Temperature-Stabilized Accelerometer**

- Goal: To achieve temperature-controlled matched structural curl for better capacitive sensing results
- Lumped thermal resistances and capacitances to represent structural thermal behavior
- Heat flux is provided by an embedded polysilicon heater

Reference: H. Lakdawala, IEEE MEMS Workshop, 2002
Cont’d: Change of Curl with Rotor Heating @ 10 Hz

- Dynamic motion of the accelerometer measured with sinusoidal rotor heating (up to 200 °C) at 10 Hz

![Image](thermal-time-constant.png)

Thermal time constant = 2.81 ms

Courtesy: H. Lakdawala

Self-Heating of a Resistor

- The linear variation of resistance with temperature is:

\[
R = R_o [1 + \alpha_R (T - T_o)]
\]

- Can we represent the self-heating resistor using a circuit representation?
  - Mixed-domain electrical-thermal simulation
  - Use a current-source driven, or a voltage-source driven resistor?

![Diagram](resistor-diagram.png)

Ref: S.D. Senturia, Chapter 11.6
### Current-Source Drive

- Joule heating power = $I^2R$, and $C_T$ represents the heat capacity
- The “voltage” across $C_T$ and $R_T$ is $(T_R - T_o)$
- Sum up the currents at the right side:
- Are the two grounds the same?

Let $T_o = 0$ for simplicity:

\[
\begin{align*}
\text{Electrical circuit} & \\
\text{Thermal circuit} & \\
\end{align*}
\]

### Cont’d

- The steady-state temperature rise is:

\[
\frac{dT_{R,ss}}{dt} = \left(1 - \frac{\alpha_h I^2 R_T}{1 - \alpha_h I^2 R_T} \right) \left( \frac{1}{R_T C_T} - \frac{\alpha_h I^2 R_T}{C_T} \right) T_{R,ss} = \frac{I^2 R_T}{C_T}
\]

\[
\therefore T_{R,ss} = \frac{R_T R_T I^2}{1 - \alpha_h I^2 R_T}
\]

- The current-source value can be determined such that $T_{R,ss}$ approaches infinity
- This is how a metal fuse works!
Voltage-Source Drive

Electrical circuit

\[ V \quad + \quad I \quad R_T \quad - \]

Thermal circuit

\[ V/R_T \quad + \quad C_T \quad - \quad R_T \quad T_R \quad T_o \]

\[ \frac{dT_o}{dt} + \frac{1}{R_T} T_o = \frac{V^2}{\left[R_o (1 + \alpha T_R)\right]} = \frac{V^2}{R_o} (1 - \alpha T_R) \]

\[ \therefore \text{Time constant } \tau = \frac{R_o C_T}{1 + \alpha V^2 R_T / R_o} \]

\[ \text{similarly, } T_{K, o} = \frac{R_o V^2 / R_T}{1 + \alpha V^2 R_T / R_o} \]

- It can work normally as a heater without thermal runaway