Chapter 8
Propagation in Transmission Lines

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At high frequencies such that the length of a transmission line is not much shorter than one wavelength, the variations of voltage, current, and impedance along the line will be rather complicated, if the load is not matched to the line. To describe the variations, the voltage reflection coefficient is a convenient parameter. On the other hand, the transformation in impedance can be used to advantage to make some devices from a segment of transmission line. By using the Smith chart, the impedance transformation can be determined graphically. Further, the impedance transformation can be used in the impedance matching to reduce the reflection from a mismatched load.

8.1. Transmission-Line Circuits

In Chapter 7, we have obtained transmission-line equations for the TEM mode. Further, the equations lead to two uncoupled second-order differential equations. That is,

\[
\begin{align*}
    \frac{d^2 V(z)}{dz^2} + k^2 V(z) &= 0 \\
    \frac{d^2 I(z)}{dz^2} + k^2 I(z) &= 0.
\end{align*}
\]

It is known that the solution is

\[
\begin{align*}
    V(z) &= V_0 e^{-jkz} \\
    I(z) &= \frac{V_0}{Z_0} e^{-jkz}.
\end{align*}
\]

It is seen that the voltage and the current along a transmission line are related to each other by the characteristic impedance as \( V(z)/I(z) = Z_0 \).

In general, there are two TEM wave propagating on a transmission line: one forward and one backward. If only the forward or the backward TEM wave propagates along a transmission line, the impedance will be the characteristic impedance \( Z_0 \). However, if both the forward and the backward TEM waves appear, the situation will be more complicated as a consequence of interference between the waves.

Consider the case where a transmission line is terminated with a load of impedance \( Z_L \) located at \( z = \ell \). Generally, a wave tends to reflect from the load as a backward wave. Assume that the amplitudes of the forward and the backward waves are \( V_0^+ \) and \( V_0^- \), respectively. Thus the total
voltage and current are given by the linear superpositions as

\[
\begin{align*}
V(z) &= V_0^+ e^{-jkz} + V_0^- e^{jkz} \\
I(z) &= \frac{V_0^+}{Z_0} e^{-jkz} - \frac{V_0^-}{Z_0} e^{jkz},
\end{align*}
\]

where coefficients \(V_0^+\) and \(V_0^-\) are referred to the voltages at the input end \((z = 0)\). It is seen that the ratio of \(V(z)\) to \(I(z)\) varies along the line.

At the load \((z = \ell)\), the load voltage \(V_L\) and load current \(I_L\) are given by

\[
\begin{align*}
V_L &= V_0^+ e^{-jk\ell} + V_0^- e^{jk\ell} \\
I_L &= \frac{V_0^+}{Z_0} e^{-jk\ell} - \frac{V_0^-}{Z_0} e^{jk\ell}.
\end{align*}
\]

Since the load impedance is given as \(Z_L = V_L/I_L\) by definition, an algebraic operation leads to that the coefficients \(V_0^+\) and \(V_0^-\) can be given in terms of \(Z_L\) and \(Z_0\) as

\[
\begin{align*}
V_0^+ &= \frac{1}{2} I_L (Z_L + Z_0) e^{jk\ell} \\
V_0^- &= \frac{1}{2} I_L (Z_L - Z_0) e^{-jk\ell}.
\end{align*}
\]

It is seen that the coefficient \(V_0^-\) depends on the difference between \(Z_L\) and \(Z_0\). When \(Z_L = Z_0\), \(V_0^- = 0\). That is, reflectionless is possible, when the load is matched to the transmission line.

Then the voltage and current along the transmission line can be given in terms of \(Z_L\) and \(Z_0\) as

\[
\begin{align*}
V(z) &= \frac{1}{2} I_L \left\{ (Z_L + Z_0) e^{-jk(z - \ell)} + (Z_L - Z_0) e^{jk(z - \ell)} \right\} \\
I(z) &= \frac{1}{2Z_0} I_L \left\{ (Z_L + Z_0) e^{-jk(z - \ell)} - (Z_L - Z_0) e^{jk(z - \ell)} \right\}.
\end{align*}
\]

By introducing the position variable \(z' = \ell - z\), which denotes the distance away from the load, one has

\[
\begin{align*}
V(z') &= \frac{1}{2} I_L \left\{ (Z_L + Z_0) e^{jkz'} + (Z_L - Z_0) e^{-jkz'} \right\} \\
I(z') &= \frac{1}{2Z_0} I_L \left\{ (Z_L + Z_0) e^{jkz'} - (Z_L - Z_0) e^{-jkz'} \right\}.
\end{align*}
\]

A little algebra leads to

\[
\begin{align*}
V(z') &= I_L (Z_L \cos k z' + jZ_0 \sin k z') \\
I(z') &= \frac{1}{Z_0} I_L (Z_0 \cos k z' + jZ_L \sin k z').
\end{align*}
\]

Apparently, at the load \(z' = 0\), \(V = I_L Z_L\) and \(I = I_L\).

It is seen that the impedance as the ratio of \(V(z)\) to \(I(z)\) varies along the line. At a distance \(z'\) away from the load, the impedance \(Z(z')\) is given by

\[
Z(z') = \frac{V(z')}{I(z')} = Z_0 \frac{Z_L + jZ_0 \tan k z'}{Z_0 + jZ_L \tan k z'}.
\]
It is seen that the impedance depends on frequency, in addition to the dependence on position. When the load is **matched** to the transmission line \((Z_L = Z_0)\), then \(Z(z') = Z_0\), independent of \(z'\). This simplicity is due to the absence of the backward wave reflected from the load. When the load is an open circuit \((Z_L \to \infty)\), then \(Z(z) = -jZ_0 \cot k z'\); while when the load is short \((Z_L \to 0)\), \(Z(z) = jZ_0 \tan k z'\). In lumped circuits \(k z' = 2\pi z'/\lambda \ll 1\), thus \(Z(z') \to Z_L\). It is noted that this result is independent of the characteristic impedance \(Z_0\). That is, in the lumped-circuit approximation, the input impedance is always identical to the load impedance, regardless of the length and characteristic impedance of the chosen transmission line.

**Reflection coefficient**

The dependences of voltage, current, and impedance on position are so complicated in high-frequency circuits. A more convenient way is to use the reflection coefficient. The ratio of \(V_0^+\) to \(V_0^-\) can be determined by the load impedance. Since \(Z_L = V_L/I_L\), one has

\[
Z_L = Z_0 \frac{V_0^+ e^{-jk\ell} + V_0^- e^{jk\ell}}{V_0^+ e^{-jk\ell} - V_0^- e^{jk\ell}}.
\]

Define the voltage reflection coefficient at the load as \(\Gamma_L = V_0^- e^{jk\ell} / V_0^+ e^{-jk\ell}\). Thus

\[
Z_L = Z_0 \frac{1 + \Gamma}{1 - \Gamma}.
\]

(The subscript \(L\) is omitted for short.) It is easy to show that in terms of \(Z_L\), the reflection coefficient \(\Gamma\) is given by

\[
\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}.
\]

Note that the formula of reflection coefficient is similar to that of the reflection coefficient for a plane wave impinging upon an interface separating two media.

If the load impedance \(Z_L\) matches the characteristic impedance \(Z_0\), no reflection occurs and hence all the impinging power is absorbed by the load. The magnitude of reflection is determined by the **mismatch** between the load impedance \(Z_L\) and the characteristic impedance \(Z_0\). Physically, the reflected waves are excited by the discontinuity in current and voltage. If \(Z_L\) is real and \(Z_L > Z_0\), the load current will be smaller than the impinging current. Then charges tend to accumulate at the input node and then a reflected wave is excited. Thus the load voltage and current increase, while the total input current decreases. This process is expected to be continued until the equilibrium state is established where the current is continuous at the load. Similar reasoning can be applied for the case \(Z_L < Z_0\).

The reflection coefficient in general is a complex quantity and can be expressed as \(\Gamma = |\Gamma| e^{i\theta}\), with \(|\Gamma| \leq 1\). If the load is a lossless inductor or capacitor, \(Z_L\) is pure imaginary and \(|\Gamma| = 1\) on a lossless line. If the load is a resistor, \(Z_L\) is real and \(-1 \leq \Gamma \leq 1\) on a lossless line. If the load is short \((Z_L \to 0)\), then \(\Gamma = -1\); and if the load is an open circuit \((Z_L \to \infty)\), then \(\Gamma = 1\). The case of \(|\Gamma| = 1\) means that all the power is reflected back to the source and hence the load does not absorb power.

It is illustrative to locate \(\Gamma\) in the \(\Gamma_r - \Gamma_i\) complex plane. As \(|\Gamma| \leq 1\), all the possible values of \(\Gamma\) are located within the unit circle in \(\Gamma\) plane. If the load is a resistor \((Z_L\) is real), then the corresponding loci of \(\Gamma\) form the \(x\) axis. Meanwhile, if the load is inductive or capacitive \((Z_L\) is pure imaginary), then \(|\Gamma| = 1\) and the corresponding loci of \(\Gamma\) form the upper or lower semicircle, respectively. Note that the \(\Gamma\) corresponding to \(Z_L = \) short, open, \(Z_0\), \(jZ_0\), and \(-jZ_0\) are located at \((-1, 0)\), \((1, 0)\), \((0, 0)\), \((0, 1)\), and \((0, -1)\), respectively.
Voltage standing wave ratio VSWR

In terms of the reflection coefficient \( \Gamma \) (\( \Gamma = \frac{V_0^- e^{jkl}}{V_0^+ e^{-jkl}} \)), the voltage and current along a transmission line are given as

\[
V(z) = V_0^+ e^{-j kz} (1 + \Gamma e^{-j 2k z'}) = V_0^+ e^{-j kz} [1 + \Gamma(z')]
\]
\[
I(z) = \frac{1}{Z_0} V_0^+ e^{-j kz} (1 - \Gamma e^{-j 2k z'}) = \frac{1}{Z_0} V_0^+ e^{-j kz} [1 - \Gamma(z')]
\]

In terms of the reflection coefficient \( \Gamma \), the load voltage and current are given as

\[
V_L = V_0^+ e^{-j kl} (1 + \Gamma)
\]
\[
I_L = \frac{V_0^+}{Z_0} e^{-j kl} (1 - \Gamma).
\]

The voltage reflection coefficient at an arbitrary position \( z' \) is then given by

\[
\Gamma(z') = \Gamma e^{-j 2k z'} = |\Gamma| e^{j(\theta - 2k z')}
\]

The values of the reflection coefficient \( \Gamma(z') \) at various \( z' \) correspond to a circle of radius \( |\Gamma| \) in the \( \Gamma_r-\Gamma_i \) complex plane, which is called the \( \Gamma \) circle. With increasing \( z' \) (away from the load gradually), this circle is traced clockwise.

The magnitudes of voltage and current along a transmission line are given as

\[
\left| \frac{1}{V_0^+} V(z) \right| = |1 + \Gamma(z')| = [1 + |\Gamma|^2 + 2 \text{Re}\{\Gamma(z')\}]^{1/2}
\]
\[
\left| \frac{Z_0}{V_0^+} I(z) \right| = |1 - \Gamma(z')| = [1 + |\Gamma|^2 - 2 \text{Re}\{\Gamma(z')\}]^{1/2}
\]

The magnitude of \( |V(z)/V_0^+| \) is equal to the length of the segment connecting the point \((-1, 0)\) and the point of \( \Gamma(z') \) on the \( \Gamma \) circle in the \( \Gamma \) plane. And the magnitude of \( |I(z)Z_0/V_0^+| \) is equal to the length of the segment connecting the point \((1, 0)\) and the point of \( \Gamma(z') \).

The value of \( |V(z)/V_0^+| \) ranges from \((1 + |\Gamma|)\) to \((1 - |\Gamma|)\). The maximum and the minimum occur at \( z' \) such that \( \Gamma(z') = |\Gamma| \) or \(-|\Gamma|\), which in turn correspond to \( \theta - 2k z' = 2n\pi \) and \((2n+1)\pi\), respectively, where \( n \) is an integer. Thus at the position of maximum or minimum, \( \Gamma(z') \) is real and the corresponding impedance \( Z(z') = Z_0(1 + \Gamma(z'))/(1 - \Gamma(z')) \) is purely resistive, regardless of the load impedance. For a resistive load with \( Z_L > Z_0, \Gamma > 0 \) and \( |V(z)| \) is a maximum at the load; while for a resistive load with \( Z_L < Z_0, \Gamma < 0 \) and \( |V(z)| \) is a minimum there. Similarly, the value of \( |I(z)Z_0/V_0^+| \) ranges from \((1 + |\Gamma|)\) to \((1 - |\Gamma|)\). The maximum and the minimum occur at \( z' \) such that \( \Gamma(z') = -|\Gamma| \) or \(|\Gamma|\), which in turn correspond to \( \theta - 2k z' = (2n+1)\pi \) and \(2n\pi\), respectively. Thus, for an arbitrary value of \( \Gamma \), the maxima of \( |V(z)| \) correspond to the minima of \( |I(z)| \), and vice versa.

The **voltage standing wave ratio** (VSWR) \( S \) is given by

\[
S = \frac{|V(z)|_{\text{max}}}{|V(z)|_{\text{min}}} = 1 + |\Gamma|,
\]

The SWR ranges from 1 to \( \infty \), which corresponds to \( |\Gamma| = 0 \) and 1, respectively. The SWR can be measured by using a probe to measure the voltage maxima and minima, where the probe protrudes
into a line through a narrow slot cut along a section of the line. From the measured SWR, the magnitude of the reflection coefficient $|\Gamma|$ can be determined from the formula

$$|\Gamma| = \frac{S - 1}{S + 1}.$$  

Note the similarity between this formula and $\Gamma = (Z_L - Z_0)/(Z_L + Z_0)$.

Further, the phase angle $\theta_\Gamma$ can be determined from the position of the voltage maximum, since $\theta_\Gamma - 2kz' = 2n\pi$ there. By measuring the distance $d$ of the voltage maximum (for $n = 0$) from the load, the phase angle is given by

$$\theta_\Gamma = 4\pi d/\lambda.$$  

In other words, the position of $\Gamma_L$ in the $\Gamma_r$-$\Gamma_i$ complex plane can be located by tracing from the intersection of the $\Gamma$ circle with the positive $\Gamma_r$ axis counterclockwise by angle of $(d/\lambda) \times 720^\circ$. If the voltage is a maximum or minimum right at a resistive load (the distance of voltage maximum $d = 0$ or $d = \lambda/4$), then the load is resistive with $Z_L > Z_0$ ($\Gamma > 0$) or $Z_L < Z_0$ ($\Gamma < 0$), respectively.

Moreover, the phase constant $\beta$ or wavelength $\lambda$ can be determined, since the distance between two successive voltage maxima or minima is a half-wavelength. Thus, by using $|\Gamma|$ and $\theta_\Gamma$, and $Z_0$, the load impedance $Z_L$ can be determined as $Z_L = Z_0(1 + \Gamma)/(1 - \Gamma)$. If $\Gamma$ is real, then the load is purely resistive. Further,

- if $\Gamma$ is positive ($\Gamma = |\Gamma|$), then $Z_L = Z_0(1 + |\Gamma|)/(1 - |\Gamma|) = Z_0 S$; while
- if $\Gamma$ is negative ($\Gamma = -|\Gamma|$), then $Z_L = Z_0(1 - |\Gamma|)/(1 + |\Gamma|) = Z_0/S$.

Consider the example where the SWR on a lossless 50-Ω line terminated with a load was found to be 3. The distance between two successive voltage minima is 20 cm and the first minimum is located at 5 cm from the load. Then one has $|\Gamma| = (S - 1)/(S + 1) = 0.5$, $\lambda = 40$ cm, and $\theta_\Gamma = 4\pi d/\lambda = 4\pi[(5 + 10)/40] = 3\pi/2$ or $-\pi/2$. Thereby, the reflection coefficient can be determined as $\Gamma = -j0.5$. From the reflection coefficient, the load impedance can be given as $Z_L = Z_0(1 - j0.5)/(1 + j0.5) = 30 - j40$ Ω.

power flux  

For a lossless line, the time-average power flux is given as

$$\mathcal{P}(z) = \frac{1}{2} \text{Re}\{V(z)I^*(z)\}$$

$$= \frac{1}{2Z_0} \text{Re}\left\{V_0^+e^{-jkz}(1 + e^{-j2kz'})[V_0^+e^{-jkz}(1 - e^{-j2kz'})]^*\right\}$$

$$= \frac{|V_0^+|^2}{2Z_0} \text{Re}\left\{1 - |\Gamma|^2 + \Gamma e^{-j2kz'} - \Gamma^*e^{j2kz'}\right\}$$

$$= \frac{|V_0^+|^2}{2Z_0} \left(1 - |\Gamma|^2\right).$$

Note that it appears that the power flux is independent of $z$ and the forward and backward waves carry respective time-average power flux independently. That is, the cross terms do not contribute to time-average power flux. However, this situation does not hold in a lossy transmission line.

8.2. Transmission-Line Circuits with Voltage Sources
Now consider the circuit connected to a generator with a voltage $V_g$ at $z = 0$. Suppose the generator has an internal impedance $Z_g$. Thus the generator voltage is given as

$$V_g = V_{in} + Z_g I_{in},$$

where the input voltage $V_{in}$ and the input current $I_{in}$ are the voltage $V(z)$ and the current $I(z)$ at the position $z = 0$, respectively. Explicitly, the input voltage and current are given as

$$
\begin{align*}
V_{in} &= V_0^+ [1 + \Gamma e^{-j2k\ell}] \\
I_{in} &= \frac{V_0^+}{Z_0} [1 - \Gamma e^{-j2k\ell}].
\end{align*}
$$

From the preceding relations, the generator voltage $V_g$ can be given in terms of the forward voltage coefficient $V_0^+$. That is,

$$
V_g = V_0^+ \left\{ [1 + \Gamma e^{-2jk\ell}] + \frac{Z_g}{Z_0} [1 - \Gamma e^{-j2k\ell}] \right\}
$$

$$
= V_0^+ \frac{1}{Z_0} \left\{ (Z_0 + Z_g) + (Z_0 - Z_g) \Gamma e^{-j2k\ell} \right\}
$$

$$
= V_0^+ \frac{Z_0 + Z_g}{Z_0} \left\{ 1 - \Gamma g \Gamma e^{-j2k\ell} \right\},
$$

where the voltage reflection coefficient $\Gamma_g$ due to the internal impedance is given by

$$
\Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0}.
$$

Note the similarity between $\Gamma_L$ and $\Gamma_g$.

Thereafter, the forward voltage coefficient $V_0^+$ can be given in terms of the generator voltage $V_g$. That is,

$$
V_0^+ = V_g Z_0 Z_g Z_0 - \Gamma g \Gamma e^{-j2k\ell}. Z_0 + Z_0 - \Gamma g \Gamma e^{-j2k\ell}.
$$

For the case with $\Gamma = 0$ or $\Gamma_g = 0$, the coefficient is then independent of the line length and becomes a simpler form of $V_0^+ = V_g Z_0 / (Z_g + Z_0)$.

In terms of the generator voltage $V_g$, the voltage and current along a transmission line is given as

$$
\begin{align*}
V(z) &= V_g \frac{Z_0}{Z_g + Z_0} \frac{1}{1 - \Gamma g \Gamma e^{-j2k\ell}} e^{-jkz} (1 + \Gamma e^{-j2kz'}) \\
I(z) &= V_0^+ \frac{1}{Z_g + Z_0} \frac{1}{1 - \Gamma g \Gamma e^{-j2k\ell}} e^{-jkz} (1 - \Gamma e^{-j2kz'}).
\end{align*}
$$

Thereby, the voltage and current at a load of impedance $Z_L$ connected to a generator voltage $V_g$ via a transmission line of length $\ell$ and characteristic impedance $Z_0$ are given as

$$
\begin{align*}
V_L &= V_g \frac{Z_0}{Z_g + Z_0} \frac{1}{1 - \Gamma g \Gamma e^{-j2k\ell}} e^{-jk\ell} (1 + \Gamma) \\
I_L &= V_0^+ \frac{1}{Z_g + Z_0} \frac{1}{1 - \Gamma g \Gamma e^{-j2k\ell}} e^{-jk\ell} (1 - \Gamma).
\end{align*}
$$
It is noted that the voltage and current depend on the characteristic impedance \( Z_0 \) and length \( \ell \) of the transmission line.

In lumped circuits \( k\ell \ll 1 \), then

\[
V_L = V(z) = V_g \frac{Z_L}{Z_g + Z_L}
\]

\[
I_L = I(z) = V_g \frac{1}{Z_g + Z_L},
\]

where we have made use of

\[
1 + \Gamma = \frac{(Z_g + Z_0)[(Z_L + Z_0) + (Z_L - Z_0)]}{(Z_L + Z_0)(Z_g + Z_0) - (Z_L - Z_0)(Z_g - Z_0)} = \frac{Z_L}{Z_0} \frac{Z_g + Z_0}{Z_g + Z_L}
\]

and

\[
1 - \Gamma = \frac{(Z_g + Z_0)[(Z_L + Z_0) - (Z_L - Z_0)]}{(Z_L + Z_0)(Z_g + Z_0) - (Z_L - Z_0)(Z_g - Z_0)} = \frac{Z_g + Z_0}{Z_g + Z_L}.
\]

It is seen that the load voltage and current are independent of the characteristic impedance \( Z_0 \) and agree with those obtained from a simple lumped-circuit analysis.

Note that the corresponding \( V_0^+ = V_g Z_0/(Z_g + Z_0)(1 - \Gamma g) = V_g Z_L + Z_0)/2(Z_g + Z_L) \), which depends on \( Z_0 \). A lumped circuit with a different characteristic impedance \( Z_0 \) results in different \( \Gamma, \Gamma_g, \) and \( V_0^+ \), but the voltage and current will remain the same.

Further, for a lumped circuit with \( Z_g = 0 \), then the \( V_L \) and \( I_L \) becomes as simple as

\[
V_L = V_g \quad \text{and} \quad I_L = V_g/Z_L.
\]

It is seen that the load voltage is simply the generator voltage, regardless of the load impedance \( Z_L \) and the characteristic impedance \( Z_0 \). Note that if the length of the transmission line is an integer multiple of one wavelength \( (\ell = n\lambda) \), then the \( V_L, I_L, V_m \) and \( I_m \) are identical to those in the lumped-circuit case.

For example, consider the case where a 100-MHz generator with an internal impedance \( Z_g = 50 \) \( \Omega \) is connected to a lossless 50-\( \Omega \) line of length \( \ell = 3.75 \) m and terminated with a load of impedance \( Z_L = 25 + j25 \)\( \Omega \). Then one has \( \Gamma_g = 0 \), \( V_0^+ = V_g Z_0/(Z_g + Z_0) = 1/2V_g \), and \( k\ell = 2\pi(3.75/3) = 5\pi/2 \) or \( \pi/2 \). The load voltage and current are \( V_L = -j\frac{1}{2}V_g(1 + \Gamma) \) and \( I_L = -j\frac{1}{2}V_g(1 - \Gamma)/Z_0 \).

**A physical interpretation of the forward voltage coefficient based on multiple reflection**

Physically, the forward wave of voltage \( V_0^+ e^{-j\ell} \) is due to the superposition of the incident wave and all the forward waves from multiple reflection between the load and the generator. The primary forward wave delivered from the voltage source is \( V_1 e^{-j\ell} \), where \( V_1 = V_g[Z_0/(Z_g + Z_0)] \). When this wave travels down the transmission line a distance \( \ell \), it hits the load and the reflected wave is \( V_1 e^{-j2\ell} \Gamma e^{-j\ell} \). After this backward wave returns at the voltage source with internal impedance \( Z_g \), a reflection from the voltage source occurs with reflection coefficient \( \Gamma_g = (Z_g - Z_0)/(Z_g + Z_0) \). Then, after one reflection from the load and another one from the voltage source, the wave propagates forward again and is expressed as \( V_2 e^{-j\ell} \), where \( V_2 = V_1 e^{-j2\ell} \Gamma e^{-j\ell} \Gamma_g = V_1 \Gamma \Gamma_g e^{-2j\ell} \). Similarly, after the set of two reflections repeats twice, the wave is forward and expressed as \( V_3 e^{-j\ell} \), where \( V_3 = V_2 \Gamma \Gamma_g e^{-2j\ell} = V_1 (\Gamma \Gamma_g e^{-2j\ell})^2 \). Thereafter, the total of forward wave is given as \( V_0^+ e^{-j\ell} = \)
(V_1 + V_2 + V_3 + \cdots) e^{-jkz}$. Thus the forward voltage coefficient is given as the summation:

\[
V_0^+ = V_g \frac{Z_0}{Z_g + Z_0} \left\{1 + \Gamma \gamma e^{-j2k\ell} + \left(\Gamma \gamma e^{-j2k\ell}\right)^2 + \cdots\right\}
\]

\[
= V_g \frac{Z_0}{Z_g + Z_0} \frac{1}{1 - \Gamma \gamma e^{-j2k\ell}}.
\]

The result is identical to that obtained previously.

**some special cases**

If the load is matched to the line, $Z_L = Z_0$ and $\Gamma = 0$. The forward voltage coefficient becomes a simpler form of $V_0^+ = \frac{1}{2} V_g Z_0/(Z_g + Z_0)$. And

\[
V_L = V_g \frac{Z_0}{Z_g + Z_0} e^{-jk\ell}
\]
\[
I_L = V_g \frac{1}{Z_g + Z_0} e^{-jk\ell}
\]
\[
V_{in} = V_g \frac{Z_0}{Z_g + Z_0}
\]
\[
I_{in} = V_g \frac{1}{Z_g + Z_0}
\]

Note that in this case, $V_L$, $I_L$, $V_{in}$ and $I_{in}$ involve only the primary forward wave.

If the voltage generator is matched to the transmission line $Z_g = Z_0$, $\Gamma \gamma = 0$. The forward voltage coefficient becomes $V_0^+ = \frac{1}{2} V_g$, regardless of the load impedance $Z_L$. Thus one has

\[
V_L = \frac{1}{2} V_g e^{-jk\ell}(1 + \Gamma)
\]
\[
I_L = \frac{1}{2Z_0} V_g e^{-jk\ell}(1 - \Gamma)
\]
\[
V_{in} = \frac{1}{2} V_g (1 + \Gamma e^{-j2k\ell})
\]
\[
I_{in} = \frac{1}{2Z_0} V_g (1 - \Gamma e^{-j2k\ell}).
\]

Note that in this case, $V_L$, $I_L$, $V_{in}$ and $I_{in}$ involve the primary forward and one backward waves.

If $Z_g = 0$ (such as at a power plant), $\Gamma \gamma = -1$, $V_0^+ = \frac{1}{2} V_g/(1 + \Gamma e^{-j2k\ell})$. Thus

\[
V_L = V_g \frac{1}{1 + \Gamma e^{-j2k\ell}} e^{-jk\ell}(1 + \Gamma)
\]
\[
I_L = V_g \frac{1}{Z_0} \frac{1}{1 + \Gamma e^{-j2k\ell}} e^{-jk\ell}(1 - \Gamma)
\]
\[
V_{in} = V_g \frac{1}{1 + \Gamma e^{-j2k\ell}}(1 + \Gamma e^{-j2k\ell})
\]
\[
I_{in} = V_g \frac{1}{Z_0} \frac{1}{1 + \Gamma e^{-j2k\ell}}(1 - \Gamma e^{-j2k\ell}).
\]

**Power in transmission lines**
For a lossless line, the time-average power flux along the line is given by

\[ P(z) = \frac{1}{2} \text{Re}\{V(z)I^*(z)\} = \frac{1}{2} \left| \frac{V_0^+}{Z_0} \right|^2 (1 - |\Gamma|^2)Z_0, \]

which does not depend on the position \( z \). Note that this power flux can be deemed as the difference between the forward and backward power fluxes. Meanwhile, the time-average load power \( P_L \) is given by

\[ P_L = \frac{1}{2}|I_L|^2R_L = \frac{1}{2} \left| \frac{V_0^+}{Z_0} \right|^2 |1 - |\Gamma|^2| R_L \]

It can be shown that for the lossless line

\[ P_L = P(z), \]

since \(|1 - |\Gamma|^2| = 4|Z_0|^2/|Z_L + Z_0|^2 \) and \( 1 - |\Gamma|^2 = 4 \text{Re}\{Z_LZ_0^*\}/|Z_L + Z_0|^2 \).

In terms of the generator voltage \( V_g \), the time-average power becomes

\[ P(z) = \frac{1}{2} \text{Re}\{V(z)I^*(z)\} = \frac{1}{2} \left| \frac{V_g}{Z_g + Z_0} \right|^2 \frac{1}{|Z_g + Z_0|^2} \frac{1 - |\Gamma|^2}{|1 - \Gamma_g e^{-j2k\ell}|^2} Z_0, \]

and the time-average load power

\[ P_L = \frac{1}{2}|I_L|^2R_L = \frac{1}{2} \left| \frac{V_g}{Z_g + Z_0} \right|^2 \frac{1}{|Z_g + Z_0|^2} \frac{1 - |\Gamma|^2}{|1 - \Gamma_g e^{-j2k\ell}|^2} R_L, \]

which look quite complicated. Note that these powers depend on the length and characteristic impedance of the transmission line. For the case with \( \Gamma = 0 \) or \( \Gamma_g = 0 \), these powers are then independent of the line length and the frequency.

For example, reconsider the previous case with a 100-MHz generator with \( V_g = 10 \text{ V} \). As the internal impedance \( Z_g = Z_0 = 50 \text{ \Omega} \), the load current \( I_L = -j\frac{1}{2}V_g(1 - \Gamma)/Z_0 \). Thus the time-average power absorbed by the load is

\[ P_L = \frac{1}{2}|I_L|^2R_L = \frac{1}{8} \left| \frac{V_g}{Z_0} \right|^2 |1 - |\Gamma|^2 R_L = \frac{1}{2} \left| \frac{V_g}{Z_g + Z_0} \right|^2 \frac{R_L}{|Z_g + Z_0|^2} = 0.2 \text{ W}. \]

The total time-average power \( P_s \) delivered from the voltage source is given by

\[ P_s = \frac{1}{2} \text{Re}\{V_g I^*_\text{in}\} = \frac{1}{2} |I_{\text{in}}|^2 (R_g + R_{\text{in}}), \]

since \( V_g = (Z_g + Z_{\text{in}})I_{\text{in}} \). For a lossless line, \( P_{\text{in}} = P_L \), where the time-average input power \( P_{\text{in}} = |I_{\text{in}}|^2R_{\text{in}} \). Thus

\[ P_s = P_g + P_L, \]

where \( P_g = \frac{1}{2}|I_{\text{in}}|^2R_g \) (\( R_g = \text{Re}\{Z_g\} \)) denotes the time-average power absorbed by the generator internal impedance. Thus the time-average supplied power is

\[ P_s = \frac{1}{2} \text{Re}\{V_g I^*_\text{in}\} = \frac{1}{2} \left| \frac{V_g}{Z_g + Z_0} \right|^2 \text{Re}\left\{ \frac{1}{1 - \Gamma_g e^{-j2k\ell}} \right\}, \]
If the load is matched to a lossless line \((Z_L = Z_0\) and \(\Gamma = 0\)), then the load power and the supplied power are given as

\[
\mathcal{P}_L = \frac{1}{2} |V_g|^2 \frac{1}{|Z_g + Z_0|^2} R_0
\]

\[
\mathcal{P}_s = \frac{1}{2} |V_g|^2 \frac{1}{|Z_g + Z_0|^2} (R_g + R_0).
\]

It is noted that both of the load and the supplied powers are independent of frequency and the length of the line.

If the voltage generator is matched to a lossless line \((Z_g = Z_0\) and \(\Gamma_g = 0\)), then the load power is

\[
\mathcal{P}_L = \frac{1}{2} |V_g|^2 \frac{1}{|Z_g + Z_0|^2} |1 - \Gamma|^2 R_L = \frac{1}{2} |V_g|^2 \frac{1}{|Z_L + Z_0|^2} R_L
\]

and the supplied power is

\[
\mathcal{P}_s = \frac{1}{2} |V_g|^2 \text{Re}\left\{ \frac{1}{Z_g + Z_0} (1 - \Gamma e^{-j2k\ell}) \right\}.
\]

It is noted that the load power is independent of frequency and the length of the line. The maximum load power occurs when \(Z_L = Z_0^*\), that is, the load impedance is complex-conjugate matched to the line, which in turn has been matched to the generator internal impedance. For a lossless line, the maximum load power occurs when \(Z_L = R_0\).

For the lumped-circuit case with \(k\ell \rightarrow 0\) (or for the case with \(\ell = n\lambda/2\)), the load power is given as

\[
\mathcal{P}_L = \frac{1}{2} |V_g|^2 \frac{1}{|Z_g + Z_0|^2} |1 - \Gamma|^2 R_L
\]

\[
= \frac{1}{2} |V_g|^2 \frac{1}{|Z_g + Z_L|^2} R_L
\]

and the supplied power

\[
\mathcal{P}_s = \frac{1}{2} |V_g|^2 \text{Re}\left\{ \frac{1}{Z_g + Z_0} \frac{1 - \Gamma}{1 - \Gamma \Gamma_g} \right\}
\]

\[
= \frac{1}{2} |V_g|^2 \frac{1}{|Z_g + Z_L|^2} (R_g + R_L),
\]

where we have made use of the relation \((1 - \Gamma)/(1 - \Gamma \Gamma_g) = (Z_g + Z_0)/(Z_g + Z_L)\) obtained previously. It is noted that in the lumped-circuit case, the load and the supplied powers are independent of the characteristic impedance \(Z_0\).

8.3. Applications of Impedance Transformation

The complicated formula of the transformation of impedance can be used to advantage to implement some devices from a segment of transmission line. Furthermore, the impedance transformation can be used to measure the characteristic impedance and the propagation constant of the line.

Impedance transformation
Remark that along a lossless line of characteristic impedance $Z_0$, the impedance at a distance $z'$ from the load of impedance $Z_L$ varies in a complicated way as

$$Z(z') = Z_0 \frac{Z_L + jZ_0 \tan kz'}{Z_0 + jZ_L \tan kz'}.$$ 

The transformation of impedance can be utilized to implement some useful lumped-circuit devices. Suppose the input is at a given position $z' = \ell$. Then the input impedance $Z_{in}$ becomes

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan k\ell}{Z_0 + jZ_L \tan k\ell}.$$ 

Note that the input impedance is a function of frequency and line length.

For the case with $\ell = \lambda/4$, $k\ell = \pi/2$. Then $Z_{in} = Z_0^2/Z_L$. Thus a quarter-wave line transforms the load impedance inversely.

**Inductors and capacitors**

By using the impedance transformation, inductors and capacitors can be implemented with open- or short-circuit microstrip lines. For an open-circuit lossless stub of length $\ell$, the input impedance is given as

$$Z_{inO} = jX_{inO} = -jR_0 \cot k\ell.$$ 

The line can be capacitive or inductive, depending on the length. If $\ell \ll \lambda$,

$$Z_{inO} \approx -jR_0 \frac{k\ell}{\omega \sqrt{LC\ell}} = \frac{1}{j\omega C\ell},$$ 

which is capacitive with capacitance $C\ell$. If $\ell = \lambda/4$, $k\ell = \pi/2$, $Z_{inO}$ becomes zero. Thus an open-circuit quarter-wave line is effectively a short-circuit at the input. When $\lambda/4 < \ell < \lambda/2$, the open-circuit line behaves as an inductor at the input.

For a short-circuit stub of length $\ell$, which is more practically implemented in a real circuit, the input impedance is given by

$$Z_{ins} = jX_{ins} = jR_0 \tan k\ell.$$ 

If $\ell \ll \lambda$,

$$Z_{ins} \approx jR_0 k\ell = j\sqrt{L/C} \sqrt{LC\ell} = j\omega L\ell,$$ 

which is inductive with inductance $L\ell$. If $\ell = \lambda/4$, $k\ell = \pi/2$, $Z_{ins}$ becomes infinite. Thus a short-circuit quarter-wave line is effectively an open-circuit at the input. When $\lambda/4 < \ell < \lambda/2$, the short-circuit line behaves as a capacitor at the input.

**Determination of characteristic impedance and propagation constant**

From the preceding formulas of $Z_{ins}$ and $Z_{inO}$, one has

$$Z_{ins} Z_{inO} = jZ_0 \tan k\ell \times (-jZ_0 \cot k\ell) = Z_0^2$$

$$Z_{ins}/Z_{inO} = jZ_0 \tan k\ell/(-jZ_0 \cot k\ell) = -\tan^2 k\ell.$$ 

Thus

$$\begin{align*}
Z_0 &= \sqrt{Z_{ins} Z_{inO}} \\
k\ell &= \tan^{-1}(\sqrt{|Z_{ins}/Z_{inO}|}).
\end{align*}$$
Thereby, the characteristic impedance $Z_0$ and the propagation constant $k$ can be determined by the product and quotient of $Z_{in0}$ and $Z_{ins}$, respectively.

**Lossy inductors and capacitors**

Any complex impedance $Z_{in} = R_{in} + jX_{in}$ can be implemented by a section of lossless line of length $\ell_m$ terminated in a purely resistive load $R_m$. That is,

$$R_{in} + jX_{in} = R_0 \frac{R_m + jR_0 \tan k\ell_m}{R_0 + jR_m \tan k\ell_m}.$$

This relation is actually a set of two coupled equations in terms of load resistance $R_m$ and length $\ell_m$. After some algebraic operations, it can be shown that the solutions for $r_m$ are

$$r_m = \frac{(|z_{in}|^2 + 1) \pm \sqrt{(|z_{in}|^2 + 1)^2 - 4r_{in}^2}}{2r_{in}},$$

where the normalized quantities $r_m = R_m/R_0$, $z_{in} = Z_{in}/R_0$, $r_{in} = R_{in}/R_0$, and $x_{in} = X_{in}/R_0$. The plus sign corresponds to $R_m > R_0$ and the minus one to $R_m < R_0$.

Besides, it can be shown that, after the normalized resistance $r_m$ is solved, the corresponding length $\ell_m$ is given by

$$\ell_m = \frac{\lambda}{2\pi} \tan^{-1} \left\{ \frac{x_{in}}{1 - r_{in}r_m} \right\}.$$

It is seen that the length is proportional to the wavelength. Remark that the voltage at the purely resistive load $R_m$ will be either a maximum or a minimum as viewed along the line.

Alternatively, the normalized resistance $r_m$ can be determined from the reflection coefficient at the input $\Gamma_{in}$ and then from the VSWR $S$. Remark that $\Gamma_{in} = (Z_{in} - R_0)/(Z_{in} + R_0)$ and $S = (1 + |\Gamma_{in}|)/(1 - |\Gamma_{in}|)$. Note that $\Gamma_{in} = \Gamma e^{-j2k\ell_m}$, where $\Gamma_L = (r_m - 1)/(r_m + 1)$.

If $r_m > 1$, one has

$$\frac{r_m - 1}{r_m + 1} = \Gamma_L = |\Gamma_L| = |\Gamma_{in}| = \frac{S - 1}{S + 1}.$$

Thus $r_m = S$. On the other hand, if $r_m < 1$, then $r_m = 1/S$. In other words, the VSWR $S$ is related to the normalized resistance $r_m$ as

$$S = \begin{cases} r_m & r_m > 1 \\ 1/r_m & r_m < 1 \end{cases}.$$

Thus the resistance $R_m$ is related to the VSWR $S$ simply as $R_m = R_0 S$ or $R_0 / S$.

Without the knowledge of $R_m$, the length $\ell_m$ can be determined graphically by locating the point $\Gamma_{in}$ on the $\Gamma_r$-$\Gamma_i$ complex plane, after the reflection coefficient $\Gamma_m$ is evaluated from the desired complex impedance $Z_{in}$. By tracing from the point $\Gamma_{in}$ counterclockwise around the $\Gamma$ circle to meet the positive or negative $\Gamma_r$ axis, the associated angle $\varphi_m$ in degree when multiplied by $\lambda/720^\circ$ is equal to the length $\ell_m$ corresponding to $R_m = R_0 S$ or $R_0 / S$, respectively. That is, $\ell_m = \lambda \varphi_m/\lambda 720^\circ$.

Once the length $\ell_m$ is determined, an alternative way to evaluate the load resistance explicitly is

$$R_m = R_0 \frac{Z_{in} + jR_0 \tan(-k\ell_m)}{R_0 + jZ_{in} \tan(-k\ell_m)},$$

Thus the are various methods to determine $R_m$.

For example, consider the case where the desired impedance is $Z_{in} = 30 - j40 \Omega$. It can be implemented by a 50-Ω line of length $\ell_m$ terminated with a resistor of resistance $R_m$. Since
Γ_{in} = (-20 - j40)/(80 - j40) = -j0.5, one has |Γ_{in}| = 1/2, S = 3, and then the required load resistance is determined to be R_m = 150 or 50/3 Ω. From the location of Γ_{in} on the complex-Γ plane, it is seen that the required length is ℓ_m = λ/8 or 3λ/8, respectively.

Conversely, a segment of transmission line can transform a complex impedance to a pure resistance. A complex impedance can also be transformed to another complex value with a prescribed resistance, as discussed later.

### 8.4. Smith Chart

Remark that the impedance transformation formula

\[ Z(z') = Z_0 \frac{Z_L + jZ_0 \tan k z'}{Z_0 + jZ_L \tan k z'}. \]

An alternative way of determining the impedance at a position \( z' \) along a transmission line is given by the following procedure.

First, use \( Z_L \) to calculate \( Γ_L \) at the load by the formula

\[ Γ_L = \frac{(Z_L - Z_0)}{(Z_L + Z_0)}. \]

Next, use \( Γ_L \) to calculate the voltage reflection coefficient \( Γ(z') \) at \( z' \) by the formula

\[ Γ(z') = Γ_L e^{-j2kz'}. \]

Then, use \( Γ(z') \) to calculate the impedance \( Z(z') \) at \( z' \) by the formula

\[ Z(z') = Z_0[1 + Γ(z')]/[1 - Γ(z')]. \]

The coefficient \( Γ(z') \) can be determined graphically on the \( Γ_r-Γ_i \) complex plane, by tracing from the point \( Γ_L \) around the Γ circle clockwise by an angle of \( ϕ = (z'/λ) \times 720° \). Further, it will be of great help if some auxiliary curves are imposed on the \( Γ_r-Γ_i \) complex plane, so that one can determine graphically the reflection coefficient from the impedance and the converse. To this end, it has been figured out to draw some constant-\( r \) circles and constant-\( x \) arcs on \( Γ_r-Γ_i \) complex plane, as discussed in what follows.

**Constant-\( r \) circles and constant-\( x \) arcs on \( Γ_r-Γ_i \) complex plane**

Define the normalized resistance \( r \) and normalized reactance \( x \) along the transmission line as

\[ r + jx = \frac{Z(z')}{Z_0} = \frac{1 + Γ(z')}{1 - Γ(z')} = \frac{1 + Γ_r + jΓ_i}{1 - Γ_r - jΓ_i}. \]

Explicitly, the normalized resistance and reactance can be written in terms of \( Γ_r \) and \( Γ_i \) as

\[
\begin{align*}
  r &= \frac{1 - Γ_r^2 - Γ_i^2}{(1 - Γ_r)^2 + Γ_i^2} \\
  x &= \frac{2Γ_i}{(1 - Γ_r)^2 + Γ_i^2}.
\end{align*}
\]

These two relations can be rearranged respectively as

\[
\begin{align*}
  \left( Γ_r - \frac{r}{1+r} \right)^2 + Γ_i^2 &= \left( \frac{1}{1+r} \right)^2 \\
  (Γ_r - 1)^2 + \left( Γ_i - \frac{1}{x} \right)^2 &= \left( \frac{1}{x} \right)^2.
\end{align*}
\]
It is of important to note that in the $\Gamma_r$-$\Gamma_i$ complex plane, the loci of constant $r$ and those of constant $x$ all form circles.

All of the circles contain the singular point $(1,0)$, which corresponds to an open-circuit impedance and at which the impedance can not be determined unambiguously. The singular point is always one of the two intersection points between any $r$-circle and any $x$-circle. The coordinates $(\Gamma_r, \Gamma_i)$ at the other intersection of an $r$-circle with an $x$-circle are thus related to the normalized impedance $r + jx$.

The centers of the $r$-circles are $(r/(1+r), 0)$ lying on the $\Gamma_r$ axis; while those of the $x$-circles are $(1, 1/x)$ lying on a line parallel to the $\Gamma_i$ axis. The radii of all the these circles increase with decreasing $r$ or $x$. Circles corresponding to positive and negative values of $x$ are the mirror image of each other with respect to the $\Gamma_r$ axis. Apparently, these two circles are orthogonal at this common intersection $(1,0)$. By reasons of symmetry (with respect to the line connecting the centers of these two circles), these two circles are also orthogonal at the other intersection.

The region in the $\Gamma_r$-$\Gamma_i$ complex plane in which one is interested is within the unit circle. Thus circles of constant $x$ are actually segments of arcs within the unit circle. Notice the curves for $r = 0$ (purely reactive) and $x = 0$ (purely resistive) correspond to the unit circle and the $\Gamma_r$ axis, respectively. And the curve for $r = 1$ or $x = \pm 1$ correspond to a circle or two arcs with the $\Gamma_i$ axis as the tangent line, respectively. Notice also the points corresponding to $\Gamma = 0$ ($r = 1, x = 0$, or matched load), $\Gamma = 1$ (open-circuit), $\Gamma = -1$ (short-circuit), and $\Gamma = \pm j$ ($r = 0, x = \pm 1$).

**Smith chart and impedance**

A pure resistor with varying resistance ($x = 0$) corresponds to the $\Gamma_r$ axis. At DC, the voltage reflection coefficient $\Gamma$ of lossless inductor and capacitor is $-1$ and $1$, respectively. The reactance of a lossless inductor or capacitor ($r = 0$) respectively follows the upper or the lower semicircles of the unit circle clockwise with increasing frequency. And, the frequency response of the impedance of a lossy inductor or capacitor with a fixed resistance corresponds to a nonzero-constant-$r$ semicircle.

**Impedance transformation using Smith chart**

Reconsider the problem of using a section of lossless line of length $\ell_m$ terminated in a purely resistive load $R_m$ to implement a complex impedance $Z_{in} = R_{in} + jX_{in}$. The problem can be solved graphically by locating the desired normalized impedance $z_{in}$ and drawing the $\Gamma$ circle on the Smith chart. The required normalized resistance $r_m$ is the impedance associated with the intersection of the $\Gamma$ circle with the positive or negative $\Gamma_r$ axis, which can be determined graphically from the Smith chart. The required length $\ell_m$ corresponds to the angles from the point $\Gamma_{in}$ to the intersections counterclockwise. It is known that the corresponding normalized resistance $r_m$ is equal to $S$ or $1/S$, where $S$ is the VSWR associated with the load $Z_L$. Thus the VSWR $S$ can be determined graphically from the Smith chart, by reading the normalized resistance $r_m$ at the intersection between the $\Gamma$ circle and the positive $\Gamma_r$ axis.

Conversely, a complex impedance $Z_L$ can be transformed to a purely resistive impedance $Z_{in} = R_{in}$. The required length $\ell$ corresponds to the angles from the point $\Gamma_L$ clockwise to the intersection of the $\Gamma$ circle with the positive or negative $\Gamma_r$ axis ($x = 0$). The corresponding normalized resistance $r_{in}$ is equal to $S$ or $1/S$, where $S$ is the VSWR associated with the load $Z_L$. Remark that the intersection of the $\Gamma$ circle with the positive or negative $\Gamma_r$ axis corresponds to a voltage maximum or minimum along the line, respectively. Thus the angles from the $\Gamma_L$ point clockwise to the positive or negative $\Gamma_r$ axis corresponds to the distances of the first voltage maximum or minimum from the load, respectively.
If one would like to transform a normalized load impedance $z_L$ to another impedance with a prescribed normalized resistance, say unity. The problem can be also solved graphically by locating the impedance $z_L$ and drawing the $\Gamma$ circle on the Smith chart. Then the required length $\ell$ corresponds to the angle from the $\Gamma_L$ point clockwise to the two intersections of the $\Gamma$ circle with the unity-resistance circle ($r = 1$).

One can obtain a prescribed normalized reactance $x$ from a short- or open-circuit line, by locating the impedance $z_{in} = jx$ on the unit circle on the Smith chart. Then the required length $\ell$ of a short- or open-circuit line corresponds to the angle from the point $\Gamma_{in}$ counterclockwise to the point $(-1, 0)$ or $(1, 0)$, respectively.

From the impedance transformation formula, it is known that a transmission line of one quarter wavelength will transform an impedance $Z_L$ to $Z_{in} = Z(z' = \lambda/4)$ as

$$Z_{in} = \frac{Z_0^2}{Z_L}.$$

In terms of normalized quantities, the relation becomes

$$z_{in} = \frac{1}{z_L}.$$

Thus the normalized admittance $y_L (= 1/z_L = Y_L/Y_0, Y_0 = 1/Z_0)$ can be determined graphically from the Smith chart. It is seen that the normalized admittance $y_L$ is equal to the normalized impedance associated with the $\Gamma$ point by rotating the $\Gamma_L$ point by an angle of $180^\circ$ around the $\Gamma$ circle. That is, the $\Gamma_{in}$ point that lies diametrically opposite to the $\Gamma_L$ point on the $\Gamma$ circle corresponds to the normalized admittance $z_{in} = y_L$. For example, for $z_L = 0.3 + j0.4$, $y_L = 1.2 - j1.6$.

In summary, the Smith chart serves several purposes as conversion between impedance and reflection coefficient, impedance transformation, conversion between impedance and admittance, VSWR, and the locations of voltage extrema.

### 8.5. Impedance Matching Networks

The low-frequency power line suffers little from mismatch, since due to low frequency the transmission line loss is very low, the generator internal impedance is close to zero, and no signal is conveyed. However, if the transmission lines are used to propagate information at a high frequency, such as in TV and network, it is desired that the reflection of the forward wave can be omitted. Otherwise, the absorbed power will be reduced. Further, if the generator internal impedance is not matched to the line, multiple reflections with various shifts in phase will distort the signal received at the load. In TV reception, this tends to result in ghost images. Thus the internal impedance of a high-frequency voltage generator is usually designed to be 50 $\Omega$, matched to the characteristic impedance of the coaxial line commonly used. Further, for maximum power absorption with an impedance-matched generator, it is desired that the load impedance can complex-conjugate match the characteristic impedance of the transmission line used. This is analogous to the impedance matching ($Z_L = Z_g^*$) in low-frequency lumped circuits using inductors and capacitors.

### 5.1 Matching network using lumped inductors and capacitors

A load of impedance $Z_L$ mismatched to a lossless line of characteristic impedance $R_0$ can be matched to the line by using two lossless capacitors $C_1$ and $C_2$, one in parallel and the other one in series to the load. Practically, an inductor $L$ is added in series to the capacitor which in turn is in series to the load. With $C_1$ being in parallel and then $C_2$ and $L$ being in series to the load, the
Impedance matching requires that

\[ R_0 = \frac{1}{R_L + jX_L} + \frac{1}{j\omega C_2} + j\omega L. \]

In which,

\[ \frac{1}{R_L + jX_L + j\omega C_1} = \frac{R_L + jX_L}{1 - \omega X_L C_1 + j\omega R_L C_1} = \frac{R_L + j[X_L - \omega(X_L^2 + R_L^2)C_1]}{(1 - \omega X_L C_1)^2 + (\omega R_L C_1)^2}. \]

Thus

\[ \begin{cases} R_0 & = \frac{1}{R_L (1 - \omega X_L C_1)^2 + (\omega R_L C_1)^2} \\ \frac{1}{\omega C_2} - \omega L & = \frac{X_L - \omega(X_L^2 + R_L^2)C_1}{(1 - \omega X_L C_1)^2 + (\omega R_L C_1)^2}. \end{cases} \]

For a given inductance \( L \), capacitances \( C_1 \) and \( C_2 \) can be determined from the above two equations. The shunt capacitance \( C_1 \) is used to adjust the resistance to the required value, and then the serial capacitance \( C_2 \) and inductance \( L \) to eliminate the reactance.

On the other hand, the impedance matching can be achieved with \( C_2 \) and \( L \) being in series and then \( C_1 \) being in parallel to the load. Thus the matching requirements are

\[ R_0 = \frac{1}{\frac{1}{R_L + jX_L} + \frac{1}{j\omega C_2} + j\omega L}. \]

By expansion, one has

\[ R_0 = \frac{-\omega^2 R_L C_1 C_2 + j\omega C_1[1 - \omega(X_L + \omega L)C_2]}{1 - \omega(X_L + \omega L)C_2 - \omega^2 C_1 C_2 + j\omega R_L C_2}. \]

Since the real part of the fraction in the preceding formula should be equal to \( R_0 \) and the imaginary part be equal to zero, one has

\[ \begin{cases} R_0 & = \omega^4 R_L C_1^2 C_2^2 \left[ \frac{1}{1 - \omega(X_L + \omega L)C_2 - \omega^2 C_1 C_2} \right]^2 + [\omega R_L C_2]^2 \end{cases} \]

\[ 0 = \omega C_1[1 - \omega(X_L + \omega L)C_2][1 - \omega(X_L + \omega L)C_2 - \omega^2 C_1 C_2] + \omega^3 R_L^2 C_1^2 C_2^2. \]

For a given inductance \( L \), capacitances \( C_1 \) and \( C_2 \) can be determined from the above two simultaneous equations.

### 5.2 Quarter-wave transformer

We then proceed to consider the impedance matching by using transmission line. Consider a purely resistive load or a complex load which has been transformed to purely resistive (by a line of suitable length in series or others). A simple method of matching to a lossless transmission line of characteristic impedance \( R_0 \) is to insert a quarter-wave line with a suitable characteristic impedance \( R'_0 \) in series with the line.

Remark that for a quarter-wave line with characteristic impedance \( Z'_0 \), the impedance transformation is \( Z_{in} = Z'_0 / Z_L \). Thus the impedance matching (\( Z_{in} = Z_0 \)) can be achieved by using a quarter-wave line of which the characteristic impedance \( R'_0 \) is chosen to be

\[ R'_0 = \sqrt{R_0 R_L}. \]
Since the length of quarter-wave line depends on wavelength, this matching method is frequency-sensitive.

5.3 Matching by a stub parallel connected at the load

Sometimes, it is helpful to use the load admittance given by $Y_L = \frac{1}{Z_L} = G_L + jB_L$, where $G_L$ is the load conductance and $B_L$ is the load susceptance. We first consider the case where $G_L = G_0$, that is, the load conductance is equal to the characteristic admittance $G_0 (= 1/R_0)$ of the lossless line. For a lumped circuit, the impedance matching can be achieved by adding a capacitor or an inductor in parallel to the load. At higher frequencies, the reactance of an inductor or the susceptance of a capacitor can be too high. Instead, the impedance matching can be implemented by using a stub connected in parallel with the load and placed at the load position.

Remark that the input impedance of a short-circuit stub of length $\ell$ and characteristic impedance $R_0$ is given as $jR_0 \tan k\ell$. Thus the length $\ell$ and the characteristic impedance $R'_0$ of a short-circuit stub should be chosen such that the susceptance of the inserted stub is given as $(jR'_0 \tan k\ell)^{-1} = -jB_L$.

The solutions for $R'_0$ and $\ell$ are not unique. If $R'_0 = R_0$, the required length $\ell$ of the short-circuit stub is then $\ell = (\lambda/2\pi) \cot^{-1} b_L$, where the normalized load susceptance $b_L = B_L R_0$. If the load is capacitive and hence the load susceptance $B_L > 0$, the length of the short-circuit stub can be chosen to be fixed to $\ell = \lambda/8$ ($\tan k\ell = 1$). Then the required characteristic impedance of the stub is chosen as $R'_0 = \frac{1}{B_L}$.

The Smith chart is of some help in visualizing the matching procedure. Consider the method with $R'_0 = R_0$. First, locate point $\Gamma_1$ corresponding to the normalized load admittance $y_L = g_L + j b_L$ on the Smith chart. Secondly, locate point $\Gamma_2$, which is the intersection of the unit circle with the constant-$x$ arc through the point $\Gamma_1$. Then, locate the point $\Gamma_3$ which is the image of point $\Gamma_2$ with respect to the $\Gamma_\pi$ axis and is associated with a normalized susceptance of $-b_L$. Finally, the angle from the point $(1,0)$ to the point $\Gamma_3$ clockwise corresponds to the required length $\ell$ of the short-circuit stub. Note that this angle is equal to that from the point $\Gamma_2$ clockwise to the point $(1,0)$.

For a load with $G_L \neq G_0$, the impedance matching can be implemented by inserting a quarter-wave transformer between the transmission line and the load together with the stub. The quarter-wave transformer should has a characteristic impedance $R'_0 = \sqrt{R_0/G_L}$.

5.4 Single-stub matching – stub with adjustable position and length

A simpler way to match a load with a general complex impedance $Z_L$ to a lossless line is to place a single stub with a suitable length $\ell$ at a suitable distance $d$ away from the load. The stub position $d$ is chosen in such a way that there are no reflection before the branch, while reflections generally exist both in the main line and the stub after the branch. The situation of no reflection implies that the current is continuous at the branch. Note that the voltages of the three sections of lines right at the branch are identical. Thereby, the continuity of current leads to

\[
\frac{1 - \Gamma e^{-j2kd}}{1 + \Gamma e^{-j2kd}} + \frac{1 - \Gamma_s e^{-j2k\ell}}{1 + \Gamma_s e^{-j2k\ell}} = 1,
\]

where $\Gamma_s$ is the reflection coefficient at the end of the stub and a common factor of the voltage divided by $Z_0$ drops out. This complex formula represents the constraints on the position $d$ and
stub length $\ell$. Note that $y_in = y(z' = d) = (1 - \Gamma e^{-j2kd})/(1 + \Gamma e^{-j2kd})$. The preceding relation is actually equivalent to the situation that the sum of the normalized admittance at the branch is unity. For a short-circuit stub, $\Gamma_s = -1$ and the constraints on $d$ and $\ell$ become

$$\frac{1 - \Gamma e^{-j2kd}}{1 + \Gamma e^{-j2kd}} = 1 + j \cot k\ell.$$ 

This represents a set of two equations in $d$ and $\ell$.

The preceding relations mean that the stub position $d$ is chosen in such a way that the normalized load conductance will become unity at that location, while the normalized susceptance is allowed to be arbitrary. That is,

$$y_in = 1 + jb_in,$$

where $b_in = \cot k\ell$. From this constraint on the normalized load conductance, the desired location $d$ can be determined analytically. Thereafter, the stub length $\ell$ is chosen in such a way that the corresponding susceptance $b_in$ can be cancelled out.

As to the desired location $d$, it can be determined by expressing the admittance $y_in$ explicitly in terms of the load admittance $y_L$ according to the admittance transformation. It is seen that $g_in$ and $b_in$ are associated with $r_L, x_L,$ and $t = \tan kd$. By requiring $g_in = 1$, it can be shown that

$$t = \begin{cases} \frac{1}{r_L - 1} \left\{ x_L \pm \sqrt{r_L[(1 - r_L)^2 + x_L^2]} \right\} & r_L \neq 1 \\ -x_L/2 \text{ or } \rightarrow \infty. & r_L = 1 \end{cases}$$

Then the desired location $d$ to achieve a unity normalized conductance is given by

$$d = \frac{\lambda}{2\pi} \tan^{-1} t,$$

where the value of the arc tangent is understood to range from zero to $\pi$.

The corresponding normalized susceptance $b_in$ is given in terms of $r_L, x_L,$ and $t$ by

$$b_in = \frac{r_L^2 t - (1 - xLt)(x_L + t)}{r_L^2 + (x_L + t)^2}.$$ 

Then the required stub length is given by $\ell = (\lambda/2\pi)\tan^{-1}(1/b_in)$ for a short-circuit stub. It is noted that both location $d$ and length $\ell$ are proportional to the operating wavelength. It could be found that the two solutions for normalized susceptance $b_in$ are always of equal magnitude but opposite signs. Then the sum of the two stub lengths corresponding to the two positions $d$ is one half of wavelength.

**Graphical solution with the Smith chart**

The Smith chart is of great help in visualizing the process of choosing the suitable positions $d$ for a unity normalized conductance. On the Smith chart we locate the point $\Gamma_L$ corresponding to $z_L$, draw the $\Gamma$ circle, mark the point $\Gamma_1$ corresponding to $y_L$, mark points $\Gamma_2$ and $\Gamma_3$ which are the two intersections of the $\Gamma$ circle with the unity-resistance circle ($r = 1$). $\Gamma_2$ and $\Gamma_3$ are located at the upper and the lower half of the complex plane, respectively. Then the angle from $\Gamma_1$ clockwise to point $\Gamma_2$ or $\Gamma_3$ corresponds to the position $d$ of the stub.

Thereafter, locate point $\Gamma_4$ which is the intersection of the unit circle with the constant-$x$ arc through the point $\Gamma_2$. Then the angle from point $\Gamma_4$ clockwise to the point $(1,0)$ and $(-1,0)$
corresponds to the required length $\ell_1$ of the short- and open-circuit stub, respectively. Similarly, locate point $\Gamma_5$ which is the intersection of the unit circle with the constant-$x$ arc through the point $\Gamma_3$. Then the angle from point $\Gamma_5$ clockwise to the point $(1, 0)$ and $(-1, 0)$ corresponds to the required length $\ell_2$ of the short- and open-circuit stub, respectively. Apparently, $\ell_1 + \ell_2 = \lambda/2$.

A physical interpretation of impedance matching

A physical interpretation of the principle of the matching network is given in what follows. The stub inserted in shunt does not cancel the reflection directly. Instead, it incurs more reflections, such as the direct reflection from the branch point and the indirect reflections from the end of the stub and the load. The cancellation of the reflection by inserting a stub is due to the destructive interference among the reflection from the load and the numerous additional reflections incurred by the stub.

**5.5 Double-stub matching**

Sometimes, it desirable that the location $d$ is fixed so that the distance $d$ does not depend on the actual load. For such a situation one should use another stub to achieve that the real part of normalized admittance $y_{in}$ be unity. Let the second stub of length $\ell_2$ be short-circuit terminated and located at the load. Then the normalized admittance $y_1 (= g_1 + j b_1)$ at the load becomes $y_1 = zL^{-1} + [j \tan k\ell_2]^{-1}$. Explicitly,

$$g_1 = \frac{r_L}{r_L^2 + x_L^2}$$

$$b_1 = \frac{-x_L}{r_L^2 + x_L^2} - \frac{1}{\tan k\ell_2}.$$

It is noted that $g_1$ does not depend on the length $\ell_2$. At $z = -d$ the normalized admittance of the parallel combination of the load and the second stub is transformed as

$$y_{in} = \frac{1 + jy_1 \tan kd}{y_1 + j \tan kd} = \frac{(1 - b_1 t) + j g_1 t}{g_1 + j(b_1 + t)}$$

$$= \frac{[g_1(1 - b_1 t) + g_1 t(b_1 + t)] + j[g_1^2 t - (1 - b_1 t)(b_1 + t)]}{g_1^2 + (b_1 + t)^2},$$

where $t = \tan kd$. The requirement of the admittance $y_{in}$ with a unity normalized conductance yields a quadratic equation in $b_1$:

$$b_1^2 + 2b_1 t + [(1 - g_1)t^2 + g_1^2 - g_1] = 0.$$

The solution is

$$b_1 = -t \pm \sqrt{g_1(1 - g_1 - t^2)}.$$

The length $\ell_2$ of the second stub in turn is chosen in such a way to implement the desired $b_1$ according to

$$\ell_2 = \frac{\lambda}{2\pi} \tan^{-1} \left\{ \frac{-x_L}{r_L^2 + x_L^2} + \tan kd \mp \sqrt{g_1(1 - g_1 - t^2)} \right\}^{-1},$$

which is determined by $Z_L$, $R_0$, and $d$.

After $x_1$ (or $\ell_2$) is solved, the associated normalized susceptance $b$ is given as

$$b = \frac{y_1^2 t + b_1 t^2 - t - 1}{y_1^2 + 2b_1 t + t^2}.$$
Again, the length \( \ell_1 \) of the first stub is given by

\[
\ell_1 = \frac{\lambda}{2\pi} \tan^{-1} \left( \frac{1}{b} \right).
\]