EE641000 Quantum Information and Computation

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Unit Six – Quantum Operations
Postulate 2: the dynamics of a closed quantum system is described by a unitary transform.

An open quantum system together with its environment becomes a closed quantum system.

- $\rho$: the density operator of the open quantum system, called the principal quantum system.
- $\rho_{env}$: the density operator of the environment.
• $U$ : a unitary operator on the state space of the closed quantum system

• $\mathcal{E}(\rho)$ : the density operator of the principal quantum system after the action of the unitary operator $U$
  
  – Closed system
  
  \[
  \mathcal{E}(\rho) = U \rho U^\dagger
  \]

  – Open system
  
  \[
  \mathcal{E}(\rho) = \text{tr}_{env} \left( U \left( \rho \otimes \rho_{env} \right) U^\dagger \right)
  \]

* Assume that the principal quantum system is prepared such that its correlation with the environment can be completely destroyed (Correlated initial state of the principal-environmental system will be discussed later)

* If the state space of the principal system has dimension $d$, we will show that it is sufficient to model the environment to have state space of dimension no greater than $d^2$
• \( \rho = \sum_{ij} \alpha_{ij} |i\rangle \langle j| \) : the density operator of the principal system
• \( \rho_{env} = |0\rangle \langle 0| \) : the environment is in the pure state \( |0\rangle \)
• \( \mathcal{E}(\rho) = P_0 \rho P_0 + P_1 \rho P_1 \) : \( P_0 = |0\rangle \langle 0| \) and \( P_1 = |1\rangle \langle 1| \) are projective operators
\[ E(\rho) = \text{tr}_2 \left( U \left( \sum_{ij} \alpha_{ij} |i\rangle \langle j| \otimes |0\rangle \langle 0| \right) U^\dagger \right) \]

\[ = \text{tr}_2 \left( \alpha_{00} |0\rangle \langle 0| \otimes |0\rangle \langle 0| + \alpha_{10} |1\rangle \langle 0| \otimes |1\rangle \langle 0| + \alpha_{01} |0\rangle \langle 1| \otimes |0\rangle \langle 1| + \alpha_{11} |1\rangle \langle 1| \otimes |1\rangle \langle 1| \right) \]

\[ = \alpha_{00} |0\rangle \langle 0| + \alpha_{11} |1\rangle \langle 1| \]

\[ P_0 \rho P_0 = \alpha_{00} |0\rangle \langle 0| \]

\[ P_1 \rho P_1 = \alpha_{11} |1\rangle \langle 1| \]

\[ E(\rho) = P_0 \rho P_0 + P_1 \rho P_1 \]
Quantum Operations Formalism
The Input-Output Formalism

\[ \mathcal{E}(\rho^A) = \text{tr}_B \left( U \left( \rho^A \otimes \rho^B \right) U^\dagger \right) \]

or

\[ \mathcal{E}(\rho^A) = \text{tr}_A \left( U \left( \rho^A \otimes \rho^B \right) U^\dagger \right) \]

- \( \rho^A \): the input density operator of quantum systems A
- \( \rho^B \): the density operators of quantum system B
- \( \mathcal{E}(\rho^A) \): the output density operator of quantum systems A or B
A General Definition of Triplet $\langle \psi | U | \varphi \rangle$

- $H_1$ and $H_2$: complex inner product spaces
- $U$: a linear operator on the tensor product space $H_1 \otimes H_2$
  $U = \sum_{i} \alpha_i T_{i}^{H_1} \otimes T_{i}^{H_2}$,

  where $T_{i}^{H_1}$ and $T_{i}^{H_2}$ are linear operators on $H_1$ and $H_2$ respectively
- $|\psi\rangle$ and $|\varphi\rangle$: two vectors in $H_2$
- $\langle \psi | U | \varphi \rangle$: a linear operator on $H_1$
  $\langle \psi | U | \varphi \rangle \triangleq \sum_{i} \alpha_i T_{i}^{H_1} \langle \psi | T_{i}^{H_2} | \varphi \rangle$
- $\langle \psi | \cdot | \varphi \rangle$: a linear map from $L(H_1 \otimes H_2, H_1 \otimes H_2)$ to $L(H_1, H_1)$
Well-defined

- \{ |j\rangle \} : an orthonormal basis of \( H_2 \)
- \{ |j\rangle \langle k| \} : a basis of \( L(H_2, H_2) \)
- Unique representation : with \( T^H_2 = \sum_{jk} \beta_{ijk} |j\rangle \langle k| \), we have

\[
U = \sum_{jk} \left( \sum_i \alpha_i \beta_{ijk} T^H_i \right) \otimes |j\rangle \langle k|
\]

Thus we have

\[
\langle \psi | U | \varphi \rangle \overset{\Delta}{=} \sum_i \alpha_i T^H_i \langle \psi | T^H_2 | \varphi \rangle = \sum_i \alpha_i T^H_i \sum_{jk} \beta_{ijk} \langle \psi | j \rangle \langle k | \varphi \rangle
\]

\[
= \sum_{jk} \left( \sum_i \alpha_i \beta_{ijk} T^H_i \right) \langle \psi | j \rangle \langle k | \varphi \rangle.
\]
A Theorem

$$\text{tr}_2(U) = \sum_i \langle i | U | i \rangle$$

- $H_1$ and $H_2$: complex inner product spaces
- $U$: a linear operator on the tensor product space $H_1 \otimes H_2$
- $\{|i\rangle\}$: an orthonormal basis of $H_2$
\[
\text{Proof}
\]

\[
\begin{align*}
\text{tr}_2(U) &= \text{tr}_2 \left( \sum_j \alpha_j T_j^{H_1} \otimes T_j^{H_2} \right) \\
&= \sum_j \alpha_j T_j^{H_1} \text{tr}(T_j^{H_2}) \\
&= \sum_j \sum_i \alpha_j T_j^{H_1} \langle i | T_j^{H_2} | i \rangle \\
&= \sum_i \sum_j \alpha_j T_j^{H_1} \langle i | T_j^{H_2} | i \rangle \\
&= \sum_i \langle i \left| \left( \sum_j \alpha_j T_j^{H_1} \otimes T_j^{H_2} \right) | i \rangle \right.
\]
\[ = \sum_{i} \langle i | U | i \rangle \]
A General Definition of Inner Product $\langle \psi | \varphi \rangle$

- $H_1$ and $H_2$: complex inner product spaces
- $|\psi\rangle$: a vector in $H_1$
- $|\varphi\rangle = \sum_i \alpha_i |v_i\rangle \otimes |w_i\rangle$: a vector in $H_1 \otimes H_2$
- $\langle \psi | \varphi \rangle$: a vector in $H_2$

\[
\langle \psi | \varphi \rangle \triangleq \sum_i \alpha_i \langle \psi | v_i \rangle w_i
\]

- $\langle \psi | \cdot \rangle$: a linear transformation from $H_1 \otimes H_2$ to $H_2$
Well-defined

- \{ |j \rangle \} : an orthonormal basis of \( H_2 \)

- Unique representation: with \( |w_i \rangle = \sum_j \beta_{ij} |j \rangle \), we have

\[
|\varphi \rangle = \sum_j (\sum_i \alpha_i \beta_{ij} |v_i \rangle) \otimes |j \rangle
\]

Thus we have

\[
\langle \psi | \varphi \rangle \triangleq \sum_i \alpha_i \langle \psi | v_i \rangle \langle v_i | w_i \rangle
\]

\[
= \sum_i \alpha_i \langle \psi | v_i \rangle \sum_j \beta_{ij} |j \rangle
\]

\[
= \sum_j \langle \psi | \left( \sum_i \alpha_i \beta_{ij} |v_i \rangle \right) |j \rangle
\]
The Operator-Sum Representation of $\mathcal{E}$

\[ \mathcal{E}(\rho) = \sum_k \langle e_k | U(\rho \otimes \rho_{env}) U^\dagger | e_k \rangle = \sum_{km} E_{km} \rho E_{km}^\dagger \]

where $E_{km} = \langle e_k | U \sqrt{\lambda_m} | \psi_m \rangle$

- $\rho$ : the density operator of the principal quantum system
- $\{|e_k\rangle\}$ : an orthonormal basis of the (finite-dimensional) state space of the environment
- $\rho_{env} = \sum_m \lambda_m |\psi_m\rangle \langle \psi_m|$ : the density operator of the environment with ensemble $\{\lambda_m, |\psi_m\rangle\}$
- $\{E_{km}\}$ : operation elements for the quantum operation $\mathcal{E}$
\[ U = \sum_i \alpha_i T_i^{pri} \otimes T_i^{env} \]
\[ \rho_{env} = \sum_m \lambda_m |\psi_m\rangle \langle \psi_m| \]

\[
\mathcal{E}(\rho) = \text{tr}_{\text{env}} \left( U (\rho \otimes \rho_{env}) U^\dagger \right) \\
= \sum_k \langle e_k | \left( \sum_i \alpha_i T_i^{pri} \otimes T_i^{env} \right) (\rho \otimes \rho_{env}) \left( \sum_j \alpha_j T_j^{pri} \otimes T_j^{env} \right)^\dagger | e_k \rangle \\
= \sum_k \sum_{ij} \alpha_i \alpha_j^* \langle e_k | \left( T_i^{pri} \rho T_j^{pri\dagger} \otimes T_i^{env} \rho_{env} T_j^{env\dagger} \right) | e_k \rangle \\
= \sum_k \sum_{ij} \alpha_i \alpha_j^* T_i^{pri} \rho T_j^{pri\dagger} \langle e_k | T_i^{env} \rho_{env} T_j^{env\dagger} | e_k \rangle 
\]
\[
\begin{align*}
&= \sum_{k} \sum_{ij} \alpha_i \alpha_j^* T^{pri}_i \rho T^{pri}_j^\dagger \sum_m \lambda_m \langle e_k | T^{env}_i | \psi_m \rangle \langle \psi_m | T^{env}_j^\dagger | e_k \rangle \\
&= \sum_{k} \sum_{m} \lambda_m \left( \sum_i \alpha_i T^{pri}_i \langle e_k | T^{env}_i | \psi_m \rangle \right) \rho \left( \sum_j \alpha_j T^{pri}_j \langle e_k | T^{env}_j | \psi_m \rangle \right)^\dagger \\
&= \sum_{k} \sum_{m} \lambda_m \langle e_k | \sum_i \alpha_i T^{pri}_i \otimes T^{env}_i | \psi_m \rangle \rho \left( \langle e_k | \sum_j \alpha_j T^{pri}_j \otimes T^{env}_j | \psi_m \rangle \right)^\dagger \\
&= \sum_{k} \sum_{m} \langle e_k | U \sqrt{\lambda_m} | \psi_m \rangle \rho \left( \langle e_k | U \sqrt{\lambda_m} | \psi_m \rangle \right)^\dagger \\
&= \sum_{k} \sum_{m} E_{km} \rho E_{km}^\dagger \\
\end{align*}
\]

where \( E_{km} = \langle e_k | U \sqrt{\lambda_m} | \psi_m \rangle \)
Lemma

Let $T$ be a linear operator on a Hilbert space $H$. If

$$\text{tr}(T \rho) = 1$$

for any density operators $\rho$ on $H$, then we have

$$T = I$$

Proof.

Note that $\text{tr}(T \rho) = 1$ for any density operator $\rho$ on $H$ if and only if $\text{tr}(T |\psi\rangle \langle \psi|) = 1$, i.e. $\langle \psi| T |\psi\rangle = 1$, for any unit vector $|\psi\rangle$ in $H$. Let $\{|e_k\rangle\}$ be an orthonormal basis of $H$. For any $i \neq j$ and any non-zero complex numbers $a, b$ such that $|a|^2 + |b|^2 = 1$, we have

$$1 = (a |e_i\rangle + b |e_j\rangle)^\dagger T (a |e_i\rangle + b |e_j\rangle)$$
which implies that
\[ a^*b\langle e_i|T|e_j\rangle + ab^*\langle e_j|T|e_i\rangle = 0. \]

By taking \( a, b \) both real, we have
\[ \langle e_i|T|e_j\rangle + \langle e_j|T|e_i\rangle = 0. \]

But by taking \( a \) real and \( b \) pure imaginary, we have
\[ \langle e_i|T|e_j\rangle - \langle e_j|T|e_i\rangle = 0. \]

Thus we conclude that \( \langle e_i|T|e_j\rangle = \langle e_j|T|e_i\rangle = 0 \) and the matrix representation of \( T \) relative to the orthonormal basis \( \{|e_k\rangle\} \) is the identity matrix which implies that \( T = I \).
Completeness Relation on Operation Elements

\[ \sum_{km} E^\dagger_{km} E_{km} = I \]

- \( \mathcal{E} \) is trace-preserving: for any density operator \( \rho \),

\[
\text{tr}(\mathcal{E}(\rho)) = \text{tr}(\text{tr}_{env}(U(\rho \otimes \rho_{env})U^\dagger)) = \text{tr}(U(\rho \otimes \rho_{env})U^\dagger) \\
= \text{tr}(U^\dagger U(\rho \otimes \rho_{env})) = \text{tr}(\rho \otimes \rho_{env}) = \text{tr}(\rho)\text{tr}(\rho_{env}) \\
= 1
\]

- \( 1 = \text{tr}(\mathcal{E}(\rho)) = \sum_{km} \text{tr}(E_{km}\rho E^\dagger_{km}) = \sum_{km} \text{tr}(E^\dagger_{km} E_{km}\rho) = \text{tr}((\sum_{km} E^\dagger_{km} E_{km})\rho) \) for any density operator \( \rho \)
Purification of the Environment

- $\rho$: the density operator of the principal quantum system
- $\rho_{env} = \sum_m \lambda_m |\psi_m\rangle\langle\psi_m|$: the density operator of the environment with ensemble $\{\lambda_m, |\psi_m\rangle\}$
- $\{|m_R\rangle\}$: an orthonormal basis of the state space of a reference system $R$, having the same cardinality as that of $\{|\psi_m\rangle\}$
- $|envR\rangle$: a pure state of the composite environment-$R$ system

\[
|envR\rangle = \sum_m \sqrt{\lambda_m} |\psi_m\rangle |m_R\rangle
\]

such that

\[
\rho_{env} = \text{tr}_R(|envR\rangle\langle envR|)
\]
Purification of the Environment (Cont’)

\[ \mathcal{E}(\rho) = \text{tr}_{\text{env}} \left( U (\rho \otimes \rho_{\text{env}}) U^\dagger \right) \]
\[ = \text{tr}_{\text{env}R} \left( (U \otimes I_R)(\rho \otimes |envR\rangle\langle envR|)(U \otimes I_R)^\dagger \right) \]

**Proof.**

Note that for an orthonormal basis \( \{|e_k\rangle\} \) of the state space of the environment, \( \{|e_k\rangle|m_R\rangle\} \) is an orthonormal basis of the state space.
of the composite environment-$R$ system. We have

$$\text{tr}_{envR} \left( (U \otimes I_R)(\rho \otimes |envR\rangle\langle envR|)(U \otimes I_R)^\dagger \right) = \sum_{km} F_{km} \rho F_{km}^\dagger$$

where

$$F_{km} = \langle e_k | (m_R | (U \otimes I_R) | envR\rangle \langle m_R | (U \otimes I_R) \sum_j \sqrt{\lambda_j} | \psi_j \rangle | j_R \rangle$$

$$= \sum_j \sqrt{\lambda_j} \langle e_k | (m_R | (U \otimes I_R) | \psi_j \rangle | j_R \rangle$$

$$= \sum_j \sqrt{\lambda_j} \langle e_k | U | \psi_j \rangle \langle m_R | j_R \rangle$$

$$= \langle e_k | U \sqrt{\lambda_m} | \psi_m \rangle$$

$$= E_{km}$$
Three Features of the Operator-Sum Representation
Physical Interpretation

- $\rho$ : the density operator of the principal quantum system
- $\rho_{env} = \sum_{m} \lambda_m |\psi_m\rangle\langle\psi_m|$ : the density operator of the environment
- $\{|e_k\rangle\}$ : an orthonormal basis of the state space of the environment
- Principle of implicit measurement : the state of the principal system will not be affected if measurement is performed on the environment
- $\{|e_k\rangle\langle e_k|\}$ : a projective measurement on the environment
- $\rho_k$ : the state of the principal system given that outcome $k$ occurs
\[ \rho_k = \text{tr}_{\text{env}} \left( \frac{(I \otimes |e_k\rangle\langle e_k|)U(\rho \otimes \rho_{\text{env}})U^\dagger(I \otimes |e_k\rangle\langle e_k|)}{\text{tr}((I \otimes |e_k\rangle\langle e_k|)U(\rho \otimes \rho_{\text{env}})U^\dagger(I \otimes |e_k\rangle\langle e_k|))} \right) \]

\[ = \text{tr}_{\text{env}} \left( \frac{\langle e_k|U(\rho \otimes \rho_{\text{env}})U^\dagger|e_k\rangle \otimes |e_k\rangle\langle e_k|}{\text{tr}(\langle e_k|U(\rho \otimes \rho_{\text{env}})U^\dagger|e_k\rangle \otimes |e_k\rangle\langle e_k|)} \right) \]

\[ = \frac{\langle e_k|U(\rho \otimes \rho_{\text{env}})U^\dagger|e_k\rangle \text{tr}(|e_k\rangle\langle e_k|)}{\text{tr}(\langle e_k|U(\rho \otimes \rho_{\text{env}})U^\dagger|e_k\rangle)\text{tr}(|e_k\rangle\langle e_k|)} \]

\[ = \frac{\sum_m E_{km} \rho E_{km}^\dagger}{\text{tr}(\sum_m E_{km} \rho E_{km}^\dagger)} \]

where \( E_{km} = \langle e_k|U \sqrt{\lambda_m} \psi_m \rangle \)

- \( \mathcal{P}(k) \): the probability that outcome \( k \) occurs

\[ \mathcal{P}(k) = \text{tr}((I \otimes |e_k\rangle\langle e_k|)U(\rho \otimes \rho_{\text{env}})U^\dagger(I \otimes |e_k\rangle\langle e_k|)) \]

\[ = \text{tr}(\sum_m E_{km} \rho E_{km}^\dagger) \]
\[ \mathcal{E}(\rho) = \sum_k \mathcal{P}(k) \rho_k = \sum_{km} E_{km} \rho E_{km}^\dagger \]

- The action of the quantum operation \( \mathcal{E} \) is equivalent to taking the state \( \rho \) as input and randomly replacing it by

\[
\frac{\sum_m E_{km} \rho E_{km}^\dagger}{\text{tr}(\sum_m E_{km} \rho E_{km}^\dagger)}
\]

with probability

\[
\text{tr}(\sum_m E_{km} \rho E_{km}^\dagger)
\]

- A quantum operation which describes a quantum noise process will be referred to as a noisy quantum channel.
\( \rho \) : the density operator of the principal system

\( \rho_{env} = |0_E\rangle\langle 0_E| : \) the environment is in the pure state \( |0_E\rangle \)

\( U = \\
|0_P0_E\rangle\langle 0_P0_E| + |0_P1_E\rangle\langle 0_P1_E| + |1_P1_E\rangle\langle 1_P0_E| + |1_P0_E\rangle\langle 1_P1_E| \\
\)
\begin{itemize}
  \item \{\ket{0_E}, \ket{1_E}\} : an orthonormal basis of the state space of the environment
  \item \[ E_0 = \braket{0_E|U|0_E} = \ket{0_P}\bra{0_P} = P_0 \]
  \item \[ E_1 = \braket{1_E|U|0_E} = \ket{1_P}\bra{1_P} = P_1 \]
  \item \[ \mathcal{E}(\rho) = P_0 \rho P_0 + P_1 \rho P_1 \]
\end{itemize}
Effect of Global Measurement

- $\rho$: the density operator of the principal quantum system
- $\rho_{env} = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$: the density operator of the environment
- $\{|e_k\}\}$: an orthonormal basis of the state space of the environment
- $\{P_m\}$: projective measurement after the unitary operation $U$
- The state of the principal system given that outcome $m$ occurs is
  \[ \text{tr}_{env} \left( \frac{P_m U (\rho \otimes \rho_{env}) U^\dagger P_m}{\text{tr}(P_m U (\rho \otimes \rho_{env}) U^\dagger P_m)} \right) \]
  with probability
  \[ \text{tr}(P_m U (\rho \otimes \rho_{env}) U^\dagger P_m) \]
Define a map

$$\mathcal{E}_m(\rho) \triangleq \text{tr}_{env}(P_m U (\rho \otimes \rho_{env}) U^\dagger P_m) = \sum_{kj} E^{(m)}_{kj} \rho E^{(m)\dagger}_{kj}$$

where

$$E^{(m)}_{kj} = \langle e_k | P_m U \sqrt{\lambda_j} | \psi_j \rangle$$

- The state of the principal system given that outcome $m$ occurs is

$$\frac{\mathcal{E}_m(\rho)}{\text{tr}(\mathcal{E}_m(\rho))}$$

with probability $\text{tr}(\mathcal{E}_m(\rho))$

- $\{\mathcal{E}_m(\rho)\}$: a kind of measurement process which generalizes the measurement described in Unit Three where $\mathcal{E}_m(\rho) = E_m \rho E_m^\dagger$ for a quantum measurement $\{E_m\}$
The Converse Problem

- \( \{ E_k \} \) : a given collection of operator elements acting on a principal quantum system and satisfying the completeness relation

\[
\sum_k E_k^\dagger E_k = I
\]

- The problem: find a system-environment model, i.e., an environment, a unitary operator \( U \) on the composite system-environment model such that \( \{ E_k \} \) is the operator elements in the operator-sum representation of the quantum operation \( \mathcal{E} \)
A System-Environment Model

- $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$: the density operator of the principal system
- $\{|e_k\}\}$: an orthonormal basis of the state space of a chosen environment, having the same cardinality as that of $\{E_k\}$
- $|0\rangle$: an arbitrarily chosen state of the environment
- $U$: a linear operator acting on the states of the form $|\psi\rangle|0\rangle$ where $|\psi\rangle$ is any state of the principal system

$$U|\psi\rangle|0\rangle = \sum_k E_k |\psi\rangle|e_k\rangle$$

and for any states $|\psi\rangle$ and $|\varphi\rangle$ of the principal system,

$$\langle\psi|\langle0|U^\dagger U|\varphi\rangle|0\rangle = \sum_{kj} (E_k |\psi\rangle|e_k\rangle)^\dagger (E_j |\varphi\rangle|e_j\rangle)$$
Thus $U$ can be extended to a unitary operator
$U = \sum_i \alpha_i T_i^{pri} \otimes T_i^{env}$ on the composite system and we have for any state $|\psi\rangle$ of the principal system

$$
\langle \langle e_k|U|0\rangle|\psi\rangle = \sum_i \alpha_i T_i^{pri} \langle e_k|T_i^{env}|0\rangle|\psi\rangle
$$

$$
= \sum_i \alpha_i T_i^{pri} |\psi\rangle \langle e_k|T_i^{env}|0\rangle
$$

$$
= \langle e_k|\left(\sum_i \alpha_i T_i^{pri} \otimes T_i^{env}\right)|\psi\rangle|0\rangle
$$

$$
= \langle e_k|U|\psi\rangle|0\rangle = \langle e_k|\sum_j E_j|\psi\rangle|e_j\rangle = E_k|\psi\rangle
$$

which says that $E_k = \langle e_k|U|0\rangle$
Axiomatic Approach to Quantum Operations
A quantum operation $\mathcal{E}$ is a map from the set of density operators of the input space $Q_1$ to the set of positive operators of the output space $Q_2$, satisfying the three axiomatic properties as follows.
Three Axioms

- Axiom I: \( \text{tr}(\mathcal{E}(\rho)) \) is the probability that the process represented by \( \mathcal{E} \) occurs, when \( \rho \) is the input density operator,

\[
0 \leq \text{tr}(\mathcal{E}(\rho)) \leq 1
\]

- Axiom II: \( \mathcal{E} \) is a convex-linear map on the set of density operators, i.e., for non-negative numbers \( \lambda_i \) with \( \sum_i \lambda_i = 1 \) and density operators \( \rho_i \),

\[
\mathcal{E}(\sum_i \lambda_i \rho_i) = \sum_i \lambda_i \mathcal{E}(\rho_i)
\]

- Axiom III: \( \mathcal{E} \) is a completely positive map, i.e., for an arbitrarily introduced system \( R \) of arbitrary dimension and the identity map \( \mathcal{I} \) on the set of all linear operators on \( R \), \( I \otimes \mathcal{E} \) is a well-defined map from the set of positive operators of the composite system \( RQ_1 \) to the set of positive operators of the composite system \( RQ_2 \)
Axiom I

\[ \text{tr}(\mathcal{E}(\rho)) \] is the probability that the process represented by \( \mathcal{E} \) occurs, when \( \rho \) is the input density operator,

\[ 0 \leq \text{tr}(\mathcal{E}(\rho)) \leq 1 \]

- This is a mathematical convenience to include the case of measurement as quantum operation, where the trace may not be preserved and \( \text{tr}(\mathcal{E}(\rho)) \) is exactly the probability of the occurrence of a particular measurement outcome when the state before measurement is \( \rho \). The state after measurement becomes \( \mathcal{E}(\rho)/\text{tr}(\mathcal{E}(\rho)) \).
Axiom II

\( \mathcal{E} \) is a convex-linear map on the set of density operators, i.e., for non-negative numbers \( p_i \) with \( \sum_i p_i = 1 \) and density operators \( \rho_i \),

\[
\mathcal{E} \left( \sum_i p_i \rho_i \right) = \sum_i p_i \mathcal{E}(\rho_i)
\]

- \( \rho = \sum_i p_i \rho_i \) : the input quantum state as a random selection from the ensemble \( \{p_i, \rho_i\} \) of quantum (mixed) states
- \( \mathcal{E}(\rho)/\text{tr}(\mathcal{E}(\rho)) = \mathcal{E}(\rho)/p(\mathcal{E}) \) : the resulting state as a random selection from the ensemble \( \{p(i|\mathcal{E}), \mathcal{E}(\rho_i)/\text{tr}(\mathcal{E}(\rho_i))\} \) of quantum (mixed) states, where \( p(i|\mathcal{E}) \) is the probability that the state prepared is \( \rho_i \), given that the process described by \( \mathcal{E} \) occurs, i.e., we demand

\[
\frac{\mathcal{E}(\rho)}{p(\mathcal{E})} = \sum_i p(i|\mathcal{E}) \frac{\mathcal{E}(\rho_i)}{\text{tr}(\mathcal{E}(\rho_i))}
\]
• A Bayesian rule:

\[ p(i|\mathcal{E}) = \frac{p(\mathcal{E}|i)p_i}{p(\mathcal{E})} = \frac{\text{tr}(\mathcal{E}(\rho_i))p_i}{p(\mathcal{E})} \]

• Justification:

\[ \mathcal{E}(\rho) = p(\mathcal{E}) \sum_i p(i|\mathcal{E}) \frac{\mathcal{E}(\rho_i)}{\text{tr}(\mathcal{E}(\rho_i))} = \sum_i p_i \mathcal{E}(\rho_i) \]
Axiom III

\( \mathcal{E} \) is a completely positive map, i.e., for an arbitrarily introduced system \( R \) of arbitrary dimension and the identity map \( \mathcal{I} \) on the set of all linear operators on \( R \), \( \mathcal{I} \otimes \mathcal{E} \) is a well-defined map from the set of positive operators of the composite system \( RQ_1 \) to the set of positive operators of the composite system \( RQ_2 \).

- It is required for a physical system that if \( \rho^{RQ_1} \) is a (mixed) state of a composite system \( RQ_1 \) and the quantum operation \( \mathcal{E} \) acts solely on the system \( Q_1 \), then the result \( \mathcal{I} \otimes \mathcal{E}(\rho^{RQ_1}) \) must also be a state (up to a normalization factor) of the composite system \( RQ_2 \).
Theorem

A map $\mathcal{E}$ from the set of density operators of the input space $Q_1$ to the set of positive operators of the output space $Q_2$ satisfies the above three axiomatic properties if and only if

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$$

for a collection of linear transformations $E_k$ from the input space $Q_1$ to the output space $Q_2$ and

$$\sum_k E_k^\dagger E_k \leq I.$$ 

Furthermore, $\mathcal{E}$ is trace-preserving, i.e., $\mathcal{E}(\rho)$ is a density operator for any density operator $\rho$ if and only if $\sum_k E_k^\dagger E_k = I$. 
Proof

From (1), it is clear that $\mathcal{E}(\rho)$ is a positive operator on $Q_2$ for any density operator $\rho$ on $Q_1$ and $\mathcal{E}$ is a linear map.

- Axiom I: for a density operator $\rho = \sum_j \lambda_j |j\rangle \langle j|$ on $Q_1$, we have

\[
0 \leq \text{tr}(\mathcal{E}(\rho)) = \sum_k \text{tr}(E_k \rho E_k^\dagger) = \sum_k \sum_j \lambda_j \text{tr}(E_k |j\rangle \langle j| E_k^\dagger)
\]

\[
= \sum_k \sum_j \lambda_j \langle j| E_k^\dagger E_k |j\rangle = \sum_j \lambda_j \langle j| (\sum_k E_k^\dagger E_k) |j\rangle
\]

\[
\leq \sum_j \lambda_j \langle j|j\rangle = 1
\]

since $\sum_k E_k^\dagger E_k \leq I$

- Axiom II: it is clear from the linearity of $\mathcal{E}$
• Axiom III : for any positive operator $B = \sum_i \alpha_i T^R_i \otimes T^{Q_1}_i$ on the composite system $RQ_1$, we define

$$(\mathcal{I} \otimes \mathcal{E})(B)$$

$$\triangleq \sum_i \alpha_i \mathcal{I}(T^R_i) \otimes (\sum_k E_k T^{Q_1}_i E_k^\dagger)$$

$$= \sum_k \sum_i \alpha_i (I_R T^R_i I_R) \otimes (E_k T^{Q_1}_i E_k^\dagger)$$

$$= \sum_k \sum_i \alpha_i (I_R \otimes E_k)(T^R_i \otimes T^{Q_1}_i)(I_R \otimes E_k^\dagger)$$

$$= \sum_k (I_R \otimes E_k)B(I_R \otimes E_k^\dagger)$$

which is clearly well-defined.
Let $|\psi\rangle$ be a state of the composite system $RQ_2$ and let $|\varphi_k\rangle = (I_R \otimes E_k^\dagger)|\psi\rangle$ for all $k$, we have

$$
\langle \psi | (I \otimes \mathcal{E})(B) | \psi \rangle = \sum_k \langle \psi | (I_R \otimes E_k)B(I_R \otimes E_k^\dagger) | \psi \rangle
$$

$$
= \sum_k \langle \varphi_k | B | \varphi_k \rangle \geq 0
$$

which implies that $(I \otimes \mathcal{E})(B)$ is a positive operator on $RQ_2$. 

Proof

- $R$: an arbitrarily introduced space, with the same dimension as $Q_1$
- $\{|i_R\rangle\}$ and $\{|i_{Q_1}\rangle\}$: orthonormal bases for $R$ and $Q_1$ respectively
- $|u\rangle = \sum_i |i_R\rangle |i_{Q_1}\rangle$: a (maximally entangled) vector in the composite system $RQ_1$
- $\mathcal{I} \otimes \mathcal{E}$: a map from the set of positive operators on $RQ_1$ to the set of positive operators on $RQ_2$ by the complete positivity of $\mathcal{E}$ from Axiom III
- $\sigma = (\mathcal{I} \otimes \mathcal{E})(|u\rangle \langle u|)$: a positive operator on $RQ_2$

$$\sigma = \sum_{ij} |i_R\rangle \langle j_R| \otimes \mathcal{E}(|i_{Q_1}\rangle \langle j_{Q_1}|)$$
– It will be seen that the positive operator $\sigma$ completely specifies the quantum operation $\mathcal{E}$

- $|v\rangle = \sum_i \alpha_i |i_{Q_1}\rangle$: a vector in $Q_1$
- $|\tilde{v}\rangle = \sum_i \alpha_i^* |i_{R}\rangle$: a vector in $R$ corresponding to the vector $|v\rangle$ in $Q_1$
- An identity: a strong connection between $\sigma$ and $\mathcal{E}$

$$
\langle \tilde{v} | \sigma | \tilde{v} \rangle = \langle \tilde{v} | \left( \sum_{ij} |i_{R}\rangle \langle j_{R}| \otimes \mathcal{E}(|i_{Q_1}\rangle \langle j_{Q_1}|) \right) | \tilde{v} \rangle \\
= \sum_{ij} \langle \tilde{v} | i_{R} \rangle \langle j_{R} | \tilde{v} \rangle \mathcal{E}(|i_{Q_1}\rangle \langle j_{Q_1}|) \\
= \sum_{ij} \alpha_i \alpha_j^* \mathcal{E}(|i_{Q_1}\rangle \langle j_{Q_1}|) \\
= \mathcal{E}(|v\rangle \langle v|)$$
• $\sigma = \sum_k \lambda_k \langle s_k | s_k \rangle :$ spectral decomposition of $\sigma$

$$\langle \tilde{v} | \sigma | \tilde{v} \rangle = \sum_k \lambda_k \langle \tilde{v} | s_k \rangle \langle s_k | \tilde{v} \rangle$$

• $E_k :$ a linear transformation from $Q_1$ to $Q_2$ defined as

$$E_k (|v\rangle) \triangleq \sqrt{\lambda_k} \langle \tilde{v} | s_k \rangle$$

for any vector $|v\rangle$ in $Q_1$

Thus for any state $|\psi\rangle$ in $Q_1$, we have

$$\mathcal{E} (|\psi\rangle\langle\psi|) = \langle \tilde{\psi} | \sigma | \tilde{\psi} \rangle = \sum_k E_k |\psi\rangle \langle \psi | E_k^\dagger$$

and then for any density operator $\rho = \sum_i \lambda_i |\psi_i\rangle \langle \psi_i |$ with ensemble \{\lambda_i, |\psi_i\rangle\}, we have
\[ \mathcal{E}(\rho) = \mathcal{E}(\sum_i \lambda_i |\psi_i\rangle\langle\psi_i|) \]

\[ = \sum_i \lambda_i \mathcal{E}(|\psi_i\rangle\langle\psi_i|) \text{ by Axiom II} \]

\[ = \sum_i \lambda_i \sum_k E_k |\psi_i\rangle\langle\psi_i| E_k^\dagger \]

\[ = \sum_k E_k (\sum_i \lambda_i |\psi_i\rangle\langle\psi_i|) E_k^\dagger \]

\[ = \sum_k E_k \rho E_k^\dagger \]
To show that
\[ \sum_k E_k^\dagger E_k \leq I, \]
we need to show that
\[ \langle \psi | (I - \sum_k E_k^\dagger E_k) | \psi \rangle \geq 0 \]
for any state \( \psi \) in \( Q_1 \). But
\[
\langle \psi | (I - \sum_k E_k^\dagger E_k) | \psi \rangle = \langle \psi | \psi \rangle - \langle \psi | \sum_k E_k^\dagger E_k | \psi \rangle
\]
\[
= 1 - \sum_k \langle \psi | E_k^\dagger E_k | \psi \rangle = 1 - \sum_k \text{tr}(E_k | \psi \rangle \langle \psi | E_k^\dagger)
\]
\[
= 1 - \text{tr}(\sum_k E_k | \psi \rangle \langle \psi |) E_k^\dagger = 1 - \text{tr}(\mathcal{E}(|\psi \rangle \langle \psi |)) \geq 0
\]
by Axiom I
Freedom in the Operator-Sum Representation
Unitary Freedom in the Operator-Sum Representation

- \( \{E_1, E_2, \ldots, E_m\} \) : operation elements of a quantum operation \( \mathcal{E} \)
- \( \{F_1, F_2, \ldots, F_n\} \) : operation elements of a quantum operation \( \mathcal{F} \)
- \( m = n \) : by appending zero operators in the shorter list of operation elements

Then \( \mathcal{E} = \mathcal{F} \) if and only if

\[
E_i = \sum_j u_{ij} F_j
\]

where \([u_{ij}]\) is an \( m \times m \) unitary matrix
Suppose that for any density operator \( \rho \), we have

\[
\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger = \sum_k F_k \rho F_k^\dagger = \mathcal{F}(\rho)
\]

- \( R \) : an introduced space, with the same dimension as \( Q_1 \)
- \( \{ |i_R\rangle \} \) and \( \{ |i_{Q_1}\rangle \} \) : orthonormal bases for \( R \) and \( Q_1 \) respectively
- \( |u\rangle = \sum_i |i_R\rangle |i_{Q_1}\rangle \) : a (maximally entangled) vector in the composite system \( RQ_1 \)
- \( \mathcal{I} \otimes \mathcal{E} = \mathcal{I} \otimes \mathcal{F} \) : a map from the set of positive operators on \( RQ_1 \) to the set of positive operators on \( RQ_2 \) by the complete positivity of \( \mathcal{E} \) from the 3rd axiomatic property
- \( \sigma = (\mathcal{I} \otimes \mathcal{E})(|u\rangle \langle u|) = (\mathcal{I} \otimes \mathcal{F})(|u\rangle \langle u|) \)
Now,

$$(\mathcal{I} \otimes \mathcal{E})(\ket{u}\bra{u}) = \sum_{ij} |i_R\rangle\langle j_R| \otimes \mathcal{E}(|i_{Q_1}\rangle\langle j_{Q_1}|)$$

$$= \sum_{ij} |i_R\rangle\langle j_R| \otimes (\sum_k E_k |i_{Q_1}\rangle\langle j_{Q_1}| E^\dagger_k)$$

$$= \sum_k (\sum_i |i_R\rangle(E_k |i_{Q_1}\rangle))(\sum_j |j_R\rangle(E_k |j_{Q_1}\rangle))^{\dagger}$$

$$= \sum_k |e_k\rangle\langle e_k|$$

$$(\mathcal{I} \otimes \mathcal{F})(\ket{u}\bra{u}) = \sum_k |f_k\rangle\langle f_k|$$

where we define

$$|e_k\rangle \triangleq \sum_i |i_R\rangle(E_k |i_{Q_1}\rangle), \quad |f_k\rangle \triangleq \sum_i |i_R\rangle(F_k |i_{Q_1}\rangle).$$
Now we have
\[ \sigma = \sum_k |e_k\rangle\langle e_k| = \sum_k |f_k\rangle\langle f_k| \]
and for any vector \( |v\rangle = \sum_i \alpha_i |i_{Q_1}\rangle \) in \( Q_1 \), we have
\[ E_k |v\rangle = \langle \tilde{v}|e_k\rangle, \quad F_k |v\rangle = \langle \tilde{v}|f_k\rangle, \]
where \( |\tilde{v}\rangle = \sum_i \alpha_i^* |i_R\rangle \) is the vector in \( R \) corresponding to the vector \( |v\rangle \)

- Unitary freedom in the ensemble for density operators: there exists an \( m \times m \) unitary matrix \([u_{kl}]\) such that
\[ |e_k\rangle = \sum_l u_{kl} |f_l\rangle \]
Thus we have \( E_k |v\rangle = \langle \tilde{v}|e_k\rangle = \sum_l u_{kl} \langle \tilde{v}|f_l\rangle = \sum_l u_{kl} F_l |v\rangle \), i.e.,
\[ E_k = \sum_l u_{kl} F_l \]
Two Examples of Quantum Operations
Trace as a Quantum Operation

- $Q$: a quantum systems with state space $H_Q$
- $\{|i\rangle\}$: an orthonormal basis of the state space $H_Q$ of $Q$
- $Q'$: a quantum systems with one-dimensional state space $H_{Q'}$
- $\{|0\rangle\}$: an orthonormal basis of the state space $H_{Q'}$ of $Q'$
- $E_i = |0\rangle \langle i|$: a linear transformation from $H_Q$ to $H_{Q'}$
- A completeness relation: $\sum_i E_i^\dagger E_i = I$
- $\mathcal{E}$: a quantum operation defined as
  
  $$\mathcal{E} (\rho) = \sum_i E_i \rho E_i^\dagger = \sum_i |0\rangle \langle i| \rho \langle i| \langle 0|$$

It is clear that

$$\mathcal{E} (\rho) = \text{tr}(\rho)|0\rangle \langle 0|$$
Partial Trace as a Quantum Operation

- **Q** and **R**: two quantum systems with state spaces \( H_Q \) and \( H_R \)
- \( \{|j\rangle\} \): an orthonormal basis of the state space \( H_R \) of \( R \)
- \( \sum_j |v_j\rangle|j\rangle \): a vector (in a unique representation format) in \( H_Q \otimes H_R \)
- **\( E_i \)**: a linear transformation from \( H_Q \otimes H_R \) to \( H_Q \) defined as
  \[
  E_i \left( \sum_j |v_j\rangle|j\rangle \right) = |v_i\rangle
  \]
- **\( E_i^\dagger \)**: the adjoint of \( E_i \), which is a linear transformation from \( H_Q \) to \( H_Q \otimes H_R \) and can be shown to be
  \[
  E_i^\dagger(|v\rangle) = |v\rangle|i\rangle
  \]
• A completeness relation: \( \sum_i E_i^\dagger E_i = I \)

• \( \mathcal{E} \): a quantum operation can be defined as

\[
\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger
\]

for all density operators \( \rho \) on the composite quantum \( QR \). In fact, \( \mathcal{E} \) is a linear map from \( L(H_Q \otimes H_R, H_Q \otimes H_R) \) to \( L(H_Q, H_Q) \)

• \( T^Q \): a linear operator on \( H_Q \), we have

\[
\mathcal{E}(T^Q \otimes |j\rangle\langle j'|) = T^Q \delta_{jj'} = \text{tr}_R(T^Q \otimes |j\rangle\langle j'|)
\]
**Proof.** Let $|v\rangle \in H_Q$. Then

$$
\mathcal{E}(T^Q \otimes |j\rangle\langle j'|)(|v\rangle) = \sum_i E_i(T^Q \otimes |j\rangle\langle j'|)E_i^\dagger(|v\rangle)
$$

$$
= \sum_i E_i(T^Q \otimes |j\rangle\langle j'|)(|v\rangle|i\rangle) = \sum_i E_i(T^Q|v\rangle \otimes \delta_{ij'}|j\rangle)
$$

$$
= \sum_i \delta_{ij}\delta_{ij'}T^Q|v\rangle = \delta_{jj'}T^Q|v\rangle = \text{tr}_R(T^Q \otimes |j\rangle\langle j'|)(|v\rangle)
$$

For each linear operator $T^{QR} = \sum_{jj'} T^{Q}_{jj'} \otimes |j\rangle\langle j'|$ on the composite system $QR$, we have

$$
\mathcal{E}(T^{QR}) = \sum_i E_iT^{QR}E_i^\dagger = \text{tr}_R(T^{QR})
$$

by linearity of $\mathcal{E}$ and $\text{tr}_R$. 

Geometric Visualization of Quantum Operations on a Qubit
**Bloch Vector Representation of Density Operators of a Qubit**

- \{ |0\rangle, |1\rangle \} : a computational basis of the state space $H$ of a qubit

- $\rho$ : a density operator of the qubit with matrix representation relative to the computational basis

\[
\begin{bmatrix}
\frac{1+r_z}{2} & \frac{r_x - ir_y}{2} \\
\frac{r_x + ir_y}{2} & \frac{1-r_z}{2}
\end{bmatrix}
\]

- $\sigma_x, \sigma_y, \sigma_z$ : Pauli operators

- Bloch vector representation :

\[
\rho = \frac{I + r_x \sigma_x + r_y \sigma_y + r_z \sigma_z}{2} = \frac{I + \vec{r} \cdot \vec{\sigma}}{2}
\]

- $\rho$ is a density operator $\implies ||\vec{r}||^2 = r_x^2 + r_y^2 + r_z^2 \leq 1$

- $\rho = I/2 \iff \vec{r} = \vec{0}$
• \( \text{tr}(\rho^2) = (1 + ||\vec{r}||^2)/2 \)

• \( \rho = |\psi\rangle\langle\psi| \) is a pure state if and only if \( ||\vec{r}|| = 1 \) 

\[
|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\varphi} \sin \frac{\theta}{2} |1\rangle
\]

with the visualizing representation

\[
(\cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta)
\]

on the Bloch sphere in Unit Four, which is equal to the Bloch vector \( \vec{r} \) in above, i.e.,

\[
[\rho] = \begin{pmatrix}
\frac{1+\cos \theta}{2} & \frac{\cos \varphi \sin \theta - i \sin \varphi \sin \theta}{2} \\
\frac{\cos \varphi \sin \theta + i \sin \varphi \sin \theta}{2} & \frac{1-\cos \theta}{2}
\end{pmatrix}
\]
The Vector Space $L(H)$ with Trace Inner Product

- $L(H)$: the vector space of all linear operators on $H$
- $(T, S) \triangleq \text{tr}(T^\dagger S)$: the Hilbert-Schmidt or trace inner product of $T$ and $S$ in $L(H)$
  - $(T, T) = \text{tr}(T^\dagger T) \geq 0$: $T^\dagger T$ is a positive operator
    * $(T, T) = \text{tr}(T^\dagger T) = 0 \iff$ all singular values of $T$ are zeros $\iff \text{rank}(T) = 0 \iff T = 0$
  - $(T, S) = \text{tr}(T^\dagger S) = \text{tr}((S^\dagger T)^\dagger) = \text{tr}(S^\dagger T) = (S, T)$: Hermitian symmetry
  - $(T, \alpha_1 S_1 + \alpha_2 S_2) = \text{tr}(T^\dagger (\alpha_1 S_1 + \alpha_2 S_2)) = \alpha_1 \text{tr}(T, S_1) + \alpha_2 \text{tr}(T, S_2) = \alpha (T, S_1) + \alpha_2 (T, S_2)$: linearity
- $L(H)$: a 4-dimensional complex inner product space with trace inner product
\[ \{I/\sqrt{2}, \sigma_x/\sqrt{2}, \sigma_y/\sqrt{2}, \sigma_z/\sqrt{2}\} : \text{an orthonormal basis of } L(H) \]
Trace-Preserving Quantum Operations as Affine Maps
from Bloch Sphere to Itself

- $\sigma_1\sigma_2 = i\sigma_3$, $\sigma_2\sigma_3 = i\sigma_1$, $\sigma_3\sigma_1 = i\sigma_2$

- $\mathcal{E} = \sum_k E_k \rho E_k^\dagger$: a trace-preserving quantum operation on a qubit (where input system = output system) with

$$E_k = \alpha_{k0} I + \sum_{i=1}^{3} \alpha_{ki} \sigma_i$$

where $\sum_k E_k^\dagger E_k = I$ which implies that

$$\sum_{k} \sum_{i=0}^{3} |\alpha_{ki}|^2 = 1, \sum_{k} \Re\{\alpha_{k0}\alpha_{k1}^*\} + \Im\{\alpha_{k2}\alpha_{k3}^*\} = 0,$$

$$\sum_{k} \Re\{\alpha_{k0}\alpha_{k2}^*\} + \Im\{\alpha_{k3}\alpha_{k1}^*\} = 0, \sum_{k} \Re\{\alpha_{k0}\alpha_{k3}^*\} + \Im\{\alpha_{k1}\alpha_{k2}^*\} = 0$$
\[ \rho = \frac{I + \vec{r} \cdot \vec{\sigma}}{2} \quad \text{and} \quad \mathcal{E} (\rho) = \frac{I + \vec{r}' \cdot \vec{\sigma}}{2} \]

\[ \vec{r} \xrightarrow{\mathcal{E}} \vec{r}' = M \vec{r} + \vec{c} \]

where

\[ M_{ij} = \sum_k \left( 2 \Re \{ \alpha_{ki} \alpha_{kj}^* \} + 2 \Im \{ \alpha_{k0}^* \sum_{p=1}^{3} \epsilon_{ijp} \alpha_{kp} \} + \left( |\alpha_{k0}|^2 - \sum_{p=1}^{3} |\alpha_{kp}|^2 \right) \delta_{ij} \right) \]

and

\[ c_j = 2i \sum_k \sum_{mn} \epsilon_{mnj} \alpha_{kn} \alpha_{km}^* \]
Examples of Single Qubit Noisy Quantum Channels
The bit flip channel flips the state of a qubit from $|0\rangle$ to $|1\rangle$ (and vice versa) with probability $p$

- $\mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$
  - $\rho$: the density operator of single qubit
  - $E_0 = \sqrt{1-p}I = \sqrt{1-p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  - $E_1 = \sqrt{p}\sigma_x = \sqrt{p} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

- Trace-preserving: $E_0^\dagger E_0 + E_1^\dagger E_1 = I$
Visualization of the Bit Flip Channel

- \( \rho = (I + \vec{r} \cdot \vec{\sigma})/2 \) and \( \mathcal{E}(\rho) = (I + \vec{r}' \cdot \vec{\sigma})/2 \)

\[
\vec{r}' = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 - 2p & 0 \\
0 & 0 & 1 - 2p
\end{bmatrix} \vec{r}
\]
Phase Flip Channel

- The phase flip channel flips the phase of the state $|1\rangle$ of a qubit with probability $p$

- $\mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$
  
  - $\rho$: the density operator of single qubit
  
  - $E_0 = \sqrt{1-p} I = \sqrt{1-p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  
  - $E_1 = \sqrt{p} \sigma_z = \sqrt{p} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

- Trace-preserving: $E_0^\dagger E_0 + E_1^\dagger E_1 = I$
Visualization of the Phase Flip Channel

- $\rho = (I + \vec{r} \cdot \vec{\sigma})/2$ and $\mathcal{E}(\rho) = (I + \vec{r}' \cdot \vec{\sigma})/2$

$$\vec{r}' = \begin{bmatrix} 1 - 2p & 0 & 0 \\ 0 & 1 - 2p & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{r}$$

- $p = 1/2 : \vec{r}' = (0, 0, r_z)$ and the collection of operator elements

$$E_0 = \frac{1}{\sqrt{2}} I, \quad E_1 = \frac{1}{\sqrt{2}} \sigma_z,$$

is unitarily equivalent to the collection of operator elements

$$P_0 = \langle 0 | 0 \rangle = \frac{1}{\sqrt{2}} E_0 + \frac{1}{\sqrt{2}} E_1, \quad P_1 = \langle 1 | 1 \rangle = \frac{1}{\sqrt{2}} E_0 - \frac{1}{\sqrt{2}} E_1$$

which is the projective measurement on the basis $\{|0\rangle, |1\rangle\}$
The bit-phase flip channel flips the state of the qubit from $|0\rangle$ to $|1\rangle$ (and vice versa) and then flips the phase of the state $|1\rangle$ of the qubit with probability $p$.

- $\sigma_y = -i\sigma_z\sigma_x$
- $\mathcal{E}(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger$
  - $\rho$ : the density operator of single qubit
  - $E_0 = \sqrt{1-p}I = \sqrt{1-p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  - $E_1 = \sqrt{p}\sigma_y = \sqrt{p} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
- Trace-preserving : $E_0^\dagger E_0 + E_1^\dagger E_1 = I$
Visualization of the Bit-Phase Flip Channel

- \( \rho = (I + \vec{r}' \cdot \vec{\sigma})/2 \) and \( \mathcal{E}(\rho) = (I + \vec{r}' \cdot \vec{\sigma})/2 \)

\[
\vec{r}' = \begin{bmatrix}
1 - 2p & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 - 2p \\
\end{bmatrix} \vec{r}
\]
The Depolarizing Channel

- With probability $p$, the single qubit is depolarized, i.e., its state is replaced by the completely mixed state $I/2$

- $\mathcal{E}(\rho) = p(I/2) + (1 - p)\rho$

- A circuit implementation of the depolarizing channel

\[
\begin{align*}
\rho & \quad \text{X} \\
I/2 & \\
(1 - p)|0\rangle\langle 0| + p|1\rangle\langle 1| & \quad \text{X}
\end{align*}
\]
Visualization of the Depolarizing Channel

- $\rho = (I + \vec{r} \cdot \vec{\sigma})/2$ and $\mathcal{E}(\rho) = (I + \vec{r}' \cdot \vec{\sigma})/2$

\[
\vec{r}' = \begin{bmatrix}
1 - p & 0 & 0 \\
0 & 1 - p & 0 \\
0 & 0 & 1 - p
\end{bmatrix} \vec{r}
\]
The Operator-Sum Representation of Depolarizing Channels

- An identity:
  \[
  \frac{I}{2} = \rho + \sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z
  \]

- \(\mathcal{E}(\rho) = (1 - (3/4)p)\rho + (p/4)(\sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z)\)
  - \(E_0 = \sqrt{1 - 3p/4}I\)
  - \(E_1 = \sqrt{p} \sigma_x / 2\)
  - \(E_2 = \sqrt{p} \sigma_y / 2\)
  - \(E_3 = \sqrt{p} \sigma_z / 2\)

- Trace-preserving: \(\sum_{k=0}^{3} E_k^\dagger E_k = I\)

- Another expression:
  \[\mathcal{E}(\rho) = (1 - p')\rho + (p'/3)(\sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z)\]
The Depolarizing Channel for $n$-Qubit Systems

- With probability $p$, the $n$-qubit system is depolarized, i.e., its state is replaced by the completely mixed state $I/(2^n)$
- $\mathcal{E}(\rho) = pI/(2^n) + (1 - p)\rho$
Amplitude Damping Channel

- $|0\rangle$ : the ground state without a quantum of energy
- $|1\rangle$ : the state with a quantum of energy
- $\mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$
  - $\rho$ : the density operator of a single qubit
  - $E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}$ : a quantum of energy was not lost to the environment such that the qubit must be more probably in the state $|0\rangle$ than in the state $|1\rangle$
  - $E_1 = \sqrt{p} \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}$ : a quantum of energy was lost to the environment with probability $\gamma$ such that the qubit must be in the state $|1\rangle$
• Trace-preserving: $E_0^\dagger E_0 + E_1^\dagger E_1 = I$
Visualization of the Amplitude Damping Channel

- $\rho = (I + \vec{r} \cdot \vec{\sigma})/2$ and $\mathcal{E}(\rho) = (I + \vec{r}' \cdot \vec{\sigma})/2$

$$
\vec{r}' = \begin{bmatrix}
\sqrt{1-\gamma} & 0 & 0 \\
0 & \sqrt{1-\gamma} & 0 \\
0 & 0 & 1-\gamma
\end{bmatrix}
\vec{r} + \begin{bmatrix}
0 \\
0 \\
\gamma
\end{bmatrix}
$$

- When describing $\gamma = 1 - e^{-t/T_1}$ as a time-varying function, the effect of amplitude damping is as a flow on the Bloch sphere, which moves every points in the unit sphere to the north pole of the unit sphere on which the state $|0\rangle$ resides.
The phase damping channel is the same as the phase flip channel

\[ \mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger \]

- \( \rho \): the density operator of single qubit

- \( E_0 = \sqrt{1-p} I = \sqrt{1-p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \)

- \( E_1 = \sqrt{p} \sigma_z = \sqrt{p} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \)

- Trace-preserving: \( E_0^\dagger E_0 + E_1^\dagger E_1 = I \)
Visualization of the Phase Damping Channel

- $\rho = (I + \vec{r} \cdot \vec{\sigma})/2$ and $\mathcal{E}(\rho) = (I + \vec{r}' \cdot \vec{\sigma})/2$

\[
\vec{r}' = \begin{bmatrix}
1 - 2p & 0 & 0 \\
0 & 1 - 2p & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \vec{r}
\]

- When describing $p = (1 - e^{-t/T_1})/2$ as a time-varying function, the effect of phase damping is as a flow on the Bloch sphere, which perpendicularly moves every points in the unit sphere to the $z$-axis of the unit sphere.