1. A function $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by the formula given for $T(x, y, z)$, where $(x, y, z)$ is an arbitrary point in $\mathbb{R}^3$. In the two cases below, determine whether $T$ is one-to-one on $\mathbb{R}^3$.

(a) $T(x, y, z) = (x + y, y + z, x + z)$.
(b) $T(x, y, z) = (x, y, x + y)$.

2. Let $V$ be a linear space. Let $S$ and $T$ be in $\mathcal{L}(V, V)$ and assume that $ST - TS = I$.

(1) Prove that $ST^n - T^n S = nT^{n-1}$ for all $n \geq 1$.

(2) If we let $V$ be the linear space of all real polynomials $p(x)$, and $T$ be the linear transformation that maps $p(x)$ onto $xp(x)$. Let $D$ denote the differentiation operator. Prove that $DT - TD = I$ and that $DT^n - T^n D = nT^{n-1}$ for $n \geq 2$.

3. Let $V$ be the linear space of all real polynomials $p(x)$. Let $R, S, T$ be the functions that map an arbitrary polynomial $p(x) = c_0 + c_1 x + \cdots + c_n x^n$ to the polynomials $r(x)$, $s(x)$, and $t(x)$, respectively, where

$$r(x) = p(0), \quad s(x) = \sum_{k=1}^{n} c_k x^{k-1}, \quad t(x) = \sum_{k=0}^{n} c_k x^{k+1}.$$ 

(a) Prove that $R, S, T$ are linear operators on $V$ and $T$ is one-to-one.
(b) If $n \geq 1$, express $(TS)^n$ and $S^n T^n$ in terms of $I$ and $R$.

**Teaching Assistants:** EECS 608, ext. 4032

- Meng-Hua Chang, mhchang@abel.ee.nthu.edu.tw
- Chao-Chung Chang, ccchang@abel.ee.nthu.edu.tw
- Chen-Wei Hsu, cwhsu@abel.ee.nthu.edu.tw
- Wen-Yao Chen, wychen@abel.ee.nthu.edu.tw