Chao-Chung Chang, Meng-Hua Chang, Chen-Wei Hsu, Wen-Yao Chen

6. \( T(x, y) = (e^x, e^y) \). For all \((x', y')\) in \( \mathbb{R}^2 \), then

\[
T((x, y) + (x', y')) = T(x + x', y + y') = (e^{x+x'}, e^{y+y'})
\]

\[
T(x, y) + T(x', y') = (e^x, e^y) + (e^{x'}, e^{y'}) = (e^{x + x'}, e^{y + y'})
\]

Since \((e^{x+x'}, e^{y+y'}) \neq (e^{x+x'}, e^{y+y'})\) in general, \( T((x, y) + (x', y')) \neq T(x, y) + T(x', y') \)

Thus \( T \) is nonlinear.

10. \( T(x, y) = (2x - y, x + y) \). For all \((x', y')\) in \( \mathbb{R}^2 \) and all scalars \( a \) and \( b \), then

\[
T(a(x, y) + b(x', y')) = T(ax + bx', ay + by')
\]

\[
= (2(ax + bx') - (ay + by'), (ax + bx') + (ay + by'))
\]

\[
= (a(2x - y) + b(2x' - y'), a(x + y) + b(x' + y'))
\]

\[
= a(2x - y, x + y) + b(2x' - y', x' + y')
\]

and

\[
aT(x, y) + bT(x', y') = a(2x - y, x + y) + b(2x' - y', x' + y')
\]

Since \( T(a(x, y) + b(x', y')) = aT(x, y) + bT(x', y') \), \( T \) is linear.

To find the null space, it’s equivalent to finding \( T(x, y) = O \).

\( \Rightarrow T(x, y) = (2x - y, x + y) = O \) \( \Rightarrow x = 0 \) and \( y = 0 \).

\( \Rightarrow N(T) = \{O\} \) and \( R(T) = \{(x, y) : (x, y) \in \mathbb{R}^2\} \)

\( \Rightarrow \) Its nullity = 0, and rank = 2.

12. Let \( ax + by = 0 \), \( a, b \in \mathbb{R} \) and \( a^2 + b^2 \neq 0 \) be a fixed line in \( \mathbb{R}^2 \) through the origin. We know \( u = \left(\frac{-b}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}\right) \) is a point on line \( ax + by = 0 \) and \( v = \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right) \) is a unit vector orthogonal to \( (\frac{-b}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}) \). Thus the set \( B = \{(\frac{-b}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}), (\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}})\} \) is an orthonormal basis for \( \mathbb{R}^2 \). To find the reflection \( T(x, y) \) of \((x, y)\) with respect to the line \( ax + by = 0 \), we consider \((x, y) = (\frac{-bx+ay}{a^2+b^2})u + (\frac{ax+by}{a^2+b^2})v \). Thus \( T(x, y) = (\frac{-bx+ay}{a^2+b^2})u - (\frac{ax+by}{a^2+b^2})v = (\frac{-(b^2-a^2)x-2ab}{a^2+b^2}, \frac{a^2-b^2)y-2abx}{a^2+b^2}) \).

Hence the transformation \( T \) is linear. Since for null space \( T(x, y) = O \), \( (\frac{-(b^2-a^2)x-2ab}{a^2+b^2}, \frac{a^2-b^2)y-2abx}{a^2+b^2}) = O, (0, 0) \) is the only solution of \((x, y)\). \( N(T) = O \) and nullity = 0. Range is all \( \mathbb{R}^2 \) and rank = 2.

15. We can take a counterexample to prove \( T \) is not linear. Let \((1, 0)\) and \((1, \pi)\) be points in \( \mathbb{R}^2 \). For \( T(r, \theta) = (r, 2\theta) \), we have \( T((1, 0) + (1, \frac{\pi}{2})) = T(\sqrt{2}, \frac{\pi}{2}) = (\sqrt{2}, \pi) \), and \( T(1, 0) + T(1, \frac{\pi}{2}) = (1, 0) + (1, \pi) = 0 \). Since \( T((1, 0) + (1, \frac{\pi}{2})) \neq T(1, 0) + T(1, \frac{\pi}{2}) \), \( T \) is not linear.
20. \( T : \mathbb{R}^3 \to \mathbb{R}^3, T(x, y, z) = (x + 1, y + 1, z - 1) \).

\( T \) is not linear, since \( T(2(0, 0, 0)) = T(0, 0, 0) = (0 + 1, 0 + 1, 0 - 1) = (1, 1, -1) \) but \( 2T((0, 0, 0)) = 2T(0, 0, 0) = 2(0 + 1, 0 + 1, 0 - 1) = (2, 2, -2) \).

24. Assume \( \dim N(T) = k < \infty \) and \( \dim T(V) = r < \infty \). Let \( \{e_1, e_2, \ldots, e_k\} \) be a basis for \( N(T) \). Since \( V \) is infinite-dimensional, there exist infinitely many \( e_{k+1}, e_{k+2}, \ldots, e_{k+n}, \ldots \) in \( V \) such that \( e_1, e_2, \ldots, e_k, e_{k+1}, \ldots, e_{k+n}, \ldots \) are linearly independent. We choose \( n > \dim T(V) = r \), then the \( n \) vectors \( T(e_{k+1}), T(e_{k+2}), \ldots, T(e_{k+n}) \) are linearly dependent. Thus, there are \( a_1, a_2, \ldots, a_n \) not all zeros, such that
\[
0 = a_1T(e_{k+1}) + a_2T(e_{k+2}) + \cdots + a_nT(e_{k+n})
\]
\( a_1e_{k+1} + a_2e_{k+2} + \cdots + a_ne_{k+n} \) is in \( N(T) \). That is \( a_1e_{k+1} + a_2e_{k+2} + \cdots + a_ne_{k+n} \) is a linear combination of \( e_1, e_2, \ldots, e_k \), a contradiction to that \( e_1, e_2, \ldots, e_k, \ldots, e_{k+n} \) are linearly independent. Hence at least one of \( T(V) \) or \( N(T) \) is infinite-dimensional.

25. Let \( p(x) = \sum_{i=0}^{n} p_i x^i, r(x) = \sum_{i=0}^{n} r_i x^i \) be two real polynomials of degree \( \leq n \), and \( a, b \in \mathbb{R} \). Since
\[
T(ap(x) + br(x)) = T(a \sum_{i=0}^{n} p_i x^i + b \sum_{i=0}^{n} r_i x^i)
\]
\[
= T \left( \sum_{i=0}^{n} (ap_i + br_i) x^i \right)
\]
\[
= \sum_{i=0}^{n} (ap_i + br_i) x^i
\]
\[
= a \sum_{i=0}^{n} p_i (x + 1)^i + b \sum_{i=0}^{n} r_i (x + 1)^i
\]
\[
= aT(p) + bT(r),
\]

\( T \) is a linear transformation.

If \( T(p(x)) = 0 \), then \( \sum_{i=0}^{n} p_i (x + 1)^i = 0 \). We know that \( \{1, 1 + x, \ldots, (1 + x)^n\} \) is a basis for \( V \) (Section 3.6), hence \( p_i = 0 \), for \( 1 \leq i \leq n \). Thus \( N(T) = \{0\} \) and \( \dim N(T) = 0 \).

The dimension of \( V \) is \( n + 1 \) which is finite. Thus by rank-nullity theorem, \( \dim N(T) + \dim T(V) = \dim V \), we have \( \dim T(V) = \dim V = n + 1 \). But \( T(V) \subset V \), we have \( T(V) = V \).

27. We find
\[
T(ax + by) = (ax + by)'' + A(ax + by)' + B(ax + by)
\]
\[
= ax'' + by'' + Aax' + Aby' + Bax + Bby
\]
\[
= a(x'' + Ax' + Bx) + b(y'' + A'y' + By)
\]
\[
= aT(x) + bT(y),
\]
so $T$ is linear. To derive its null space, we need $T(y) = y'' + Ay' + By = 0$. Let $y_h = e^{\lambda x}$, then $\lambda^2 e^{\lambda x} + A \lambda e^{\lambda x} + B e^{\lambda x} = 0 \Rightarrow e^{\lambda x}(\lambda^2 + A\lambda + B) = 0. \Rightarrow \lambda^2 + A\lambda + B = 0$. Then the solution is $\lambda = \frac{-A \pm \sqrt{A^2 - 4B}}{2}$.

(a) If $A^2 - 4B = 0$, $T(e^{-Ax/2}) = 0$ and $T(xe^{-Ax/2}) = 0$. So the null space $N(T) = L\{e^{-Ax/2}, xe^{-Ax/2}\}$ with nullity 2, and the range $T(V) = \{y''(t) + Ay'(t) + By(t) : y(t) \in V\}$ with rank infinity since $\dim V = \infty$ and by Exercise 24.

(b) If $A^2 - 4B > 0$, $T(e^{-\frac{A+\sqrt{A^2-4B}}{2}x}) = 0$ and $T(e^{-\frac{A-\sqrt{A^2-4B}}{2}x}) = 0$. So the null space $N(T) = L\{e^{-\frac{A+\sqrt{A^2-4B}}{2}x}, e^{-\frac{A-\sqrt{A^2-4B}}{2}x}\}$ with nullity 2, and the range $T(V) = \{y''(t) + Ay'(t) + By(t) : y(t) \in V\}$ with rank infinity since $\dim V = \infty$ and by Exercise 24.

(c) If $A^2 - 4B < 0, \lambda = \frac{-A \pm j\sqrt{4B - A^2}}{2}$. So $T(e^{-\frac{A+j\sqrt{4B - A^2}}{2}x}) = 0$ and $T(e^{-\frac{A-j\sqrt{4B - A^2}}{2}x}) = 0$. This means if $y = c_1 e^{-\frac{A+j\sqrt{4B - A^2}}{2}x} + c_2 e^{-\frac{A-j\sqrt{4B - A^2}}{2}x}$ for arbitrary $c_1$ and $c_2$, then $T(y) = 0$. But in this example, $y$ must be real, so we take $c_1$ as $a + \frac{b}{j}$ and $c_2$ as $a - \frac{b}{j}$ where $a, b$ are two arbitrary real numbers. Then

$$y = \left( a + \frac{b}{j} \right) e^{-\frac{A+j\sqrt{4B - A^2}}{2}x} + \left( a - \frac{b}{j} \right) e^{-\frac{A-j\sqrt{4B - A^2}}{2}x}$$

$$= ae^{-A/2}(e^{\frac{j\sqrt{4B - A^2}}{2}} + e^{-\frac{j\sqrt{4B - A^2}}{2}}) + be^{-A/2}(e^{\frac{j\sqrt{4B - A^2}}{2}} - e^{-\frac{j\sqrt{4B - A^2}}{2}})/j$$

$$= 2ae^{-A/2} \cos \frac{\sqrt{4B - A^2}}{2} + 2be^{-A/2} \sin \frac{\sqrt{4B - A^2}}{2}$$

Hence $N(T) = L\{\cos \frac{\sqrt{4B - A^2}}{2}, \sin \frac{\sqrt{4B - A^2}}{2}\}$ with nullity 2 and the range $T(V) = \{y''(t) + Ay'(t) + By(t) : y(t) \in V\}$ with rank infinity since $\dim V = \infty$ and by Exercise 24.

28. Since

$$T(\alpha f_1 + \beta f_2) = \int_a^b (\alpha f_1(t) + \beta f_2(t)) \sin(x - t)dt$$

$$= \alpha \int_a^b f_1(t) \sin(x - t)dt + \beta \int_a^b f_2(t) \sin(x - t)dt$$

$$= \alpha T(f_1)(x) + \beta T(f_2)(x),$$

$T$ is linear. Note that

$$T(f) = \int_a^b f(t) \sin(x - t)dt$$

$$= \sin x \int_a^b f(t) \cos tdt - \cos x \int_a^b f(t) \sin tdt$$

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Thus \( T(f) \in L(\sin x, \cos x) \). Consider the equation \( \int_a^b \cos (t + k) \sin t dt = 0 \). We have

\[
\int_a^b (\cos t \cos k - \sin t \sin k) \sin t dt = 0
\]

\[\Rightarrow \cos k \int_a^b \cos t \sin t dt - \sin k \int_a^b \sin^2 t dt = 0\]

\[\Rightarrow \cos k \int_a^b \frac{1}{2} \sin 2tdt - \sin k \int_a^b \frac{1}{2} - \cos 2t dt = 0\]

\[\Rightarrow \frac{1}{2} \cos k \frac{\cos 2t}{-2} \bigg|_a^b - \frac{1}{2} \sin k(t - \frac{\sin 2t}{2}) \bigg|_a^b = 0\]

\[\Rightarrow -\frac{1}{4} \cos k(\cos 2b - \cos 2a) - \frac{1}{2} \sin k(b - a - \frac{\sin 2b - \sin 2a}{2}) = 0\]

\[\Rightarrow -\cos k(\cos 2b - \cos 2a) = 2 \sin k(b - a - \frac{\sin 2b - \sin 2a}{2})\]

\[\Rightarrow \tan k = \frac{\sin k}{\cos k} = \frac{\cos 2a - \cos 2b}{2(b - a) - (\sin 2b - \sin 2a)} \text{ if } 2(b - a) \neq (\sin 2b - \sin 2a)\]

We have to examine when \( 2b - \sin 2b = 2a - \sin 2a \). Let \( f(b) = 2b - \sin 2b \), we have \( f'(b) = 2 - 2\cos 2b \geq 0 \) with equality iff \( b = n\pi \) where \( n \) is an integer. Thus, \( f(b) \) is a nondecreasing function and \( f'(b) = 0 \) when \( b = n\pi \) where \( n \) is an integer. Therefore, \( 2b - \sin 2b = 2a - \sin 2a \) only when \( a = b \).

Next, we consider the equation \( \int_a^b \cos (t + k) \cos t dt = 0 \). Similarly, we have \( \tan k = \frac{2(b - a) + \sin 2b - \sin 2a}{\cos 2a - \cos 2b} \) if \( \cos 2b \neq \cos 2a \). Note that \( \cos 2b = \cos 2a \) when \( b = \pm a + n\pi \) where \( n \) is an integer. We next show that we cannot have \( \int_a^b \cos(t + k) \cos t dt = 0 \) and \( \int_a^b \cos(t + k) \cos t dt = 0 \) simultaneously for any \( k \) when \( b \neq \pm a + n\pi \) where \( n \) is an integer. Otherwise, we could have

\[
\Rightarrow \frac{\cos 2a - \cos 2b}{2(b - a) - (\sin 2b - \sin 2a)} = \frac{2(b - a) + \sin 2b - \sin 2a}{\cos 2a - \cos 2b}
\]

\[\Rightarrow (\cos 2a - \cos 2b)^2 = 4(b - a)^2 - (\sin 2b - \sin 2a)^2\]

\[\Rightarrow \cos^2 2a - 2 \cos 2a \cos 2b + \cos^2 2b = 4(b - a)^2 - \sin^2 2b + 2 \sin 2b \sin 2a - \sin^2 2a\]

\[\Rightarrow 4(b - a)^2 = -2 \cos(2a - 2b)\]

\[\Rightarrow 2(b - a)^2 = -\cos(2(b - a))\]

Let \( x = b - a \), then we have \( 2x^2 + \cos 2x = 0 \). Define \( f(x) = 2x^2 + \cos 2x \). Then \( f'(x) = 4x - 2 \sin 2x \) and \( f''(x) = 4(1 - \cos 2x) \). Note that \( f''(x) \geq 0 \) for all \( x \) and therefore, \( f(x) \) is a convex function. So the local minimum of \( f(x) \) is the global minimum of it. Since \( f'(x) = 0 \Leftrightarrow x = 0 \), the global minimum happens when \( x = 0 \). But \( f(0) = 1 \), so \( 2(b - a)^2 \neq -\cos(2(b - a)) \), a contradiction, when \( b \neq \pm a + n\pi \) where \( n \) is an integer.

Let us consider two different cases:

(a) \( b \neq \pm a + n\pi \) where \( n \) is an integer.
Let \( k_1 \) and \( k_2 \) satisfy the following equations

\[
\int_a^b \cos(t + k_1) \sin t \, dt = 0 \quad \text{and} \quad \int_a^b \cos(t + k_2) \cos t \, dt = 0
\]

respectively. Then by the previous result, we have \( \int_a^b \cos(t + k_1) \cos t \, dt = C_1 \neq 0 \) and \( \int_a^b \cos(t + k_2) \sin t \, dt = C_2 \neq 0 \) respectively. Thus, \( \int_a^b \cos(t + k_1) \cos t \, dt = C_1 \neq 0 \) and \( \int_a^b \cos(t + k_2) \sin t \, dt = C_2 \neq 0 \) respectively. Hence \( L(\cos x, \sin x) \subseteq T(V) \). We conclude that \( T(V) = L(\cos x, \sin x) \) and \( \{\cos x, \sin x\} \) is a basis of \( T(V) \) since it is a linearly independent set. Therefore, the rank=2. The null space \( N(T) = \{f \in V | \int_a^b f(x) \cos x \, dx = \int_a^b f(x) \sin x \, dx = 0\} \).

(b) \( b = \pm a + n\pi \) but \( b \neq a \) where \( n \) is an integer. In this case, we have

\[
\int_a^b \cos t \cos t \, dt = \int_a^b \cos^2 t \, dt
\]

\[
= \int_a^b \left( \frac{1 + \cos 2t}{2} \right) \, dt
\]

\[
= \left( \frac{t}{2} + \frac{\sin 2t}{4} \right) \bigg|_a^b
\]

\[
= \frac{b - a}{2} + \frac{\sin 2b - \sin 2a}{4}
\]

\[
= \frac{1}{4}(2b - 2a + \sin 2b - \sin 2a)
\]

\[
= C_1 \neq 0 \quad \text{(The reason is similar to that of } 2b - 2a - \sin 2b + \sin 2a) \]

and

\[
\int_a^b \cos t \sin t \, dt = \frac{1}{2} \int_a^b \sin 2t \, dt
\]

\[
= \frac{1}{4} \cos 2t \bigg|_a^b
\]

\[
= -\frac{1}{4}(\cos 2b - \cos 2a)
\]

\[
= 0.
\]

So

\[
T\left( \frac{\cos t}{C_1} \right) = \sin x.
\]

In a similar way, we have

\[
\int_a^b \sin t \cos t \, dt = 0
\]

and

\[
\int_a^b \sin t \sin t \, dt = \frac{1}{4}(2b - 2a - \sin 2b + \sin 2a) = C_2 \neq 0.
\]

Hence

\[
T\left( \frac{\sin t}{C_2} \right) = \cos x.
\]
Therefore, \( L(\cos x, \sin x) \subseteq T(V) \). We conclude that \( T(V) = L(\cos x, \sin x) \) and \( \{\cos x, \sin x\} \) is a basis of \( T(f) \) since it is a linearly independent set. Therefore, the rank=2. Since \( L\{\cos 2x, \sin 2x, \cos 3x, \sin 3x, \ldots\} \subseteq N(T) \), the nullity is infinity.

30. (a) To prove \( S \) is a subspace of \( V \), we only need to check closure axioms because \( S \) is a subset of \( V \).

i. If \( f_1 \) and \( f_2 \) are two elements of \( S \), then
\[
\int_{-\pi}^{\pi} (f_1(t) + f_2(t)) dt = \int_{-\pi}^{\pi} f_1(t) dt + \int_{-\pi}^{\pi} f_2(t) dt = 0 + 0 = 0.
\]
\[
\int_{-\pi}^{\pi} (f_1(t) + f_2(t)) \cos nt dt = \int_{-\pi}^{\pi} f_1(t) \cos nt dt + \int_{-\pi}^{\pi} f_2(t) \cos nt dt = 0 + 0 = 0.
\]
\[
\int_{-\pi}^{\pi} (f_1(t) + f_2(t)) \sin nt dt = \int_{-\pi}^{\pi} f_1(t) \sin nt dt + \int_{-\pi}^{\pi} f_2(t) \sin nt dt = 0 + 0 = 0.
\]

ii. If \( f \) are a elements of \( S \), then
\[
\int_{-\pi}^{\pi} c f(t) dt = c \int_{-\pi}^{\pi} f(t) dt = c \cdot 0 = 0.
\]
\[
\int_{-\pi}^{\pi} c f(t) \cos nt dt = c \int_{-\pi}^{\pi} f(t) \cos nt dt = c \cdot 0 = 0.
\]
\[
\int_{-\pi}^{\pi} c f(t) \sin nt dt = c \int_{-\pi}^{\pi} f(t) \sin nt dt = c \cdot 0 = 0.
\]

Thus, \( S \) is a subspace of \( V \).

(b) i. If \( f(x) = \cos nx \) where \( n = 2, 3, \ldots \), then
\[
\int_{-\pi}^{\pi} f(t) dt = \int_{-\pi}^{\pi} \cos nt dt = \sin nt |_{-\pi}^{\pi} = \sin n\pi - \sin (-n\pi) = 0 + 0 = 0.
\]
\[
\int_{-\pi}^{\pi} f(t) \cos nt dt = \int_{-\pi}^{\pi} \cos nt \cos nt dt = \frac{1}{2} (\int_{-\pi}^{\pi} \cos (n + 1) t dt + \int_{-\pi}^{\pi} \cos (n - 1) t dt) = \frac{1}{2} (0 + 0) = 0.
\]
\[
\int_{-\pi}^{\pi} f(t) \sin nt dt = \int_{-\pi}^{\pi} \cos nt \sin nt dt = \frac{1}{2} (\int_{-\pi}^{\pi} \sin (n + 1) t dt - \int_{-\pi}^{\pi} \sin (n - 1) t dt) = \frac{1}{2} (0 + 0) = 0.
\]

ii. If \( f(x) = \sin nx \) where \( n = 2, 3, \ldots \), then
\[
\int_{-\pi}^{\pi} f(t) dt = \int_{-\pi}^{\pi} \sin nt dt = -\cos nt |_{-\pi}^{\pi} = -\cos n\pi - \cos (-n\pi) = -\cos n\pi - \cos n\pi = 0.
\]
\[
\int_{-\pi}^{\pi} f(t) \cos nt dt = \int_{-\pi}^{\pi} \sin nt \cos nt dt = \frac{1}{2} (\int_{-\pi}^{\pi} \sin (n + 1) t dt + \int_{-\pi}^{\pi} \sin (n - 1) t dt) = \frac{1}{2} (0 + 0) = 0.
\]
\[
\int_{-\pi}^{\pi} f(t) \sin nt dt = \int_{-\pi}^{\pi} \sin nt \sin nt dt = -\frac{1}{2} (\int_{-\pi}^{\pi} \cos (n + 1) t dt - \int_{-\pi}^{\pi} \cos (n - 1) t dt) = -\frac{1}{2} (0 - 0) = 0.
\]

so \( S \) contains the functions \( f(x) = \cos nx \) and \( f(x) = \sin nx \) for each \( n = 2, 3, \ldots \).

(c) We note that \( \int_{-\pi}^{\pi} \cos mx \sin nx dx = \frac{1}{2} (\int_{-\pi}^{\pi} \sin (m + n)x - \sin (m - n)x dx) = \frac{0 - 0}{2} = 0 \forall m, n \). Also \( \int_{-\pi}^{\pi} \cos mx \sin nx dx = \int_{-\pi}^{\pi} \sin mx \cos nx dx = 0 \forall m \neq n \). Hence \( \cos mx \) and \( \sin nx \) are linearly independent. It is clear that \( W = L\{\cos 2x, \sin 2x, \cos 3x, \sin 3x, \ldots\} \subseteq S \) and \( \dim W = +\infty \). Thus \( \dim S = +\infty \).

(d) For any \( f(x) \in V \), the image \( g(x) = T(f) \) of \( f \) is
\[
g(x) = \int_{-\pi}^{\pi} \{1 + \cos(x-t)\} f(t) dt
\]
\[
= \int_{-\pi}^{\pi} f(t) dt + \cos x \int_{-\pi}^{\pi} \cos tf(t) dt + \sin x \int_{-\pi}^{\pi} \sin tf(t) dt
\]
\[
\in L(1, \cos x, \sin x).
\]

Thus \( T(V) \subseteq L(1, \cos x, \sin x) \). 

6
Consider \( f(t) = \frac{1}{2\pi} \cos \frac{t}{\pi}, \sin \frac{t}{\pi} \), we have

\[
\int_{-\pi}^{\pi} \frac{1}{2\pi} dt = 1, \quad \int_{-\pi}^{\pi} \frac{1}{2\pi} \cos t dt = 0, \quad \int_{-\pi}^{\pi} \frac{1}{2\pi} \sin t dt = 0,
\]

\[
\int_{-\pi}^{\pi} \cos \frac{t}{\pi} dt = 0, \quad \int_{-\pi}^{\pi} \cos \frac{t}{\pi} \cos t dt = 1, \quad \int_{-\pi}^{\pi} \cos \frac{t}{\pi} \sin t dt = 0,
\]

\[
\int_{-\pi}^{\pi} \sin \frac{t}{\pi} dt = 0, \quad \int_{-\pi}^{\pi} \sin \frac{t}{\pi} \cos t dt = 0, \quad \int_{-\pi}^{\pi} \sin \frac{t}{\pi} \sin t dt = 1.
\]

Thus, we have \( T\left(\frac{1}{2\pi}\right) = 1, T(\cos x) = \cos x, T(\sin x) = \sin x \), which implies that \( L(1, \cos x, \sin x) \subseteq T(V) \). We conclude that \( T(V) = L(1, \cos x, \sin x) \) and \( \{1, \cos x, \sin x\} \) is a basis of \( T(V) \) since it is a linearly independent set.

(e) \( f(x) \in N(T) \) if and only if \( \int_{-\pi}^{\pi} f(t) dt + \cos x \int_{-\pi}^{\pi} \cos t f(t) dt + \sin x \int_{-\pi}^{\pi} \sin t f(t) dt = 0 \). Since \( 1, \cos x \) and \( \sin x \) are linearly independent in \( V \), we have \( \int_{-\pi}^{\pi} f(t) dt = 0, \int_{-\pi}^{\pi} \cos t f(t) dt = 0 \) and \( \int_{-\pi}^{\pi} \sin t f(t) dt = 0 \). Thus \( N(T) = S \).

(f) If \( T(f) = cf, c \neq 0, f \neq 0 \), then \( cf \) is in \( T(V) \). Hence \( f = c_1 + c_2 \cos x + c_3 \sin x \).

so

\[
T(f) = 2\pi c_1 T(\frac{1}{2\pi}) + \pi c_2 T(\frac{\cos x}{\pi}) + \pi c_3 T(\frac{\sin x}{\pi})
\]

\[
= 2\pi c_1 + \pi c_2 \cos x + \pi c_3 \sin x.
\]

Thus, \( 2\pi c_1 + \pi c_2 \cos x + \pi c_3 \sin x = c(c_1 + c_2 \cos x + c_3 \sin x) \Rightarrow (2\pi - c)c_1 + (\pi - c)c_2 \cos x + (\pi - c)c_3 \sin x = 0 \Rightarrow (2\pi - c)c_1 = (\pi - c)c_2 = (\pi - c)c_3 = 0 \).

If \( c_1 \neq 0 \), then \( c = 2\pi, c_2 = c_3 = 0 \), and \( f(x) = c_1 \) where \( c_1 \neq 0 \) but otherwise arbitrary. If one of \( c_2 \) and \( c_3 \) is non-zero, then \( c = \pi, c_1 = 0 \) and \( f(x) = c_2 \cos x + c_3 \sin x \) where \( c_2, c_3 \) are not both 0 but otherwise arbitrary.