3. (a) If \( \| x + y \| = \| x - y \| \), then
\[
(x + y, x + y) = (x - y, x - y)
\]
\[
\Rightarrow (x, x) + (x, y) + (y, x) + (y, y) = (x, x) - (x, y) - (y, x) + (y, y),
\]
\[
\Rightarrow (x, x) + 2(x, y) + (y, y) = (x, x) - 2(x, y) + (y, y), \quad \text{(By Axiom 1)}
\]
\[
\Rightarrow 4(x, y) = 0,
\]
\[
\Rightarrow (x, y) = 0.
\]

(b) If \( (x, y) = 0 \), then
\[
\| x + y \| = (x + y, x + y)^{1/2}
\]
\[
= \{ (x, x) + 2(x, y) + (y, y) \}^{1/2}
\]
\[
= \{ (x, x) + (y, y) \}^{1/2}
\]
\[
= \{ (x, x) - 2(x, y) + (y, y) \}^{1/2}
\]
\[
= (x - y, x - y)^{1/2}
\]
\[
= \| x - y \|.
\]

8. (b)
\[
\| x + y \|^2 - \| x - y \|^2 = (x + y, x + y) - (x - y, x - y)
\]
\[
= (x, x) + (x, y) + (y, x) + (y, y) - \{ (x, x) - (x, y) - (y, x) + (y, y) \}
\]
\[
= 2(x, y) + 2(y, x).
\]

(c)
\[
\| x + y \|^2 + \| x - y \|^2 = (x + y, x + y) + (x - y, x - y)
\]
\[
= (x, x) + (x, y) + (y, x) + (y, y) + (x, x) - (x, y) - (y, x) + (y, y)
\]
\[
= 2(x, x) + 2(y, y)
\]
\[
= 2\| x \|^2 + 2\| y \|^2.
\]

11. (a) Given \( (f, g) = \int_1^e (\log(x)) f(x) g(x) dx \). If \( f(x) = \sqrt{x} \), then
\[
\| f \| = (f, f)^{1/2}
\]
\[
= (\int_1^e (\log x) f(x) f(x) dx)^{1/2}
\]
\[
= (\int_1^e (\log x) x dx)^{1/2}.
\]
Let \( u = \log x \), then \( du = \frac{1}{x} \, dx \). Let \( v = \frac{x^2}{2} \), then \( dv = x \, dx \). Thus

\[
( \int (\log x) \, x \, dx )^{1/2} = ( \int u \, dv )^{1/2} = (uv - \int v \, du)^{1/2}
= (\log x \frac{x^2}{2} - \int \frac{x^2}{2} \, dx)^{1/2}.
\]

So

\[
( \int_1^e (\log x) \, x \, dx )^{1/2} = (\log x \frac{x^2}{2} \bigg|_1^e - \int_1^e \frac{x^2}{2} \, dx )^{1/2}
= (\frac{e^2}{2} - \frac{x^2}{4} |_1^e)^{1/2}
= (\frac{e^2}{2} - \frac{e^2 - 1}{4})^{1/2}
= (\frac{e^2 + 1}{4})^{1/2}
= \frac{1}{2} \sqrt{e^2 + 1}.
\]

(b) First we evaluate \( \int_1^e \log x \, dx \). Let \( u = x^2 \log x \), then \( du = (2x \log x + x) \, dx \). Let \( v = -x^{-1} \), then \( dv = \frac{1}{x^2} \, dx \). Thus

\[
\int \log x \, dx = \int (\frac{1}{x^2}) x^2 \log x \, dx
= \int uv \, dx
= uv - \int v \, du
= -x \log x + \int \frac{1}{x} (2x \log x + x) \, dx.
\]

So

\[
\int_1^e \log x \, dx = -x \log x \bigg|_1^e + \int_1^e \frac{1}{x} (2x \log x + x) \, dx
= -e + \int_1^e 2 \log x \, dx + \int_1^e dx
= -e + \int_1^e 2 \log x \, dx + e - 1
= \int_1^e 2 \log x \, dx - 1.
\]
Thus, \( f_1^e \log x \, dx = 1 \). Now we want to find a linear polynomial \( g(x) = a + bx \) nonzero and orthogonal to \( f(x) = 1 \), i.e., \( (f, g) = 0 \). Since

\[
(f, g) = \int_1^e \log x(a + bx) \, dx = a \int_1^e \log x \, dx + b \int_1^e x \log x \, dx = a + b \left( \frac{e^2 + 1}{4} \right) \quad \text{(by (a)),}
\]

we have \( (f, g) = 0 \) when \( a = -b \left( \frac{e^2 + 1}{4} \right) \). So \( g(x) = b(x - \frac{e^2 + 1}{4}) \), \( b \) is an arbitrary real number.

12. \( (f, g) = \int_{-1}^1 f(t)g(t) \, dt \).

Since \( u_1(t) = 1 \) and \( u_2(t) = t \), we have the following results:

\[
(u_1, u_1) = \int_{-1}^1 1 \cdot 1 \, dt = t \bigg|_{-1}^{1} = 2,
\]

\[
(u_2, u_2) = \int_{-1}^1 t \cdot t \, dt = \frac{t^3}{3} \bigg|_{-1}^{1} = \frac{2}{3}, \quad \text{and}
\]

\[
(u_1, u_2) = \int_{-1}^1 1 \cdot t \, dt = \frac{t^2}{2} \bigg|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0.
\]

Then \( \|u_1\| = (u_1, u_1)^{1/2} = \sqrt{2} \), \( \|u_2\| = (u_2, u_2)^{1/2} = \sqrt{\frac{2}{3}} \), and that \( u_1 \) and \( u_2 \) are orthogonal.

By the fact that \( u_3(t) = 1 + t = u_1(t) + u_2(t) \) and \( (u_1, u_2) = 0 \), we have

\[
(u_1, u_3) = (u_1, u_1 + u_2) = (u_1, u_1) + (u_1, u_2) = (u_1, u_1) = \|u_1\|^2,
\]

\[
(u_2, u_3) = (u_2, u_1 + u_2) = (u_2, u_1) + (u_2, u_2) = (u_2, u_2) = \|u_2\|^2,
\]

and

\[
(u_3, u_3) = (u_1 + u_2, u_1 + u_2) = (u_1, u_1) + (u_2, u_1) + (u_2, u_2) = 2 + 0 + 0 + \frac{2}{3} = \frac{8}{3}.
\]

The last equation implies

\[
\|u_3\| = (u_3, u_3)^{1/2} = \sqrt{\frac{8}{3}}.
\]

Let \( \theta_{ij} \) be the angle between \( u_i \) and \( u_j \), for \( 1 \leq i, j \leq 3 \) and \( i \neq j \). Then

\[
\cos \theta_{12} = \frac{(u_1, u_2)}{\|u_1\| \|u_2\|} = 0,
\]

\[
\cos \theta_{13} = \frac{(u_1, u_3)}{\|u_1\| \|u_3\|} = \frac{\|u_1\|^2}{\|u_1\| \|u_3\|} = \frac{\|u_1\|}{\|u_3\|} = \sqrt{\frac{2}{3}} \quad \text{and}
\]

\[
\cos \theta_{23} = \frac{(u_2, u_3)}{\|u_2\| \|u_3\|} = \frac{\|u_2\|^2}{\|u_2\| \|u_3\|} = \frac{\|u_2\|}{\|u_3\|} = \sqrt{\frac{2}{3}} = \frac{1}{2}.
\]
Thus
\[ \theta_{12} = \cos^{-1} 0 = \frac{\pi}{2}, \quad \theta_{13} = \cos^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{6}, \quad \text{and} \quad \theta_{23} = \cos^{-1} \frac{1}{2} = \frac{\pi}{3}. \]

14. Let \( P \) be the linear space of all real polynomials and \( O \) be the zero element of \( P \), that is, \( O(t) = 0 \).

(a) \( (f, g) = f(1)g(1) \).
   Let \( f(t) = t - 1 \), then \( (f, f) = f(1)f(1) = 0 \cdot 0 = 0 \). Since \( f \not= O \), the nonnegativity property is violated.

(b) \( (f, g) = |\int_0^1 f(t)g(t) \, dt| \).
   Let \( c < 0 \) and \( f(t) \neq O(t) \). Then \( c(f, f) = c|\int_0^1 f(t)f(t) \, dt| < 0 \), and \( (cf, f) = |\int_0^1 cf(t)f(t) \, dt| > 0 \). Thus \( c(f, f) \neq (cf, f) \) and the linearity property is violated.

(c) \( (f, g) = \int_0^1 f'(t)g'(t) \, dt \).
   Let \( f(t) \) be a nonzero polynomial of degree 0, say \( f(t) = 1 \). Then \( (f, f) = \int_0^1 f'(t)g'(t) \, dt = \int_0^1 0 \cdot 0 \, dt = 0 \). Thus the nonnegativity property is violated.

(d) \( (f, g) = (\int_0^1 f(t) \, dt)(\int_0^1 g(t) \, dt) \).
   Let \( f(t) = t - \frac{1}{2} \). Then \( (f, f) = [\int_0^1 (t - \frac{1}{2}) \, dt|^2 = (\frac{t^2}{2} - \frac{t}{2})|_0^1 = 0 - 0 = 0 \). Thus the nonnegativity property is violated.

15. (a) Let \( f \) and \( g \) be two elements of set \( V \). Thus \( \int_0^\infty e^{-t}f(t)^2 \, dt \) and \( \int_0^\infty e^{-t}g(t)^2 \, dt \) converge. Since

\[ \lim_{M \to \infty} \left( \int_0^M e^{-t}|f(t)g(t)| \, dt \right)^2 = \lim_{M \to \infty} \left| \int_0^M e^{-t}|f(t)||g(t)| \, dt \right|^2 \]
\[ \leq \lim_{M \to \infty} \left( \int_0^M e^{-t}|f(t)||f(t)| \, dt \cdot \int_0^M e^{-t}|g(t)||g(t)| \, dt \right) \]
(b) Since the set of all functions continuous on a given interval is a linear space and \( V \) is a subset of it, we only need to check the closure axioms.

i Let \( f \) and \( g \) be two elements of set \( V \). For \( f + g \),

\[ \int_0^\infty e^{-t}(f(t) + g(t))^2 \, dt = \int_0^\infty e^{-t}(f(t)^2 + g(t)^2 + 2f(t)g(t)) \, dt \]
\[ = \int_0^\infty e^{-t}f(t)^2 \, dt + \int_0^\infty e^{-t}g(t)^2 \, dt + 2\int_0^\infty e^{-t}f(t)g(t) \, dt. \]
Since \( \int_0^\infty e^{-t}f(t)g(t)dt \) converges, and \( \int_0^\infty e^{-t}f(t)^2dt \) and \( \int_0^\infty e^{-t}g(t)^2dt \) converge, \( \int_0^\infty e^{-t}(f(t) + g(t))^2dt \) converges. Hence \( f + g \) is an element of \( V \) and Axiom for closure under addition holds.

ii Let \( f \) be the element of set \( V \) such that \( \int_0^\infty e^{-t}f(t)^2dt \) converges, and \( c \) be a real scalar. Since \( \int_0^\infty e^{-t}(af(t))^2dt = \int_0^\infty e^{-t}a^2f(t)^2dt = a^2 \int_0^\infty e^{-t}f(t)^2dt \) converges, \( af \) is an elements of \( V \). Hence Axiom for closure under scalar multiplication holds.

Hence \( V \) is a linear space. Then we need to check if \((f, g)\) is an inner product for \( V \). Let \( x, y, \) and \( z \) be elements of \( V \), and \( c \) be a real scalar.

i Since \( (x, y) = \int_0^\infty e^{-t}x(t)y(t)dt = \int_0^\infty e^{-t}y(t)x(t)dt = (y, x) \), axiom for commutativity holds.

ii Since

\[
(ax + \beta y, z) = \int_0^\infty e^{-t}(ax + \beta y)(t)z(t)dt \\
= \int_0^\infty e^{-t}(ax(t)z(t) + \beta y(t)z(t))dt \\
= \alpha \int_0^\infty e^{-t}x(t)z(t)dt + \beta \int_0^\infty e^{-t}y(t)z(t)dt \\
= \alpha(x, z) + \beta(y, z),
\]

axiom for linearity holds.

iii We note that zero function \( 0(t) \) is the zero element \( O \) in \( V \) since \( x(t) + 0(t) = x(t) \) for all \( x \). Then for all \( x \neq O \), \( (x, x) = \int_0^\infty e^{-t}x(t)x(t)dt = \int_0^\infty e^{-t}x(t)^2dt > 0 \) since \( \int_0^\infty e^{-t}x(t)^2dt \) converges. Hence axiom for positivity holds.

(c) We prove \((f, g) = \frac{m!}{2^m} \) for \( f = e^{-t} \) and \( g = t^n \), where \( n = 0, 1, 2, \ldots \) by induction. When \( n = 0 \),

\[
(f, g) = \int_0^\infty e^{-t} \cdot e^{-t} \cdot 1dt \\
= \int_0^\infty e^{-2t}dt \\
= -\frac{1}{2} e^{-2t}|_0^\infty \\
= \frac{1}{2} = \frac{0!}{2^{(0+1)}}.
\]

Let \((f, g) = \frac{k!}{2^{k+1}} \) when \( n = k \).

When \( n = k + 1 \),

\[
(f, g) = \int_0^\infty e^{-t} \cdot e^{-t} \cdot t^{k+1}dt \\
= \int_0^\infty t^{k+1} \cdot e^{-2t}dt \\
= -\frac{1}{2} t^{k+1} e^{-2t}|_0^\infty + \frac{k + 1}{2} \int_0^\infty e^{-2t}t^k dt \\
= 0 + \frac{k + 1}{2} \cdot \frac{k!}{2^{k+1}} = \frac{(k + 1)!}{2^{k+2}}.
\]
Hence \((f, g) = \frac{n!}{2^{n+1}}\) where \(g(t) = t^n\) and \(f(t) = e^{-t}\) by induction.

16. (a) \(\sum_{n=1}^{\infty} x_n y_n\) converges absolutely \(\iff\) \(\sum_{n=1}^{\infty} |x_n y_n|\) converges.

Consider two new sequences \(x' = \{x_n\}\) and \(y' = \{|y_n|\}\) both in \(V\).

\[\sum_{n=1}^{\infty} |x_n y_n| = (x', y')\]

Since \(\sum_{n=1}^{\infty} |x_n y_n| = \sum_{n=1}^{\infty} |x_n||y_n|\),

Then using Cauchy-Schwarz inequality for the inner product space \(R^M\) with standard inner product,

\[\left(\sum_{n=1}^{M} |x_n y_n|\right)^2 \leq (\sum_{n=1}^{M} |x_n|^2)(\sum_{n=1}^{M} |y_n|^2) \leq (\sum_{n=1}^{\infty} x_n^2)(\sum_{n=1}^{\infty} y_n^2), \forall M.\]

By taking \(M \to \infty\), we have \(\left(\sum_{n=1}^{\infty} x_n y_n\right)^2 \leq (\sum_{n=1}^{\infty} x_n^2)(\sum_{n=1}^{\infty} y_n^2) < \infty.\)

Thus \(\sum_{n=1}^{\infty} |x_n y_n|\) converges and \(\sum_{n=1}^{\infty} x_n y_n\) converges absolutely.

(b) Since the set of all sequences of real numbers is a linear space and \(V\) is a subset of it, we only need to check the closure axioms.

i. Let \(x = \{x_n\}\) and \(y = \{y_n\}\) be two sequences in \(V\). Consider \(x+y = \{x_n+y_n\}\),

\[\sum_{n=1}^{M} (x_n + y_n)^2 = \sum_{n=1}^{M} (x_n^2 + 2x_n y_n + y_n^2) = \sum_{n=1}^{M} x_n^2 + 2 \sum_{n=1}^{M} x_n y_n + \sum_{n=1}^{M} y_n^2\]

From (a) we know that \(\sum_{n=1}^{M} x_n y_n\) converges absolutely as \(M \to \infty\) In addition, \(\sum_{n=1}^{\infty} x_n^2\) and \(\sum_{n=1}^{\infty} y_n^2\) converge. Thus \(\sum_{n=1}^{\infty} (x_n + y_n)^2\) converges and \(x+y\) is in \(V\).

ii. Let \(x = \{x_n\}\) in \(V\), and \(y = cx = \{cx_n\}\) where \(c\) is a real scalar.

Then \(\sum_{n=1}^{\infty} (cx_n)^2 = c^2 \sum_{n=1}^{\infty} x_n^2\) converges.

Thus \(cx\) is in \(V\).

Hence \(V\) is a linear space. Next, we test if \(V\) is a linear space with \((x, y)\) as an inner product. Consider all choices of \(x, y, z\) in \(V\) and all real scalars \(c\):

i. \((x, y) = \sum_{n=1}^{\infty} x_n y_n = \sum_{n=1}^{\infty} y_n x_n = (y, x)\).

ii. \((x, y+z) = \sum_{n=1}^{\infty} x_n (y_n + z_n) = \sum_{n=1}^{\infty} (x_n y_n + x_n z_n) = \sum_{n=1}^{\infty} x_n y_n + \sum_{n=1}^{\infty} x_n z_n = (x, y) + (x, z)\).

iii. \(c(x, y) = c \sum_{n=1}^{\infty} x_n y_n = \sum_{n=1}^{\infty} (cx_n) y_n = (cx, y)\).

iv. Since \((x, x) = \sum_{n=1}^{\infty} x_n^2, (x, x) \geq 0\) and \((x, x) = 0 \iff x = O.\)

Thus \((x, x) > 0\) if \(x \neq O\).

Hence the four axioms all hold, \(V\) is a linear space with \((x, y)\) as an inner product.

(c) \((x, y) = \sum_{n=1}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{n+1}) = (1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \cdots) = 1.\)

(d) Recall that \(e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\)

Then \((x, y) = \sum_{n=1}^{\infty} (2^{-n})(\frac{1}{n!}) = \sum_{n=1}^{\infty} \frac{2^{-n}}{n!} = -1 + (1 + \frac{2-1}{1!} + \frac{2-2}{2!} + \cdots) = -1 + e^{2^{-1}} = e^{\frac{1}{2}} - 1.\)