10. Let $V$ denote the nonempty set of all rational functions $f/g$, with the degree of $f \leq$ the degree of $g$ (including $f = 0$). Let $x, y, z$ and $w$ be four arbitrary elements in $V$ and $x = \frac{f_1}{g_1}, y = \frac{f_2}{g_2}, z = \frac{f_3}{g_3}, w = \frac{f}{g}$ where $f, f_1, f_2, f_3, g, g_1, g_2, g_3$ are all polynomials. Let $a, b$ be two scalars. For simplicity, we define $\deg(h)$ as the degree of a polynomial $h$. Now we show that $V$ is a linear space indeed.

(a) $x + y = \frac{f_1}{g_1} + \frac{f_2}{g_2} = \frac{f_1g_2 + f_2g_1}{g_1g_2}$. Note that $(g_1g_2)$ and $(f_1g_2 + f_2g_1)$ are also polynomials, and hence we denote them as $g', f'$, respectively. So $x + y = \frac{f'}{g'}$. Besides, $\deg(g') = \deg(g_1) + \deg(g_2)$ and $\deg(f') \leq \max(\deg(f_1) + \deg(g_2), \deg(f_2) + \deg(g_1))$. But $\deg(f_1) + \deg(g_2) \leq \deg(g_1) + \deg(g_2)$ $\deg(f_2) + \deg(g_1) \leq \deg(g_2) + \deg(g_1) = \deg(g_1) + \deg(g_2)$, we have

$$\deg(f') \leq \max(\deg(f_1) + \deg(g_2), \deg(f_2) + \deg(g_1))$$
$$\leq \max(\deg(g_1) + \deg(g_2), \deg(g_1) + \deg(g_2))$$
$$= \deg(g_1) + \deg(g_2)$$
$$= \deg(g')$$

Thus, Axiom 1 holds.

(b) We find $aw = a(\frac{f}{g}) = \frac{af}{g}$, But $af$ is still a polynomial, so $aw$ is a rational function. Besides, $\deg(af) \leq \deg(f) \leq \deg(g)$ and hence Axiom 2 holds.

(c)

$$x + y = \frac{f_1}{g_1} + \frac{f_2}{g_2}$$
$$= \frac{f_1g_2 + f_2g_1}{g_1g_2}$$
$$= \frac{f_2g_1 + f_1g_2}{g_1g_2}$$
$$= \frac{f_2}{g_2} + \frac{f_1}{g_1}$$
$$= y + x$$

So Axiom 3 holds.
(d)

\[(x + y) + z = \left(\frac{f_1}{g_1} + \frac{f_2}{g_2}\right) + \frac{f_3}{g_3} = \frac{f_1g_2 + f_2g_1}{g_1g_2} + \frac{f_3}{g_3} = \frac{f_1g_2g_3 + f_2g_1g_3 + f_3g_1g_2}{g_1g_2g_3} = \frac{f_1}{g_1} + \left(\frac{f_2}{g_2} + \frac{f_3}{g_3}\right) = x + (y + z)\]

So Axiom 4 holds.

(e) We find the zero rational function 0 is in \(V\) and is a zero element of \(V\) since \(w + 0 = \frac{f}{g} + 0 = \frac{f+0g}{g} = \frac{f}{g} = w\). Hence Axiom 5 holds.

(f) The rational function \((-1)w = \frac{-f}{g}\) with \(\text{deg}(-f) = \text{deg}(f) \leq \text{deg}(g)\) is clearly in \(V\). Since \(w + (-1)w = \frac{f}{g} + (-1)\frac{f}{g} = \frac{f - f}{g} = \frac{0}{g} = 0\), Axiom 6 holds.

(g) \(a(bw) = a(b\frac{f}{g}) = \frac{af}{g} = (ab)\frac{f}{g} = (ab)w\). Hence Axiom 7 holds.

(h) \(a(x + y) = a\left(\frac{f_1}{g_1} + \frac{f_2}{g_2}\right) = a\frac{f_1g_2 + f_2g_1}{g_1g_2} = \frac{af_1g_2 + af_2g_1}{g_1g_2} = \frac{af_1}{g_1} + \frac{af_2}{g_2} = ax + ay\). Hence Axiom 8 holds.

(i) \((a + b)w = (a + b)\frac{f}{g} = \frac{(a + b)f}{g} = \frac{af + bf}{g} = \frac{af}{g} + \frac{bf}{g} = a\frac{f}{g} + b\frac{f}{g} = aw + bw\). Hence Axiom 9 holds.

(j) \(1w = 1\frac{f}{g} = \frac{f}{g} = w\). Hence Axiom 10 holds.

17. Let \(V = \{ f : |f(x)| \leq M_f \text{ for all } x, M_f \text{ depends on } f \}\). Let \(f, f_1, f_2, f_3\) be four arbitrary elements in \(V\) and \(|f| \leq M_f, |f_1| \leq M_{f_1}, |f_2| \leq M_{f_2}, |f_3| \leq M_{f_3}\) with \(M_f, M_{f_1}, M_{f_2}, M_{f_3} \geq 0\). Let \(a, b\) be two scalars. For clarity, we let the domain of this function be \(X\). Now we show that \(V\) is a linear space indeed.

(a) If we denote the sum of \(f_1\) and \(f_2\) as \(f'\) and the sum of \(M_{f_1}\) and \(M_{f_2}\) as \(M_{f'}\), then

\[|f'| = |f_1 + f_2| \leq |f_1| + |f_2| \leq M_{f_1} + M_{f_2} = M_{f'}\]

Thus, \(f'\) is in \(V\) and Axiom 1 holds.

(b) If we denote the product of \(f\) and \(a\) as \(f''\) and the product of \(M_f\) and \(|a|\) as \(M_{f''}\), then \(|f''| = |af| = |a||f| \leq |a|M_f = M_{f''}\). Hence Axiom 2 holds.

(c) \((f_1 + f_2)(x) = f_1(x) + f_2(x) = f_2(x) + f_1(x) = (f_2 + f_1)(x)\). Hence Axiom 3 holds.
(d) \[
((f_1 + f_2) + f_3)(x) = (f_1 + f_2)(x) + f_3(x) \\
= f_1(x) + f_2(x) + f_3(x) \\
= f_1(x) + (f_2(x) + f_3(x)) \\
= f_1(x) + f_2 + f_3(x) \\
= (f_1 + f_2 + f_3)(x)
\]

Hence Axiom 4 holds.

(e) The zero function \(0\) is in \(V\) since \(|0(x)| \leq 0 \forall x \in X\). Also, \(f(x) + 0 = f(x)\), where \(0\) is the zero function. Hence Axiom 5 holds.

(f) The function \((-1)f\) is clearly in \(V\) by (b). Since \((f + (-1)f)(x) = f(x) + (-1)f(x) = 0\). Hence Axiom 6 holds.

(g) \(a(bf)(x) = a(bf(x)) = abf(x) = (ab)f(x)\). Hence Axiom 7 holds.

(h) \(a(f_1 + f_2)(x) = a(f_1(x) + f_2(x)) = af_1(x) + af_2(x) = (af_1 + af_2)(x)\). Hence Axiom 8 holds.

(i) \((a + b)f(x) = (a + b)f(x) = af(x) + bf(x) = (af + bf)(x)\). Hence Axiom 9 holds.

(j) \((1f)(x) = 1f(x) = f(x)\). Hence Axiom 10 holds.

20. Let \(V = \{a \sin x + b \cos x; a, b \in R\}\).

(1) For \(a_1, b_1, a_2\) and \(b_2 \in R\),
\[
(a_1 \sin x + b_1 \cos x) + (a_2 \sin x + b_2 \cos x) = (a_1 + a_2) \sin x + (b_1 + b_2) \cos x,
\]
for all \(x \in R\). Thus \((a_1 + a_2) \sin x + (b_1 + b_2) \cos x \in V\).

(2) For every \(r \in R\) and \(a \sin x + b \cos x \in V\), we have
\[
r(a \sin x + b \cos x) = ra \sin x + rb \cos x,
\]
for all \(x \in R\). Thus \(r(a \sin x + b \cos x) \in V\).

(3) \((a_1 \sin x + b_1 \cos x) + (a_2 \sin x + b_2 \cos x) = (a_1 + a_2) \sin x + (b_1 + b_2) \cos x = (a_2 + a_1) \sin x + (b_2 + b_1) \cos x = (a_2 \sin x + b_2 \cos x) + (a_1 \sin x + b_1 \cos x),\) for all \(x \in R\). Thus Axiom 3 holds.

(4) For \(a_1 \sin x + b_1 \cos x, a_2 \sin x + b_2 \cos x\) and \(a_3 \sin x + b_3 \cos x\) in \(V\),
\[
[(a_1 \sin x + b_1 \cos x) + (a_2 \sin x + b_2 \cos x)] + (a_3 \sin x + b_3 \cos x) \\
= (a_1 + a_2) \sin x + (b_1 + b_2) \cos x + (a_3 \sin x + b_3 \cos x) \\
= (a_1 + a_2 + a_3) \sin x + (b_1 + b_2 + b_3) \cos x, \text{ for all } x \in R
\]
and
\[
(a_1 \sin x + b_1 \cos x) + [(a_2 \sin x + b_2 \cos x) + (a_3 \sin x + b_3 \cos x)] \\
= (a_1 \sin x + b_1 \cos x) + (a_2 + a_3) \sin x + (b_2 + b_3) \cos x \\
= (a_1 + a_2 + a_3) \sin x + (b_1 + b_2 + b_3) \cos x, \text{ for all } x \in R
\]

The associative law for addition holds.
(5) Let $O$ be the zero function, i.e. $O(x) = 0$ for all $x \in R$, then $O(x) = 0 \cdot \sin x + 0 \cdot \cos x \in V$. For any $a \sin x + b \cos x$ in $V$, $(a \sin x + b \cos x) + O(x) = a \sin x + b \cos x + 0 = a \sin x + b \cos x, \forall x \in R$. Thus the zero function is a zero element in $V$.

(6) For every $a \sin x + b \cos x$ in $V$, $(-1)(a \sin x + b \cos x)$ is also in $V$ by (2) and $(a \sin x + b \cos x) + (-1)(a \sin x + b \cos x) = (a - a) \sin x + (b - b) \cos x = O(x)$ for all $x \in R$. Thus Axiom 6 holds.

(7) For $r, s \in R$ and $a \sin x + b \cos x \in V$, $r(s(a \sin x + b \cos x)) = r(sa \sin x + sb \cos x) = rsa \sin x + rsb \cos x = (rs)a \sin x + (rs)b \cos x = (rs)(a \sin x + b \cos x)$. The associative law for multiplication by numbers holds.

(8) For $(a_1 \sin x + b_1 \cos x)$ and $(a_2 \sin x + b_2 \cos x) \in V$, and $r \in R$,

$$r[(a_1 \sin x + b_1 \cos x) + (a_2 \sin x + b_2 \cos x)] = r[(a_1 + a_2) \sin x + (b_1 + b_2) \cos x] = r(a_1 + a_2) \sin x + r(b_1 + b_2) \cos x = (ra_1 + ra_2) \sin x + (rb_1 + rb_2) \cos x = (ra_1 \sin x + rb_1 \cos x) + (ra_2 \sin x + rb_2 \cos x) = r(a_1 \sin x + b_1 \cos x) + r(a_2 \sin x + b_2 \cos x), \forall x \in R.$$

Thus the Axiom 8 holds.

(9) For $r$ and $s \in R$, $a \sin x + b \cos x \in V$, $(r + s)(a \sin x + b \cos x) = (r + s)a \sin x + (r + s)b \cos x = r(a \sin x + b \cos x) + s(a \sin x + b \cos x) \forall x \in R$, thus Axiom 9 holds.

(10) For every $a \sin x + b \cos x \in V$, $1(a \sin x + b \cos x) = a \sin x + b \cos x \forall x \in R$.

24. (d) If $a = 0$, there is nothing to prove. If $a \neq 0$, multiplying $a^{-1}$ to both sides of $ax = O$, we have $a^{-1}ax = a^{-1}O$. Using Theorem 3.3 (b), we have $x = a^{-1}ax = a^{-1}O = O$.

(e) Because $a \neq 0$, we can multiply $a^{-1}$ to both sides of $ax = ay$ to get $a^{-1}ax = a^{-1}ay$. Thus $x = y$.

(f) Adding the negative element of $bx$, $-1(bx)$, to both sides of $ax = bx$, we have $ax - bx = bx - bx$. The left hand side of the equality is equal to $(a - b)x$, and the right hand side is equal to $O$, so we have $(a - b)x = O$. By (d), $a - b = 0$, hence $a = b$.

(g) $x + y + (-x) + (-y) = x + (-x) + y + (-y)$, by Axiom 3

$= O + O$, by Axiom 6

$= O,$

hence $-x + y = (-x) + (-y)$

(h) i. $x + x = 1x + 1x = (1 + 1)x = 2x$.

ii. $x + x + x = (x + x) + x = 2x + x = 2x + 1x = (2 + 1)x = 3x$. 

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iii. We prove the general case, \( \sum_{i=1}^{n} x = nx \), by mathematical induction. For \( n = 1 \), \( 1x = x \) is true.

We assume the equality holds for \( n = k \), that is,
\[
\sum_{i=1}^{k} x = kx.
\]

For the case \( n = k + 1 \),
\[
\sum_{i=1}^{k+1} x = (\sum_{i=1}^{k} x) + x = kx + x = (k + 1)x.
\]

Hence the equality holds.

25. (a) Let \( V \) be the set of all functions \( f \) integrable on \([0,1]\) such that \( \int_{0}^{1} f(x)dx = 0 \). Let \( f, f_1, f_2 \) and \( f_3 \) be elements in \( V \) and \( a \) and \( b \) be real numbers.

(1) Since \( \int_{0}^{1} (f_1(x) + f_2(x))dx = \int_{0}^{1} f_1(x)dx + \int_{0}^{1} f_2(x)dx = 0 + 0 = 0 \), \( f_1 + f_2 \) is in \( V \).

(2) Since \( \int_{0}^{1} (af(x))dx = a(\int_{0}^{1} f(x)dx) = a0 = 0 \), \( af \) is in \( V \).

(3) Since \( (f_1 + f_2)(x) = f_1(x) + f_2(x) = f_2(x) + f_1(x) = (f_2 + f_1)(x) \) for all \( x \in [0, 1] \), \( f_1 + f_2 = f_2 + f_1 \).

(4) Since \( (f_1(x) + f_2(x)) + f_3(x) = f_1(x) + f_2(x) + f_3(x) = f_1(x) + (f_2(x) + f_3(x)) \) for all \( x \in [0, 1] \), \( f_1 + f_2 + f_3 = f_1 + (f_2 + f_3) \).

(5) Let \( O(x) = 0 \) for all \( x \). Since \( \int_{0}^{1} O(x)dx = 0 \), \( O(x) \in V \). Since \( f(x) + O(x) = f(x) + 0 = f(x) \) for all \( x \in [0, 1] \), \( f + O = f \).

(6) By (2), \( (-1)f \) is in \( V \). Since \( f(x) + (-1)f(x) = 0 \) for all \( x \in [0, 1] \), \( f + (-f) = O \).

(7) Since \( a(bf(x)) = abf(x) = (ab)f(x) \) for all \( x \in [0, 1] \), \( a(bf) = (ab)f \).

(8) Since \( a(f_1(x) + f_2(x)) = af_1(x) + af_2(x) \) for all \( x \in [0, 1] \), \( a(f_1 + f_2) = af_1 + af_2 \).

(9) Since \( (a + b)f(x) = af(x) + bf(x) \) for all \( x \in [0, 1] \), \( (a + b)f = af + bf \).

(10) Since \( 1f(x) = f(x) \) for all \( x \in [0, 1] \), \( 1f = f \).

(b) Let \( V \) denote the nonempty set of all functions \( f \) integrable on \([0,1]\) with \( \int_{0}^{1} f(x)dx \geq 0 \). Let \( a < 0 \) be a real number. And choose an \( f \in V \), e.g., \( f(x) = x \), such that \( \int_{0}^{1} f(x)dx > 0 \), then \( \int_{0}^{1} af(x)dx = a \int_{0}^{1} f(x)dx < 0 \).
So, \( af \notin V \) and Axiom 2 fails to hold. Then Axioms 6, 7, 8, 9, which relate to the scalar multiplication, fail to hold.

(c) Let \( V = \{ f(x); \lim_{x \to \infty} f(x) = 0 \} \).

1. If \( f_1(x) \) and \( f_2(x) \in V \), then \( \lim_{x \to \infty} (f_1 + f_2)(x) = \lim_{x \to \infty} f_1(x) + \lim_{x \to \infty} f_2(x) = 0 + 0 \),

   thus \( f_1(x) + f_2(x) \in V \).

2. \( \lim_{x \to \infty} (af)(x) = a \lim_{x \to \infty} f(x) = a \cdot 0 = 0 \), thus \( af(x) \in V \).

3. For \( f_1(x) \) and \( f_2(x) \in V \), \( f_1(x) + f_2(x) = f_1(x) + f_2(x) \) for all \( x \in R \).

4. For \( f_1(x), f_2(x) \) and \( f_3(x) \in V \), \( (f_1(x) + f_2(x)) + f_3(x) = f_1(x) + (f_2(x) + f_3(x)) \)

   for all \( x \in R \). Thus Axiom 4 holds.

5. Define \( O(x) = 0 \), for all \( x \in R \). Since \( \lim_{x \to \infty} O(x) = 0 \), \( O(x) \) is in \( V \). Since

   \( f(x) + O(x) = f(x) + 0 = f(x) \) for any \( f(x) \in V \), \( O(x) \) is a zero element in \( V \).

6. For any \( f(x) \in V \), we have \( (-1)f(x) \) in \( V \) by (2). Since \( f(x) + (-1)f(x) = f(x) - f(x) = 0 = O(x) \), Axiom 6 holds.

7. For every \( f \) in \( V \), and all real numbers \( r \) and \( s \), \( r(sf(x)) = (rs)f(x) \) for all \( x \) in \( R \). Thus Axiom 7 holds.

8. For all \( f_1 \) and \( f_2 \) in \( V \), and \( r \in R \), \( r(f_1(x) + f_2(x)) = rf_1(x) + rf_2(x) \) for all \( x \in R \), thus Axiom 8 holds.

9. For all \( f \in V \) and all real numbers \( r \) and \( s \), we have \( (r+s)f(x) = rf(x) + sf(x) \)

   for all \( x \in R \). Thus Axiom 9 holds.

10. For all \( f \in V \), \( 1f(x) = f(x) \) for all \( x \in R \), thus Axiom 10 holds.

(d) Let \( f_1, f_2 \) and \( f_3 \) satisfy the linear second-order differential equation. In other words, \( f_i'' + P(x)f_i' + Q(x)f_i = 0 \), for \( i = 1, 2 \) and 3. Let \( a \) and \( b \) be real numbers.

1. Axiom 1.

   Let \( g = f_1 + f_2 \).

   Then \( g'' + P(x)g' + Q(x)g \)

   \( = (f_1'' + f_2'') + (P(x)f_1' + P(x)f_2') + (Q(x)f_1 + Q(x)f_2) \)

   \( = (f_1'' + P(x)f_1' + Q(x)f_1) + (f_2'' + P(x)f_2' + Q(x)f_2) \)

   \( = 0 + 0 \)

   \( = 0 \).


   Let \( g = af \)

   Then \( g'' + P(x)g' + Q(x)g \)

   \( = af'' + aP(x)f' + aQ(x)f \)

   \( = a(f'' + P(x)f' + Q(x)f) \)

   \( = a0 \)

   \( = 0 \).

3. Axiom 3.

   Since \( (f_1 + f_2)(x) = f_1(x) + f_2(x) = f_2(x) + f_1(x) = (f_2 + f_1)(x) \),

   \( f_1 + f_2 = f_2 + f_1 \).


   Since \( (f_1(x) + f_2(x)) + f_3(x) = f_1(x) + f_2(x) + f_3(x) = f_1(x) + (f_2(x) + f_3(x)) \),

   \( (f_1 + f_2) + f_3 = f_1 + (f_2 + f_3) \).
(5) Axiom 5.
Let \( O(x) \) be the zero function. Then \( O''(x) = O'(x) = O(x) = 0 \). Thus
\[ O'' + P(x)O' + Q(x)O = 0 + 0 + 0 = 0 \] and \( O(x) \) is in \( V \).
Since \( f(x) + 0 = f(x) \), \( f + O = f \).

(6) Axiom 6.
By (2), \((-1)f\) is in \( V \). Since \( f(x) + (-1)(f(x)) = 0 \), \( f + (-1)f = O \).

(7) Axiom 7.
Since \( a(bf(x)) = abf(x) = (ab)f(x) \), \( a(bf) = (ab)f \).

(8) Axiom 8.
Since \( a(f_1(x) + f_2(x)) = af_1(x) + af_2(x) \), \( a(f_1 + f_2) = af_1 + af_2 \).

(9) Axiom 8.
Since \((a + b)f(x) = af(x) + bf(x)\), \((a + b)f = af + bf\).

(10) Axiom 10.
Since \( 1f(x) = f(x) \), \( 1f = f \).