Optimal reconstruction in multirate transmultiplexer systems under channel noise: Wiener separation filtering approach

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Received 18 February 1999; received in revised form 28 October 1999

Abstract

The problem of optimal signal reconstruction in transmultiplexer systems under channel noise is considered in this work. The objective of signal reconstruction in transmultiplexer systems is to develop an appropriate separation filter bank design such that the output signals can resemble the original input signals as closely as possible. Based on some multirate digital signal processing techniques, such as polyphase decomposition, noble identity and signal blocking, a transmultiplexer system can be transformed into an equivalent one of single sampling rate, but with blocked input/output signals. It is shown that for a given combining filter bank, a causal and stable separation filter bank is developed by using calculus of variation method and matrix spectral factorization technique, to achieve optimal signal reconstruction of noisy transmultiplexer systems from the viewpoint of Wiener–Hopf equation in the frequency domain. For practical application, a suboptimal FIR separation filter bank is also presented to approximate the proposed IIR Wiener separation filter bank via the least-squares method and maintain the reconstruction performance. Three examples of a two-band, a five-band and a speech transmission transmultiplexer system are provided to illustrate the reconstruction performance of the proposed method. © 2000 Elsevier Science B.V. All rights reserved.

Zusammenfassung


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0165-1684(0)0158-9
Nomenclature

\begin{itemize}
\item \(x_t(n), \Delta x(n)\) scalar function in time domain
\item \(x(n), \Delta x(n)\) vector function in time domain
\item \(T(z), H(z), F(z)\) scalar function in frequency domain
\item \(H(z), F(z)\) vector or matrix function in frequency domain
\item \(\text{tr}(A)\) trace of matrix \(A\)
\item \(E\Delta x^T(n)\Delta x(n)\) expectation operation of \(x^T(n)\Delta x(n)\)
\item \(\lambda(A(z))\) order of the transfer function \(A(z)\)
\item \(O(\cdot)\) computational complexity
\end{itemize}

Résumen

Dans ce travail, nous considérons le problème de la reconstruction optimale de signaux dans des systèmes transmultiplexeurs soumis à du bruit de canal. L'objectif de la reconstruction de signaux dans des systèmes transmultiplexeurs est de développer une conception de bancs de filtres de séparation appropriés, de sorte que les signaux de sortie puissent ressembler le plus possible aux signaux d'entrée. Par des techniques de traitement de signaux numériques à cadences multiples, telle que la décomposition polynomiale, l'identité de Noble et la mise en blocs de signaux, un système transmultiplexeur peut être transformé en un système équivalent à cadence d'échantillonnage unique, mais avec des signaux d'entrée/sortie par blocs. Nous montrons que pour un banc de filtre combinant donné, un banc de filtres causal et stable est développé en utilisant des calculs de méthode variationnelle et une technique de factorisation de la matrice spectrale, pour atteindre une reconstruction de signaux optimale dans des systèmes transmultiplexeurs bruits du point de vue de l'équation de Wiener-Hopf dans le domaine fréquentiel. Pour des applications pratiques, nous présentons aussi un banc de filtre de séparation FIR sous-optimale, qui approche le banc de filtres de séparation IIR de Wiener par la méthode des moindres carrés et maintient les performances de reconstruction. Trois exemples d'un système transmultiplexeur à deux bandes, à quatre bandes et de transmission de voix sont fournis pour illustrer les performances de reconstruction de la méthode proposée. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Optimal deconvolution, Transmultiplexer, Wiener filter

1. Introduction

Transmultiplexer is an important multirate signal transmission system, which is traditionally used for signal interconversion between the time-division multiplexing (TDM) format and the frequency-division multiplexing (FDM) format [1]. In the TDM format several signals are interleaved with each other, whereas in the FDM format individual signals are allocated to separate bands of the spectrum of transmission channel. The purpose of transmultiplexer systems is to transmit many data sequences (TDM \(\rightarrow\) FDM) simultaneously through a single channel and reconstruct them on the other side (FDM \(\rightarrow\) TDM). By using sophisticated filter design techniques, cross-talks among different transmission bands are allowed and the transmitted signals can be reconstructed at the receiver. An extreme example is the code-division multiple-access (CDMA) system, in which the spectrum of each user's signal is spread over the whole channel bandwidth by its signature code at transmitter and each signal is reconstructed by the orthogonality of signature codes at receiver. The related topic is very important in digital signal transmission systems, especially in multiuser communications [17,18] and has received widespread attention covering the theory, design and implementation of transmultiplexers [1-5,9-15].

A schematic diagram of conventional transmultiplexer systems is shown in Fig. 1. At the transmitter, the \(M\) input signals are upsampled, passed through a set of combining filters (i.e., combining filter bank), and
then multiplexed into a composite signal. At the receiver, the composite signal is passed through a set of
separation filters (i.e., separation filter bank), whose outputs are downsampled to recover the original input
signals.

A general approach of conventional transmultiplexer design is focused on crosstalk elimination without
considering channel noise. Such crosstalk phenomenon is basically caused by the down-sampling operations
and the fact that the separation filters are not ideal. Although ideal filters cannot be realized in practice, the
crosstalk in transmultiplexer systems can still be canceled by incorporating proper design of separation filters
and thereby perfect reconstruction (PR) is realized. A necessary and sufficient condition for crosstalk-free
transmultiplexers has been presented earlier [15]. Moreover, because the functions in transmultiplexers are
the same as those in multirate filter banks, but in reverse order, the concept of multirate filter banks has been
applied to realizing transmultiplexers [3,4]. The mathematical relationships between transmultiplexers and
multirate filter banks are discussed in the literature [4]. A number of conventional transmultiplexers are
synthesized based on a set of combining filters and separation filters that are formed from the frequency-
shifted versions (i.e., modulated filter bank) of a low-pass prototype [13]. Substantial progress has been made
in recent years towards the development of noise-free transmultiplexer systems [3,10,11,13,14].

In practical transmultiplexer systems, because of the quantization coding noise in digital transmission and
external noise in transmission channel, the performance of PR transmultiplexer systems always deteriorates
when channel noise appears. Hence, it is appealing to develop a feasible transmultiplexer design method
which can optimally eliminate the influence of channel noise. Recently, a block state-space model has been
introduced by Lin and Chen [5], wherein transmultiplexer systems unify the multirate signals and channel
noise. A multirate Kalman filter-based separation filter bank design has been developed to achieve an
optimal reconstruction in noisy transmultiplexer systems. However, in many signal processing applications
the transfer function form of Wiener filter is more preferable than the state-space form of Kalman filter.

The Wiener filtering technique cannot be directly used to handle this noise suppression problem because of
the multirate downsampling/upsampling operations. Some suitable transformation methods have to be
performed beforehand. Fortunately, an equivalent polyphase structure for transmultiplexer systems without
downsampling/upsampling operations (see Fig. 4) has been developed [3,13] by using the noble identities
and the blocking of channel noise. Thus, the original multirate signal reconstruction problem becomes
a linear time-invariant deconvolution filtering problem. A multivariate Wiener filtering (smoothing) method
can hence be applied to handling this optimal deconvolution problem. However, the conventional Wiener
filtering method usually produces a noncausal filtering design. In this study, calculus of variation method,
matrix spectral factorization technique and Cauchy's residue theory are used to develop a causal separation
filter bank in terms of mean-square error (MSE). For practical application, a suboptimal FIR separation
filter bank is also presented via the least-squares method to approximate the proposed IIR Wiener separation filter bank and maintain the reconstruction performance. Computational complexity (the order of the filter bank) and the sensitivity to computational accuracy are also demonstrated. Three simulation examples of a two-band, a five-band noisy transmultiplexers and an application case for speech transmission are presented to illustrate the proposed design procedure and evaluate the performance.

In this work, italicized upper- and lower-case letters indicate scalars (or scalar functions) in frequency and time domain, respectively. Boldfaced upper- and lower-case letters denote vectors (or matrices) in frequency and time domain, respectively. The rest of this paper is organized as follows: A review of multirate transmultiplexer systems is presented in Section 2. The condition of PR is introduced here. In Section 3, the Wiener filter-based transmultiplexer system is discussed. An equivalent polyphase structure of noisy transmultiplexer system is developed. The solution of Wiener separation filter bank design problem is studied in Section 4. A suboptimal FIR separation filter bank and computational complexity discussion are also given. Simulation examples are presented in Section 5. The sensitivity to computational accuracy is also presented via a speech transmission case. The conclusion is finally given in Section 6.

2. Review of transmultiplexer systems and problem description

2.1. Review of transmultiplexer systems

Consider a noise-free transmultiplexer system as shown in Fig. 1. An M-band transmultiplexer is examined with a sampling rate $M$. At the transmitter, implicit modulation occurs in the upsampling stage since the spectrum of each input signal is replicated with period $2\pi/M$. The combining filter bank allocates input signals $x_i(n)$, for $i = 0, 1, \ldots, M - 1$ to separate portions of equal channel bandwidth for transmission by selecting a set of $M$ center frequencies. The outputs of the combining filters $F_i(z)$, for $i = 0, 1, \ldots, M - 1$ are multiplexed into one composite signal and then transmitted through a single channel, i.e., the combining filter bank converts the signals from the TDM format to the FDM format. At the receiver, the composite signal is passed through a parallel structure of separation filter bank $H_i(z)$, for $i = 0, 1, \ldots, M - 1$, whose outputs are individually downsampled to yield the reconstruction signals $\hat{x}_i(n)$, for $i = 0, 1, \ldots, M - 1$, i.e., the separation filter bank converts the signals from the FDM format back to the TDM format. The upsampling and downsampling are performed synchronously at the same rate and in-phase with each other.

The relation between the output $\hat{X}_i(z)$ and the input $X_i(z)$ is written as [10]

$$\hat{X}_i(z) = \frac{1}{M} \sum_{k=0}^{M-1} X_k(z) \sum_{l=0}^{M-1} F_k(z^{1/M} W^{-l}) H_l(z^{1/M} W^{-l})$$

(2.1)

for $i = 0, 1, \ldots, M - 1$, where $W = e^{-j2\pi/M}$. Replacing $z^M$ by $z$, we get

$$\hat{X}_i(z) = \frac{1}{M} \sum_{k=0}^{M-1} X_k(z^M) \sum_{l=0}^{M-1} F_k(z W^{-l}) H_l(z W^{-l})$$

(2.2)

for $i = 0, 1, \ldots, M - 1$. Note that all transfer functions are now expressed at the channel sampling rate. Denote the transfer function between output signal $\hat{X}_i(z^M)$ and input signal $X_k(z^M)$ as $(1/M) T_{ki}(z^M)$, then we have

$$T_{ki}(z^M) = \sum_{l=0}^{M-1} F_k(z W^{-l}) H_i(z W^{-l})$$

(2.3)

for $i, k = 0, 1, \ldots, M - 1$. The term $T_{ki}(z^M)$, $k \neq i$, denotes the crosstalk function which represents the contribution of the input $X_k(z^M)$ to the output $\hat{X}_i(z^M)$. In the conventional transmultiplexer design, in order
to eliminate crosstalk and get an identical input/output transfer function, the equality
\[ F(z)H^T(z) = T(z^M)I_M \]  
\[ (2.4) \]
must be satisfied \([10,11]\), where \(I_M\) is an \(M \times M\) identity matrix and
\[
F(z) = \begin{bmatrix}
F_0(z) & F_0(zW^{-1}) & \cdots & F_0(zW^{-M+1}) \\
F_1(z) & F_1(zW^{-1}) & \cdots & F_1(zW^{-M+1}) \\
\vdots & \vdots & \ddots & \vdots \\
F_{M-1}(z) & F_{M-1}(zW^{-1}) & \cdots & F_{M-1}(zW^{-M+1})
\end{bmatrix},
\]
\[
H(z) = \begin{bmatrix}
H_0(z) & H_0(zW^{-1}) & \cdots & H_0(zW^{-M+1}) \\
H_1(z) & H_1(zW^{-1}) & \cdots & H_1(zW^{-M+1}) \\
\vdots & \vdots & \ddots & \vdots \\
H_{M-1}(z) & H_{M-1}(zW^{-1}) & \cdots & H_{M-1}(zW^{-M+1})
\end{bmatrix}.
\]
Then, the transfer function from \(X_i(z)\) to \(\hat{X}_i(z)\) is obtained as
\[ \hat{X}_i(z) = \frac{1}{M} T(z)X_i(z), \]
\[ (2.5) \]
where \(T(z)\) denotes the intersymbol interference (ISI) in the transmission. The ISI term can be eliminated if and only if
\[ T(z) = cz^{-d}, \]
\[ (2.6) \]
where \(c\) is a constant and \(d\) is a positive integer. In this situation, PR is achieved and there are no amplitude and phase distortions, i.e., the output sequences are scaled and delayed versions of the input sequences. Most of the previous research has focused on solving \((2.4)\) from the PR point of view.

However, PR can be achieved only in the noise-free channel case. If channel noise exists in the transmission path (see Fig. 2), the performance of the PR design method would be deteriorated. In practical transmission systems, channel noise is inevitable because of the quantization process prior to transmission and the corruption of environment disturbance in transmission path. Therefore, an effective method for eliminating the channel noise is very important in practical transmultiplexer design. The design difficulty encountered is that the signals in transmultiplexer systems have different sampling rates due to the downsampling/upsampling operations. Conventional filtering techniques cannot be directly applied to treating this noisy signal reconstruction problem. Therefore, it is appealing to have a new method for designing a transmultiplexer system which can achieve optimal signal reconstruction under channel noise as well as allow easy implementation.

2.2. Problem description

In the noisy transmultiplexer system depicted in Fig. 2, let us denote the input and output signal vectors as
\[
x(n) = \begin{bmatrix} x_0(n) \\ x_1(n) \\ \vdots \\ x_{M-1}(n) \end{bmatrix} \quad \text{and} \quad \hat{x}(n) = \begin{bmatrix} \hat{x}_0(n) \\ \hat{x}_1(n) \\ \vdots \\ \hat{x}_{M-1}(n) \end{bmatrix},
\]
\[ (2.7) \]
The reconstruction (smoothing) error is defined as

$$\Delta x(n) = q^{-\ell} x(n) - \hat{x}(n),$$

(2.8)

where \(q^{-1}\) denotes the back-shift operator and \(\ell\) denotes the smoothing lag which is chosen by the designer. If \(\ell = 0\), this is a filtering problem. The advantage of a fixed-lag smoother over a filter is that the reconstruction error decreases as \(\ell\) increases due to the additional information. In practice, a lag of two or three times of the dominant system time constant often guarantees a good performance.

For a given combining filter bank \(F_i(z)\), for \(i = 0, 1, \ldots, M - 1\) of the noisy transmultiplexer system, the design problem involves in developing a realizable separation filter bank \(H_i(z)\), for \(i = 0, 1, \ldots, M - 1\) such that the following MSE is minimized.

$$J = E \Delta x^T(n) \Delta x(n).$$

(2.9)

where \(E\) denotes the expectation operation.

3. Wiener filter-based transmultiplexer systems

Owing to the sampling rate conversions, some transformations must be performed beforehand to obtain an equivalent system with a single sampling rate prior to employing the Wiener filtering technique for the optimal signal reconstruction. Hence, our attention is focused on the frequency-domain-based description of transmultiplexer systems.

The polyphase representations of the combining and separation filters are written as [13]

$$H_i(z) = \sum_{k=0}^{M-1} z^{-k} E_{ik}(z^M) \quad \text{(type-1 polyphase)},$$

(3.1)

$$F_i(z) = \sum_{k=0}^{M-1} z^{-(M-1-k)} R_{ik}(z^M) \quad \text{(type-2 polyphase)}$$

(3.2)
for \( i = 0, 1, \ldots, M - 1 \). Then, we can rewrite the separation filter bank as

\[
\begin{bmatrix}
    H_0(z) \\
    H_1(z) \\
    \vdots \\
    H_{M-1}(z)
\end{bmatrix} = \begin{bmatrix}
    E_{00}(z^M) & E_{01}(z^M) & \cdots & E_{0,M-1}(z^M) \\
    E_{10}(z^M) & E_{11}(z^M) & \cdots & E_{1,M-1}(z^M) \\
    \vdots & \vdots & \ddots & \vdots \\
    E_{M-1,0}(z^M) & E_{M-1,1}(z^M) & \cdots & E_{M-1,M-1}(z^M)
\end{bmatrix} \begin{bmatrix}
    1 \\
    z^{-1} \\
    \vdots \\
    z^{-(M-1)}
\end{bmatrix}
\]

\[= z^{-1}E(z^M)\Theta_1(z), \tag{3.3}\]

where \( \Theta_1(z) = [1, z^{-1}, \ldots, z^{-(M-1)}]^T \) and \( E(z) \) is the \( M \times M \) type-1 polyphase component matrix for the separation filter bank. Similarly, we can rewrite the combining filter bank as

\[
\begin{bmatrix}
    F_0(z) \\
    F_1(z) \\
    \vdots \\
    F_{M-1}(z)
\end{bmatrix} = z^{-(M-1)} z^{-(M-2)} \cdots 1
\]

\[= \begin{bmatrix}
    R_{00}(z^M) & R_{01}(z^M) & \cdots & R_{0,M-1}(z^M) \\
    R_{10}(z^M) & R_{11}(z^M) & \cdots & R_{1,M-1}(z^M) \\
    \vdots & \vdots & \ddots & \vdots \\
    R_{M-1,0}(z^M) & R_{M-1,1}(z^M) & \cdots & R_{M-1,M-1}(z^M)
\end{bmatrix}
\]

\[= \Theta_2(z)R(z^M), \tag{3.4}\]

where \( \Theta_2(z) = [z^{-(M-1)}, \ldots, z^{-1}, 1] \) and \( R(z) \) is the \( M \times M \) type-2 polyphase component matrix for the combining filter bank.

Based on the polyphase decompositions, noble identities, and blocking of channel noise \( w(n) \) shown in Fig. 3, the noisy transmultiplexer system in Fig. 2 can be transformed into an equivalent transmultiplexer system shown in Fig. 4. Now, the overall system becomes a linear discrete time-invariant one with inputs \( x_i(n) \) and channel noise \( w(nM - i) \), for \( i = 0, 1, \ldots, M - 1 \).

**Remark 1.** The above equivalent transmultiplexer system is based on Fig. 59.5 in [13], but wherein the channel noise is disregard.

Consider the equivalent transmultiplexer system in Fig. 4, the input/output relation can be expressed as

\[
\hat{X}(z) = E(z)\Gamma(z)R(z)X(z) + E(z)W(z), \tag{3.5}
\]

![Fig. 3. M-fold blocking of channel noise.](image-url)
where \( W(z) \) denotes the \( z \)-transform of the \( M \)-fold blocking version of channel noise
\[
w(n) = \begin{bmatrix} w(nM) & w(nM - 1) & \cdots & w(nM - M + 1) \end{bmatrix}^T
\]
and \( \Gamma(z) \) denotes the transfer function between \( R(z) \) and \( E(z) \) [14], i.e.,
\[
\Gamma(z) = \begin{bmatrix} 0 & 1 \\ z^{-1}I_{M-1} & 0 \end{bmatrix}.
\]
From (2.8), we have
\[
\Delta X(z) = z^{-N}X(z) - \hat{X}(z),
\]
where \( \Delta X(z) \) denotes the \( z \)-transform of \( \Delta x(n) \). Substituting (3.5) into (3.6), we get
\[
\Delta X(z) = (z^{-N}I_M - E(z)\Gamma(z)R(z))X(z) - E(z)W(z).
\] (3.7)
According to the Parseval’s theorem, the MSE in (2.9) can be rewritten as
\[
J = \frac{1}{2\pi j} \text{tr} \oint_{|z|=1} \Phi_{\Delta x \Delta x}(z) \frac{dz}{z},
\] (3.8)
where \( \Phi_{\Delta x \Delta x}(z) \) denotes the power spectrum matrix of \( \Delta x(n) \), \( \oint_{|z|=1} dz \) denotes the integration around unit circle on the \( z \)-plane, and \( \text{tr} \mathbf{A} \) denotes the trace of matrix \( \mathbf{A} \). Let us denote the power spectra matrices of \( x(n) \) and \( w(n) \) as \( \Phi_{xx}(z) \) and \( \Phi_{ww}(z) \), respectively. Then we get
\[
\Phi_{\Delta x \Delta x}(z) = (z^{-N}I_M - E(z)\Gamma(z)R(z))\Phi_{xx}(z) - E(z)\Gamma(z)R(z)^* + E(z)\Phi_{ww}(z)E^*(z),
\] (3.9)
where ‘*’ denotes the complex conjugate and transpose operation. Substituting (3.9) into (3.8), we get
\[
J = \frac{1}{2\pi j} \text{tr} \oint_{|z|=1} \left\{(z^{-N}I_M - E(z)\Gamma(z)R(z))\Phi_{xx}(z)(z^{-N}I_M - E(z)\Gamma(z)R(z))^* + E(z)\Phi_{ww}(z)E^*(z)\right\} \frac{dz}{z}.
\] (3.10)
For a given $R(z)$, the design problem lies in how to develop a realizable $E(z)$ to minimize the cost function $J$ so as to achieve optimal signal reconstruction. Once the optimal polyphase matrix $E(z)$ is obtained, the corresponding separation filter bank $[H_0(z) \ H_1(z) \ \cdots \ H_{M-1}(z)]^\top$ can be easily obtained from the polyphase matrix representation in (3.3).

**Remark 2.**
- In practice, the power spectrum matrix $\Phi_{xx}(z)$ may not be precisely known, but can be approximated by a weighting function $W_g(z)$, then the performance function in (3.10) is changed as
  \[
  J_1 = \frac{1}{2\pi} \text{tr}\left\{ (z^{-\prime}I_M - E(z)\Gamma(z)R(z))W_g(z)(z^{-\prime}I_M - E(z)\Gamma(z)R(z))^* + E(z)\Phi_{ww}(z)E^*(z) \right\} \frac{dz}{z}. \tag{3.11}
  \]
  Since the performance is influenced by the weighting function $W_g(z)$, a proper choice of $W_g(z)$ is important. If channel noise is white and its variance is known, $W_g(z)$ can be obtained as follows. Denote $y(z) = \Gamma(z)R(z)X(z) + W(z)$ as the $z$-transform of $y(n) = [y(nM) \ y(nM-1) \ \cdots \ y(nM-M+1)]^\top$, which is the $M$-fold blocking of the received signal $y(n)$ as shown in Fig. 2. The spectrum $\Phi_{yy}(e^{j\omega}) = \Gamma(e^{j\omega})R(e^{j\omega})\Phi_{xx}(e^{j\omega})R^*(e^{j\omega})\Gamma^*(e^{j\omega}) + \Phi_{ww}(e^{j\omega})$ can be measured at the receiver. Then, $\Phi_{xx}(e^{j\omega})$ can be evaluated at almost all $\omega$ and $W_g(z)$ can be chosen to approximate $\Phi_{xx}(e^{j\omega})$. Furthermore, the spectral estimation has been discussed in the literature [19].
- If the power spectrum matrices $\Phi_{xx}(z)$ and $\Phi_{ww}(z)$ is unavailable but the signal-to-noise ratio is known, the cost function in (3.10) is changed as
  \[
  J_2 = \frac{1}{2\pi} \text{tr}\left\{ \rho(z^{-\prime}I_M - E(z)\Gamma(z)R(z))(z^{-\prime}I_M - E(z)\Gamma(z)R(z))^* + E(z)E^*(z) \right\} \frac{dz}{z}, \tag{3.12}
  \]
  where $\rho$ denotes the ratio of signal power to noise power.

**Remark 3.**
- In the case of $x(n)$ and $w(n)$ being correlated and their correlation matrix being $\Phi_{ww}(z)$, (3.9) must be modified as
  \[
  \Phi_{xxw}(z) = (z^{-\prime}I_M - E(z)\Gamma(z)R(z))\Phi_{xx}(z)(z^{-\prime}I_M - E(z)\Gamma(z)R(z))^* + E(z)\Phi_{ww}(z)E^*(z).
  \]
  - If $w(n)$ is nonstationary and can be described by a dynamical model whose tap coefficients are of stochastic processes, the MSE in (2.9) must be modified as $J = EE_w(z)\Delta x^\top(n)\Delta x(n)$, where $E_w$ denotes the expectation with respect to the tap coefficients of the noise model.

4. Optimal separation filter bank and complexity discussion

4.1. Optimal separation filter bank

This subsection describes how to find a realizable $E(z)$ to minimize the cost function $J$ in (3.10). This is a multivariate Wiener–Hopf optimization problem [7]. In the following, we use the subscript ‘+’ and ‘−’ to denote the causal (realizable) and noncausal part of a transfer function. For example, denote
\( A(z) = A_+(z) + A_-(z) \) as the causal/noncausal decomposition, where \( A_+(z) \) is analytic for \( |z| > 1 \) and \( A_-(z) \) is analytic for \( |z| < 1 \).

Based on the calculus of variation technique, let

\[
E(z) = E\_-(z) = E^*_{\alpha}(z) + \varepsilon \gamma(z),
\]

where \( E^*_{\alpha}(z) \) denotes the optimal solution of \( E_+(z) \), \( \gamma(z) \) is any realizable matrix with all poles in \( |z| < 1 \), and \( \varepsilon \) is an arbitrarily small real number. The MMSE must satisfy

\[
\frac{\partial J}{\partial \varepsilon}_{\varepsilon=0} = 0.
\]

Then, the optimal \( E^*_{\alpha}(z) \) which minimizes \( J \) in (3.10) is derived as

\[
E^*_{\alpha}(z) = z\{z^{-\nu + 1}\Phi_{\alpha}(z)R^\alpha(z)\Gamma^*(z)A_-(z)\} + \Delta^{-1}(z),
\]

where \( A_+(z) \) and \( A_-(z) \) denote the following spectral factorization [6–8]:

\[
A_+(z)A_-(z) = \Gamma(z)R(z)\Phi_{\alpha}(z)R^\alpha(z)\Gamma^*(z) + \Phi_{\alpha}(z).
\]

The detail derivations are presented in the Appendix.

Once the optimal polyphase matrix \( E^*_{\alpha}(z) \) of the separation filter bank is derived, the optimal separation filter bank is obtained as

\[
[H_0^{\alpha}(z) \quad H_1^{\alpha}(z) \quad \ldots \quad H_{M-1}^{\alpha}(z)]^T = E^*_{\alpha}(z)\Theta_{\alpha}(z).
\]

where \( \Theta_{\alpha}(z) \) is given in (3.3). Substituting \( E^*_{\alpha}(z) \) into (3.10) and using the Cauchy’s residue theorem, the MMSE is obtained as

\[
J_{\alpha} = \frac{1}{2\pi i} \int_{|z|=1} \{z^{-\nu}I_M - E^*_{\alpha}(z)\Gamma(z)R(z)\Phi_{\alpha}(z)(z^{-\nu}I_M - E^*_{\alpha}(z)\Gamma(z)R(z))^*\}
\]

\[
+ E^*_{\alpha}(z)\Phi_{\alpha}(z)E^*_{\alpha}(z) \frac{dz}{z}.
\]

The design algorithm for the proposed optimal Wiener filter-based transmultiplexer systems is presented as follows.

**Design algorithm:**

**Step 0:** Estimate the power spectra of input signals and channel noise \( \Phi_{\alpha}(z) \) (or \( W_p(z) \)) and \( \Phi_{\alpha}(z) \), respectively.

**Step 1:** Select a combining filter bank \( F_i(z), i = 0, 1, \ldots, M - 1 \) and compute the polyphase matrix \( R(z) \) according to (3.4).

**Step 2:** Perform spectral factorization in (4.4) to find \( A_+(z) \) and \( A_-(z) \).

**Step 3:** Decompose \( \{z^{-\nu + 1}\Phi_{\alpha}(z)R^\alpha(z)\Gamma^*(z)A_-(z)\} \) to find the realizable part \( \{z^{-\nu + 1}\Phi_{\alpha}(z)R^\alpha(z)\Gamma^*(z)A_-(z)\} \).

**Step 4:** Calculate the optimal \( E^*_{\alpha}(z) \) as

\[
E^*_{\alpha}(z) = z\{z^{-\nu + 1}\Phi_{\alpha}(z)R^\alpha(z)\Gamma^*(z)A_-(z)\} + \Delta^{-1}(z).
\]

**Step 5:** Recover the optimal separation filter bank \( [H_0^{\alpha}(z) \quad H_1^{\alpha}(z) \quad \ldots \quad H_{M-1}^{\alpha}(z)]^T \) from (4.5).

The spectral factorization in Step 2 and the causal/noncausal decomposition in Step 3 are employed to find a realizable \( E(z) \). Hence, the optimal separation filter bank is a set of causal and stable filters and can be realized by a recursive structure. Without these applications, only a noncausal \( E(z) \) is obtained.
Remark 4.
- If the performance in (3.11) with an approximate signal power spectrum $W_s(z)$ is considered, the optimal $E_i^p(z)$ becomes

$$E_i^{opt}(z) = z^{-\sigma+1}W_s(z)R_\Phi(z)\Gamma^*(z)A_2^{-1}(z), \quad A_2^{-1}(z).$$  \hspace{1cm} (4.7)

where $A_2(z)$ denotes the following spectral factorization:

$$A_2(z) = \Gamma(z)R(z)W_s(z)R_\Phi(z)\Gamma^*(z) + \Phi_{sw}(z).$$  \hspace{1cm} (4.8)

- If $x(n)$ and $w(n)$ are correlated, the optimal $E_i^p(z)$ is obtained as

$$E_i^{opt}(z) = z^{-\sigma+1}(\Phi_{xx}(z)R_\Phi(z)\Gamma^*(z) + \Phi_{sw}(z))A_1^{-1}(z) + A_1^{-1}(z).$$  \hspace{1cm} (4.9)

where $A_1(z)$ denotes the following spectral factorization:

$$A_1(z) = \Gamma(z)R(z)\Phi_{xx}(z)R_\Phi(z)\Gamma^*(z) + \Phi_{sw}(z)\Gamma(z)R(z)\Phi_{sw}(z)$$

$$+ \Phi_{sw}(z)R_\Phi(z)\Gamma^*(z).$$  \hspace{1cm} (4.10)

- Moreover, the proposed design method is suitable for both IIR and FIR combining filters while the Kalman filter-based method [5] is applicable only to the FIR combining filters.

- The optimal Wiener-based filter bank systems can be designed via a similar procedure, but interchanging the separation filter bank with the combining filter bank in Fig. 2.

4.2. Suboptimal FIR separation filter bank

The order of the proposed IIR Wiener separation filter bank may be high because the filter bank is designed at the channel sampling rate. In this subsection, we use an $N$-order FIR filter bank to approximate the IIR one via the least-squares method. In general, we can arbitrarily choose the order of the FIR filter bank which is only determined by the performance specification. In our experiment, an $N$-order FIR filter bank, i.e., $N = N(h(z))$ (the same order as the combining filter bank), is good enough with little performance decline.

Let

$$\tilde{H}_i^{opt}(z) = h_{i0} + h_{i1}z^{-1} + \cdots + h_{iN}z^{-N}$$  \hspace{1cm} (4.11)

be the FIR filter to approximate $H_i^{opt}(z)$, for $i = 0, 1, \ldots, M - 1$, where $h_{ik}$, for $k = 0, 1, \ldots, N$ is the coefficients of the FIR filter to be estimated. The linear regression can be written as

$$\begin{bmatrix}
  h_{00} & h_{01} & \cdots & h_{0N} \\
  h_{10} & h_{11} & \cdots & h_{1N} \\
  \vdots & \vdots & \ddots & \vdots \\
  h_{(M-1)0} & h_{(M-1)1} & \cdots & h_{(M-1)N}
\end{bmatrix}
\begin{bmatrix}
  1 \\
  z^{-1} \\
  \vdots \\
  z^{-N}
\end{bmatrix} =
\begin{bmatrix}
  H_0^{opt}(z) \\
  H_1^{opt}(z) \\
  \vdots \\
  H_{M-1}^{opt}(z)
\end{bmatrix}.  \hspace{1cm} (4.12)$$
Let the measurements be observed at \( p (p \geq N) \) uniformly spaced points on the unit circle \(|z| = 1\) and let 
\( \eta = e^{-j\pi p} \). Then, (4.12) is rewritten as

\[
\begin{bmatrix}
  h_{00} & h_{01} & \cdots & h_{0N} \\
  h_{10} & h_{11} & \cdots & h_{1N} \\
  \vdots & \vdots & \ddots & \vdots \\
  h_{(M-1)0} & h_{(M-1)1} & \cdots & h_{(M-1)N}
\end{bmatrix}
\begin{bmatrix}
  1 & 1 & \cdots & 1 \\
  1 & e^{j\pi} & \cdots & e^{j(p-1)\pi} \\
  \vdots & \vdots & \ddots & \vdots \\
  1 & e^{jN\pi} & \cdots & e^{j(N(p-1)\pi)}
\end{bmatrix}
\]

(4.13)

\[
= \begin{bmatrix}
  H_0^{\text{opt}}(1) & H_0^{\text{opt}}(e^{j\eta}) & \cdots & H_0^{\text{opt}}(e^{j(p-1)\eta}) \\
  H_1^{\text{opt}}(1) & H_1^{\text{opt}}(e^{j\eta}) & \cdots & H_1^{\text{opt}}(e^{j(p-1)\eta}) \\
  \vdots & \vdots & \ddots & \vdots \\
  H_{M-1}^{\text{opt}}(1) & H_{M-1}^{\text{opt}}(e^{j\eta}) & \cdots & H_{M-1}^{\text{opt}}(e^{j(p-1)\eta})
\end{bmatrix}
\]

We abbreviate the above equation as

\( \mathcal{H} = \Psi \). (4.14)

Then, the least-squares estimate of \( \mathcal{H} \) is obtained as [19]

\( \hat{\mathcal{H}} = \Psi \mathcal{E}^*(\mathcal{E} \mathcal{E}^*)^{-1}. \) (4.15)

Though the proposed optimal separation filter bank is IIR, an FIR filter bank with appropriate order can be used to approximate the IIR filter bank with little performance decline. This claim will be evidenced in the simulations.

4.3. Computational complexity discussion

In this subsection, the orders of the FIR/IIR filter banks (computational complexity) are compared with the proposed method, PR, PR followed by Wiener filters which is shown in Fig. 5, and the Kalman filter-based method [5]. For convenience, we denote \( \mathcal{N}(A(z)) \) as the highest order of the transfer function \( A(z) \) and denote \( O(\mathcal{N}(\cdot)) \) as the complexity which increases with \( \mathcal{N}(\cdot) \). However, if we can exactly account the number of the computation operations, for example, there are \( M \times N \) multiplication operations in an
Table 1: Computational complexity comparison

<table>
<thead>
<tr>
<th>Method</th>
<th>Computational complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed IIR</td>
<td>$M^3 \times A'(\Phi_{xx}(z))$</td>
</tr>
<tr>
<td>Proposed FIR</td>
<td>$M \times N$</td>
</tr>
<tr>
<td>PR</td>
<td>$M \times N$</td>
</tr>
<tr>
<td>PR + Wiener (IIR)</td>
<td>$M \times (N + A'(\Phi_{xx}(z)))$</td>
</tr>
<tr>
<td>Kalman</td>
<td>$O(M \times A'(\Phi_{xx}(z))^2)$</td>
</tr>
</tbody>
</table>

N-order, M-band transmultiplexer, $M \times N$ is directly used to denote the complexity. For simplifying, only the multiplication operations are considered since they dominate the computation time. For fair comparison, we let the combining filter banks be the same in the four design methods.

For an N-order, M-band PR transmultiplexer system, the number of multiplication operations is approximated as $M \times N$. For the proposed method, the order of the IIR separation filter bank evaluated from (4.3) and (4.5) is $M^2 \times A'(\Phi_{xx}(z)) + 1$, where one $M$ is because $F_{2PR}^{(z)}$ is designed at the channel sampling rate and the other $M$ comes from the recovery of the separation filter bank in (4.5). Then, the number of multiplication operations is approximated as $M^3 \times A'(\Phi_{xx}(z))$. However, an N-order FIR filter bank can be used to approximate the proposed IIR filter bank via the least-square method. The N-order FIR separation filter bank has the same computational complexity (i.e., $M \times N$) as the PR method.

For the PR followed by Wiener filters, the computational complexity is the combination of PR and Wiener filters. In Fig. 5, the channel noise observed at the output of the separation filter bank is approximated as a white noise since its spectrum is flat in the transmission band after filtered and decimated by the PR separation filter bank and the downsamplers, sequentially. Then, the orders of the IIR Wiener filters are all $A'(\Phi_{xx}(z))$ and the system's computational complexity is $M \times N + M \times A'(\Phi_{xx}(z))$.

The computational complexity of the Kalman filter-based method is dominated by an iterative Riccati equation [5] whose dimension is $M \times A'(\Phi_{xx}(z))$. Its computational complexity is denoted as $O(M \times A'(\Phi_{xx}(z))^2)$.

The summary result is shown in Table 1. It is noted that the computational complexities of the proposed IIR separation filter bank, the PR followed by Wiener filters and the Kalman filter-based method depend on the order of the signal model $\Phi_{xx}$ which is taken into consideration in the three design methods.

5. Simulation examples

The performance of the proposed Wiener filter-based transmultiplexer system is illustrated in this section. The three simulated transmultiplexer systems are a two-band, a five-band, and a speech transmission system. The performances of the PR transmultiplexer [9,13], the PR followed by Wiener filters, and Kalman filter-based transmultiplexer [5] are also presented and compared with that of the proposed method. Before further discussion, an appropriate performance metric is defined as

$$\text{SNR}_r = \frac{1}{M} \sum_{i=0}^{M-1} \text{SNR}_r^i,$$  \hspace{1cm} (5.1)

where

$$\text{SNR}_r^i = 10 \log_{10} \frac{\sum_n x_i^2(n)}{\sum_n (x_i(n - \ell) - \bar{x}_i(n))^2},$$
Table 2  
The proposed separation filter bank in Example 1

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Num. of $H^o(z)$</td>
<td>0.0486</td>
<td>0.0035</td>
<td>-0.2209</td>
<td>-0.1230</td>
<td>0.4713</td>
<td>0.3790</td>
<td>-0.6233</td>
<td>-0.6799</td>
</tr>
<tr>
<td>Num. of $H^f(z)$</td>
<td>0.0486</td>
<td>-0.0035</td>
<td>-0.2209</td>
<td>0.1230</td>
<td>0.4713</td>
<td>-0.3790</td>
<td>-0.6233</td>
<td>0.6799</td>
</tr>
<tr>
<td>Den.</td>
<td>1.0000</td>
<td>0</td>
<td>-5.1954</td>
<td>0</td>
<td>15.2543</td>
<td>0</td>
<td>-30.5317</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
</tr>
<tr>
<td>0.4726</td>
<td>0.7071</td>
<td>0.0624</td>
<td>-0.368</td>
<td>-0.3688</td>
<td>-0.2080</td>
<td>0.5662</td>
<td>0.6075</td>
<td>0.4984</td>
</tr>
<tr>
<td>0.4726</td>
<td>-0.7071</td>
<td>-0.0624</td>
<td>0.368</td>
<td>0.3688</td>
<td>0.2080</td>
<td>0.5662</td>
<td>-0.6075</td>
<td>-0.4984</td>
</tr>
<tr>
<td>45.9767</td>
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<td>-53.9117</td>
<td>0</td>
<td>50.4278</td>
<td>0</td>
<td>-37.7030</td>
<td>0</td>
<td>22.4633</td>
</tr>
<tr>
<td>17</td>
<td>18</td>
<td>19</td>
<td>20</td>
<td>21</td>
<td>22</td>
<td>23</td>
<td>24</td>
<td>25</td>
</tr>
<tr>
<td>-0.6698</td>
<td>0.2994</td>
<td>0.4654</td>
<td>-0.1253</td>
<td>-0.2242</td>
<td>0.0338</td>
<td>0.0692</td>
<td>-0.0045</td>
<td>-0.0121</td>
</tr>
<tr>
<td>0.6698</td>
<td>0.2994</td>
<td>-0.4654</td>
<td>-0.1253</td>
<td>0.2242</td>
<td>0.0338</td>
<td>-0.0692</td>
<td>0.0045</td>
<td>0.0121</td>
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<tr>
<td>0</td>
<td>10.4082</td>
<td>0</td>
<td>-3.6180</td>
<td>0</td>
<td>-0.8542</td>
<td>0</td>
<td>0.1134</td>
<td>0</td>
</tr>
</tbody>
</table>

which denotes the reconstruction signal-to-noise ratio (SNR) of the $i$th input signal and $\xi$ is the smoothing lag chosen by the designer to achieve a better reconstruction performance. Since SNR$_{\xi}$ is significantly influenced by the variance of channel noise, the channel SNR is defined as

$$SNR_{\xi} = 10 \log_{10} \frac{\sum_{n=0}^{\infty} x^2(n)}{\sum_{n=0}^{\infty} w^2(n)}. \quad (5.2)$$

**Example 1** (Two-band transmultiplexer). In the noisy two-band transmultiplexer system given in Fig. 2, i.e., $M = 2$, let the combining filters be $F_0(z) = \frac{1}{2}(1 + z^{-1})$ and $F_1(z) = \frac{1}{2}(1 - z^{-1})$, then the separation filters for PR transmultiplexer are $H_0(z) = \frac{1}{2}(1 + z^{-1})$ and $H_1(z) = \frac{1}{2}(1 - z^{-1})$. The input signals are modeled as an AR process of order 6 driven by zero-mean white noises as

$$AR(6) = \frac{1}{1 - 3.02z^{-1} + 5.75z^{-2} - 6.4762z^{-3} + 5.1942z^{-4} - 2.4644z^{-5} + 0.7371z^{-6}}.$$  

The smoothing lag $\xi$ is chosen to be 6, i.e., $\hat{\xi}_i(n) = x_i(n - 6)$, $i = 0, 1$. The coefficients of the numerators and denominators of the IIR separation filter bank, which is designed by using the proposed design algorithm at SNR$_{\xi} = 25$ dB, are presented in Table 2. The order of the proposed IIR filters is 25 according to (4.3) and (4.5). The spectra of the proposed separation filter bank, the PR and the input signal are shown in Fig. 6. It is noted that the 2-band PR separation filter bank has two equally spaced spectra in the transmission band, but our proposed separation filter bank has a sharper spectra because the signal model is taken into consideration in the design procedure. Fig. 7 shows the SNR$_{\xi}$ versus SNR$_{\xi}$ performance for the proposed transmultiplexer, PR [9,13], PR followed by Wiener filters and the Kalman filter-based one [5]. Fig. 8 shows the MSE performance. A significant performance improvement by the proposed method is clearly observed.

It is noted that the Kalman filter-based transmultiplexer has almost the same performance as the proposed Wiener filter-based one. This result is reasonable because the Kalman filter in steady-state case is equivalent to the Wiener filter. Our simulations also confirm this fact. However, the implementation of a Wiener-based transmultiplexer is simpler than that of a Kalman-based one.

**Example 2** (Five-band transmultiplexer). The five-band transmultiplexer in [9] is used in this simulation example. The tap numbers of all the combining filters are 55. The input signals are modeled as an AR process of order 16 driven by zero-mean white noises. To account for the sensitivity to the imprecise signal spectrum
Fig. 6. Spectra in Example 1.

Fig. 7. SNR_s vs. SNR_e in two-band transmultiplexer system.

\( W_s(z) \), an approximate signal model of order 12 is also simulated in this example. The coefficients of the AR processes for the exact and approximate input signal models are presented in Table 3.

Fig. 9 shows the spectra of the proposed separation filter bank, the PR and the signal model. In Fig. 9, the proposed separation filter bank is designed under SNR_e = 25 dB and smoothing lag \( \zeta = 12 \). The five-band PR separation filter bank has five equally spaced spectra in the transmission band, but our proposed separation filter bank has sharper spectra because the signal model is taken into consideration in the design procedure.
The exact and approximate signal models in Example 2

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Phi_\alpha(z)): AR(16)</td>
<td>1.0000</td>
<td>-2.1152</td>
<td>1.2862</td>
<td>-0.0886</td>
<td>0.2192</td>
<td>-0.5219</td>
<td>0.2883</td>
<td>0.2115</td>
</tr>
<tr>
<td>(W_\alpha(z)): AR(12)</td>
<td>1.0000</td>
<td>-2.1309</td>
<td>1.3136</td>
<td>-0.0606</td>
<td>0.1173</td>
<td>-0.4824</td>
<td>0.3778</td>
<td>0.1403</td>
</tr>
<tr>
<td>(\Phi_\beta(z)): AR(12)</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
</tr>
<tr>
<td>(W_\beta(z)): AR(12)</td>
<td>-0.3923</td>
<td>-0.2456</td>
<td>0.5335</td>
<td>-0.6205</td>
<td>0.2584</td>
<td>-0.2476</td>
<td>0.1330</td>
<td>0.0843</td>
</tr>
<tr>
<td>(W_\gamma(z)): AR(12)</td>
<td>-0.4296</td>
<td>-0.1656</td>
<td>0.8412</td>
<td>-0.7121</td>
<td>0.2306</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

The orders of the proposed IIR separation filter banks for the exact and approximate signal models are 401 and 301, respectively. However, in this simulation an FIR filter bank of order 54 is used to approximate the proposed IIR separation filter bank via the least-square method. Fig. 10 shows the SNR, versus SNR, performance for the proposed IIR/FIR separation filter banks, PR, PR followed by Wiener filters and the Kalman filter-based one. Obviously, the performance of the proposed FIR separation filter bank is almost unchanged when compared with that of the IIR separation filter bank. Moreover, our proposed methods perform much better than the PR method.

For practical application, a weighting function \(W_\beta(z)\) of order 12 is chosen to approximate the signal power spectrum \(\Phi_\alpha(z)\), as presented in Table 3. Its SNR, performance is also shown in Fig. 10. It shows that the performance is slightly deteriorated by the use of an imprecise power spectrum, but is still better than that of the PR method.

Example 3 (Speech transmission). In this simulation, the five-band transmultiplexer [9] as in Example 2 is used to transmit the speech signal. Fig. 11 shows the speech signal transmitted in channel #1. The speech signal is identified by an AR process of order 12 via the linear prediction method. For the practical application, the proposed IIR separation filter bank is approximated by a 55-tap FIR filter bank via the least-squares
method. Then, the proposed FIR separation filter bank has the same order as the PR filter bank. As shown in Fig. 12, for the speech transmission, our proposed method still performs 3–7 dB better than the PR method.

In practical filter design, the coefficients of the filter must have finite precision since the memory space is finite. To account for the sensitivity to computational accuracy, all the coefficients of the proposed FIR separation filter bank are rounded at the third decimal. The coefficients of the PR transmultiplexer are also rounded at the third decimal. Their SNRc versus SNRc performances are also shown in Fig. 12. Obviously, our proposed method performs better than the PR method in these cases.
Fig. 11. The speech signal in Example 3.

Fig. 12. SNR<sub>e</sub> vs. SNR<sub>c</sub> in Example 3.

6. Conclusion

An optimal signal reconstruction method for Wiener filter-based transmultiplexer systems for multirate signal transmission through a noisy channel has been proposed in this study. It has been shown that for a given combining filter bank and the (approximate) power spectra of the input signals and noise, an optimal
and realizable IIR Wiener separation filter bank can be obtained to reconstruct the original input signals. A procedure for designing such filter bank is also presented in this work. In order to reduce the computational complexity, a suboptimal FIR separation filter bank is also presented with little performance decline to approximate the IIR Wiener separation filter bank via the least-squares method.

The motivation for developing these optimal reconstruction filters for transmultiplexer systems has stemmed from the fact that quantization coding noise in digital transmission and external noise in transmission channel are inevitable. The channel noise is disregarded in the conventional PR transmultiplexer system design. We show that the proposed IIR/FIR separation filter banks provide a superior improvement for the reconstruction performance when compared with the PR method. Finally, the optimal signal reconstruction in noisy multirate filter bank systems can also be designed via a similar methodology.

Appendix

Substituting (4.1) into (3.10), the MSE becomes

\[
J = A_1 + \sigma A_2 + \varepsilon^2 A_3, \tag{A.1}
\]

where

\[
A_1 = \frac{1}{2\pi j} \text{tr} \oint_{j|z| = 1} \left\{ z^{-M} - E^{\text{opt}}(z) R(z) \right\} \Phi_{xx}(z) \left\{ z^{-M} - E^{\text{opt}}(z) Y(z) R(z) \right\}^* \frac{dz}{z}
\]

\[+ E^{\text{opt}}(z) \Phi_{ww}(z) \frac{dz}{z} \]

\[
A_2 = \frac{1}{2\pi j} \text{tr} \oint_{j|z| = 1} \left\{ - Y(z) \Gamma(z) R(z) \Phi_{xx}(z) (z^{-M} - E^{\text{opt}}(z) \Gamma(z) R(z))^* \right. \\
\left. - (z^{-M} - E^{\text{opt}}(z) \Gamma(z) R(z)) \Phi_{xx}(z) R^*(z) \Gamma^*(z) Y^*(z) \right. \\
\left. + Y(z) \Phi_{ww}(z) E^{\text{opt}}(z) + E^{\text{opt}}(z) \Phi_{ww}(z) Y^*(z) \right\} \frac{dz}{z}
\]

and

\[
A_3 = \frac{1}{2\pi j} \text{tr} \oint_{j|z| = 1} \left\{ Y(z) \Gamma(z) R(z) \right\} \Phi_{xx}(z) (Y(z) \Gamma(z) R(z))^* + Y(z) \Phi_{ww}(z) Y^*(z) \frac{dz}{z}.
\]

Based on the calculus of variation technique [7,16], the MMSE must satisfy the following condition

\[
\left. \frac{\partial J}{\partial H} \right|_{\epsilon = 0} = 0.
\]

Consequently, we get \(A_2 = 0\). Then

\[
\frac{1}{2\pi j} \text{tr} \oint_{j|z| = 1} \left\{ E^{\text{opt}}(z) \Phi_{yy}(z) - z^{-M} \Phi_{xx}(z) R^*(z) \Gamma^*(z) \right\} \right. \\
\left. Y^*(z) \frac{dz}{z} = 0, \tag{A.2}
\]

where \(\Phi_{yy}(z) = \Gamma(z) R(z) \Phi_{xx}(z) R^*(z) \Gamma^*(z) + \Phi_{ww}(z)\) and \(y(n)\) is the input of the separation filter bank shown in Fig. 4. The other terms in \(A_2\) are equal to the terms in (A.2) by substituting \(z\) for \(z^{-1}\) and transposing since the
matrices in (A.2) are all real rational functions of z, for example, \( R^*(z) = R^T(z^{-1}) \). The integration of (A.2) is equal to that of the other terms in \( A_2 \).

Performing the matrix spectral factorization [6–8]

\[
A_+(z)A_-(z) = \Phi_{qz}(z),
\]

(A.3)

where \( A_+(z) \) is free of poles and zeros in \( |z| > 1 \) and \( A_-(z) \) is free of poles and zeros in \( |z| < 1 \) and substituting (A.3) into (A.2), we get

\[
\frac{1}{2\pi j} \text{tr} \int_{|z|=1} \left\{ E^p_{qz}(z) A_-(z) A_-(z) - z^{-1} \Phi_{qz}(z) R^*(z) F^*(z) \right\} Y^*(z) \frac{dz}{z} = 0.
\]

(A.4)

Denote the terms in the brace of (A.4) as \( Q(z) \), i.e.,

\[
Q(z) = z^{-1} E^p_{qz}(z) A_+(z) A_-(z) - z^{-1} \Phi_{qz}(z) R^*(z) F^*(z).
\]

(A.5)

Multiplying the above equation from the right by \( \Delta^{-1}(z) \) and performing the causal/noncausal decomposition, we get

\[
Q(z) \Delta^{-1}(z) = z^{-1} \Phi_{qz}(z) A_+(z) A_-(z) - z^{-1} \Phi_{qz}(z) R^*(z) F^*(z) \Delta^{-1}(z)
\]

\[
= z^{-1} E^p_{qz}(z) A_+(z) A_-(z) - \left\{ z^{-1} \Phi_{qz}(z) R^*(z) F^*(z) \Delta^{-1}(z) \right\} +
\]

\[
- \left\{ z^{-1} \Phi_{qz}(z) R^*(z) F^*(z) \Delta^{-1}(z) \right\} -.
\]

(A.6)

After rearranging, (A.4) becomes

\[
\frac{1}{2\pi j} \text{tr} \int_{|z|=1} Q(z) \Delta^{-1}(z) A_-(z) Y^*(z) \frac{dz}{z} = 0.
\]

(A.7)

By Cauchy’s residue theorem, if (A.7) holds, all the poles of the term \( Q(z) \Delta^{-1}(z) \) in (A.6) must lie outside the unit circle on the z-plane. Thus,

\[
z^{-1} E^p_{qz}(z) A_+(z) - \left\{ z^{-1} \Phi_{qz}(z) R^*(z) F^*(z) \Delta^{-1}(z) \right\} = 0
\]

(A.8)

since the left-hand side of (A.8) has poles in \( |z| < 1 \).

References


