Optimal Signal Reconstruction in Noisy Filter Bank Systems: Multirate Kalman Synthesis Filtering Approach

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Abstract—A multirate Kalman synthesis filter is proposed in this paper to replace the conventional synthesis filters in a noisy filter bank system to achieve optimal reconstruction of the input signal. Based on an equivalent block representation of subband signals, a state-space model is introduced for an M-band filter bank system with subband noises. The composite effect of the input signal, analysis filter bank, decimators, and interpolators is represented by a multirate state-space model. The input signal is embedded in the state vector, and the corrupting noises in subband paths are generally considered as additive noises. Hence, the signal reconstruction problem in the M-band filter bank systems with subband noises becomes a state estimation procedure in the resultant multirate state-space model. The multirate Kalman filtering algorithm is then derived according to the multirate state-space model to achieve optimal signal reconstruction in noisy filter bank systems. Based on the optimal state estimation theory, the proposed multirate Kalman synthesis filter provides the minimum-variance reconstruction of the input signal. Two numerical examples are also included. The simulation results indicate that the performance improvement of signal reconstruction in noisy filter bank systems is remarkable.

I. INTRODUCTION

MULTIRATE systems based on filter banks are widely used in many application areas, for example, digital audio systems, subband coding of speech and image signals, and analog voice privacy systems [1]. During the last several years there has been substantial progress in multirate system research. This includes design of decimation and interpolation filters, analysis/synthesis filter banks (see [1] and the references therein). The overall performance of such systems is dependent on the combination of many different factors associated with the individual filters in the filter banks, the characteristics of the overall filter banks and the properties of the total analysis/synthesis systems.

Recently, FIR filter bank systems capable of exactly reconstructing the input signal without aliasing, spectral magnitude distortion and/or spectral phase distortion have received high attention in the literature [2]–[12]. Research in this area has focused on developing the reconstruction theory, identifying and classifying the various solutions, developing filter design procedure and new algorithms, and the realization of efficient structures for implementation. Two classes of approaches have been developed to treat the FIR filter bank design problem. Most theoretical developments and design procedures are based on frequency domain analysis of the systems [2]–[9]. The work in [9] used the quadratic-constrained least-squares minimization approach in the design of filter banks. In the alternative approach, filter bank systems are primarily analyzed in the time domain [10]–[12].

In the above research, the subband paths of filter banks are free of noise and perfect reconstruction is exploited (Fig. 1). However, in practical filter bank systems, subband noises due to quantization [13], roundoff [14] and/or corruption in subband paths always exist and cannot be attenuated effectively with the conventional filter bank design procedure. The performance of filter bank systems based on the conventional perfect reconstruction design method will be deteriorated when channel noises appear in the subband paths. In order to improve the performance of filter bank systems with subband noises, a new design procedure is developed to reconstruct the input signal. Recently, the multirate model for stochastic signal processes has been discussed in [15] for signal estimation/interpolation under external noise. Furthermore, a multirate state space model has also been developed in [16] for transmultiplexer systems to improve the signal reconstruction in a transmultiplexer system with a noisy channel. In this paper, a multirate Kalman synthesis filter is proposed to...
II. REVIEW OF FILTER BANK THEORY AND PROBLEM STATEMENT

In this section, the perfect reconstruction filter bank theory is briefly reviewed. An $M$-band filter bank system without subband noises is shown in Fig. 1. A discrete-time signal $x(k)$ passes through a set of digital filters $H_i(z)$ called analysis filters, with desired bandpass frequency responses respectively. The analysis filters $H_1(z), H_2(z), \ldots, H_M(z)$ split the signal $x(k)$ into $M$ frequency bands. The subband signals $u_i(k)$ are thus approximately bandlimited. Then, each subband signal $u_i(k)$ is decimated by a factor $R$. If the decimation factor is equal to the number of bands (i.e., $R = M$), it is called a maximally decimated structure, and hence, the sum of samples $w_i(k)$ per unit time is the same as that of $x(k)$. At the synthesis part, these subband signals are upsampled, filtered by the synthesis filters $G_i(z)$, and then recombined. In this manner, an approximation $\hat{x}(k)$ of the signal $x(k)$ is generated. Such recombination is subject to several types of errors. First, there is an aliasing error created due to decimation of $u_i(k)$. Besides, there are spectral amplitude and phase distortions.

The subsampling and upsampling cause aliasing and imaging effects. The following most general expression for $\hat{X}(z)$ can be derived [2]:

$$\hat{X}(z) = \frac{1}{M} \sum_{l=1}^{M} X(zW^{l-1}) \sum_{i=1}^{M} H_i(zW^{l-1})G_i(z). \quad (1)$$

The term corresponding to $l = 1$ represents the genuine output which would result if the subsampling and upsampling were absent. The terms corresponding to $2 \leq l \leq M$ are undesired aliasing terms. If we wish to stop the aliasing effect completely, $H_i(z)$ and $G_i(z), i = 1, 2, \ldots, M$ should be chosen such that these aliasing terms are equal to zero for any possible input signal $x(k)$. Assuming that aliasing is somehow canceled, the following relation holds:

$$\frac{\hat{X}(z)}{X(z)} = T(z) = \frac{1}{M} \sum_{i=1}^{M} H_i(z)G_i(z) \quad (2)$$

where $T(z)$ represents the overall transfer function, or the distortion function. If $H_i(z)$ and $G_i(z)$ are chosen such that $T(z)$ is allpass, then there is no amplitude distortion. On the other hand, if $T(z)$ is designed to be a linear phase FIR function, there is no phase distortion. Finally, if $T(z)$ is a pure delay (and if aliasing has already been canceled), then it is a perfect–reconstruction filter bank. For the purpose of achieving perfect reconstruction, the following equation must be satisfied [3]

$$H(z)g(z) = [cz^{-n_0} \ 0 \ \cdots \ \ 0]^T \quad (3)$$

where $g(z) = [G_1(z) \ G_2(z) \ \cdots \ G_M(z)]^T$, $H(z)$ has elements $H_{n,k}(z) = H_k(zW^n)$, and $n_0$ is an integer. Several new methods have been developed to solve the above equation from a frequency domain or time domain viewpoint to achieve perfect reconstruction in the filter bank design procedure [1]-[12].

All of the above design methods for filter bank systems have assumed the subband signals $w_i(k)$ are free of any corrupting
noise in the subband paths; therefore, the perfect reconstruction design can be achieved. However, in practical filter bank systems, corrupting noises are inevitable in the subband paths due to the quantization effect of coding algorithm and/or stain source in each subband transmission path. These noises deteriorate over the overall performance of filter bank systems. Hence, the output signal is not the same as the input signal when the subband noises exist. From several simulation results of the paper, it is shown that the performance of the filter bank systems with perfect reconstruction property decays quickly under the subband noises. Therefore, a filter bank system design with consideration of the attenuation of subband noises is an important topic in practical applications.

The corrupting noises in the subband paths of filter bank systems are considered in this paper. A multirate state-space model is introduced at first to unify the input signal, analysis filters, decimators, channel noises, and interpolators together as a single dynamic system with a single time scale. Then, a multirate Kalman synthesis filter is proposed to replace the conventional synthesis filters to achieve optimal signal reconstruction in noisy filter bank systems. From Fig. 2, the subband sequence \( \{a_i(k), k = 0, 1, 2, \ldots \} \) is obtained from the convolution of input signal \( x(k) \) and the impulse response \( h_i(k) \) of the \( i \)-th analysis filter \( H_i(z) \). After subsampling, noise corruption and upsampling, the received noisy signal \( \tilde{y}_i(k) \) in the \( i \)-th subband path of the filter bank system is given by

\[
\tilde{y}_i(k) = \begin{cases} 
  u_i(k) + n_i(k), & \forall k = jR \\
  0, & \forall k \neq jR 
\end{cases} 
\]

for \( i = 1, 2, \ldots, M; j = 0, 1, 2, \ldots \) (4)

where \( n_i(k) \) denotes the corrupting noise in the \( i \)-th subband path, and the decimation factor \( R > 1 \) is an integer. From the vector point of view, let us denote the received subband signal vector \( \tilde{y}(k) \) in the filter bank system as follows:

\[
\tilde{y}(k) = \begin{bmatrix} \tilde{y}_1(k) & \tilde{y}_2(k) & \cdots & \tilde{y}_M(k) \end{bmatrix}^T
\]

\[
u(k) = \begin{bmatrix} u_1(k) & u_2(k) & \cdots & u_M(k) \end{bmatrix}^T
\]

\[
n(k) = \begin{bmatrix} n_1(k) & n_2(k) & \cdots & n_M(k) \end{bmatrix}^T.
\]

where

\[
\tilde{y}(k) = u(k) + n(k) \text{ for } k = 0, R, 2R, \ldots
\] (5)

Our design method for filter bank systems with noisy subband paths is to equip an optimal synthesis filter to replace the conventional synthesis filters. The synthesis filter uses the received noisy subband signal \( \tilde{y}(k) \) in (5) as input and then generates the estimate \( \hat{x}(k) \) of the input signal. The estimate \( \hat{x}(k) \) can be optimal in some statistical sense by using prior knowledge of both input signal \( x(k) \) and subband noise \( n_i(k) \).

Before further analysis design of optimal synthesis filter of filter bank systems with noisy subband paths, a multirate state-space model is introduced in the next section.

III. MULTIRATE STATE-SPACE DESCRIPTION OF FILTER BANK SYSTEM

In this section, a multirate state-space model for an \( M \)-band filter bank system is established. The input signal is embedded in the state vector, and the subband noises are regarded as additive noises. Hence, an optimal signal reconstruction procedure is transformed to an optimal state estimation problem in the multirate state-space model.

A. Basic Signal Model

The analysis filters in a filter bank system may have different coefficient lengths. Let \( L_i \) be the length of \( H_i(z) \), for \( i = 1, 2, \ldots, M \). For the block representation below, all analysis filters must be treated as with the same length; therefore, we choose \( L = \max\{L_1, L_2, \ldots, L_M\} \). A set of analysis filters with different lengths can be unified to be of length \( L \) by adding zeros to each of the filters with length shorter than \( L \).

Since the subband noises are considered in our filter bank design and the optimal estimation technique will be employed to treat the input signal reconstruction problem, some knowledge of the input signal \( x(k) \) is needed for the purpose of estimation. In this paper, we use AR model to describe input signal \( x(k) \) because it is widely used in the fields of statistical signal processing. Now, suppose the input signal \( x(k) \) is modeled as the following AR process:

\[
x(k) + a_1x(k-1) + \cdots + a_px(k-p) = \nu(k)
\] (7)

where the driving source \( \nu(k) \) is a zero-mean, white Gaussian noise with time-varying covariance \( \sigma^2 \in(k) = \sigma^2 \in(k-i) \), and the \( p \)-dimensional initial signal vector \( x_0 = [x(-p) \cdots x(-2) x(-1)]^T \) is known to have mean \( E[x_0] = \mu_0 \) and covariance matrix \( E[(x_0 - \mu_0)(x_0 - \mu_0)^T] = \Sigma_0 \). The parameters \( a_i, i = 1, 2, \cdots, p \) are given or can be obtained from parameter estimation algorithms.

In the derivation of the multirate state-space model, the order \( p \) of the AR model (7) must be equal to the length \( L \) of the analysis filter bank. In the case that \( p \) is less than \( L \), the order of the AR model can be augmented to \( L \) by adding some terms with zero coefficients to the model, i.e.

\[
x(k) + a_1x(k-1) + \cdots + a_px(k-p) + \cdots + a_Lx(k-L) = \nu(k)
\] (8)

with

\[
a_i = 0 \text{ for } i = p+1, p+2, \cdots, L.
\] (9)
The initial signal vector \( \mathbf{x}_0 \) can also be augmented to the one \( \mathbf{x}_0 = [x(-L) \cdots x(-2) \ x(-1)]^T \) with dimension \( L \times 1 \), which is assumed to have mean \( E[\mathbf{x}_0] = \mathbf{m}_0 \) and covariance matrix \( E[(\mathbf{x}_0 - \mathbf{m}_0)(\mathbf{x}_0 - \mathbf{m}_0)^T] = \mathbf{P}_0 \). On the other hand, if \( p \) is larger than \( L \), the length \( L \) can be increased to \( p \) by adding zeros to each of the analysis filters.

The AR signal process in (7) can be rewritten in the following state-space model [15], [17]:

\[
\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{b}\nu(k+1)
\]

where the state vector \( \mathbf{x}(k) = [x(k-L+1) \ x(k-L+2) \cdots x(k)]^T \) and its dimension is \( L \times 1 \). The parameter matrix/vector \( \mathbf{A} \) and \( \mathbf{b} \) are, respectively, as follows:

\[
\mathbf{A} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \cdots & \vdots \\
0 & \cdots & 0 & 1 & 0 \\
-\mathbf{a}_L & \cdots & -\mathbf{a}_2 & -\mathbf{a}_1 & 1
\end{bmatrix}
\]

\[
\mathbf{b} = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}
\]

Remark: If the input signal is modeled as an ARMA process, i.e.

\[
x(k) + a_1x(k-1) + \cdots + a_Lx(k-L) = \nu(k) + b_1\nu(k-1) + \cdots + b_q\nu(k-q)
\]

it can also be written in the form of (10). The parameter matrix/vector \( \mathbf{A} \), \( \mathbf{b} \), and state vector \( \mathbf{x}(k) \) are of the following forms:

\[
\mathbf{A} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \cdots & \vdots \\
0 & \cdots & 0 & 1 & 0 \\
-\mathbf{a}_L & \cdots & -\mathbf{a}_2 & -\mathbf{a}_1 & 1
\end{bmatrix}
\]

\[
\mathbf{b} = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \cdots & \vdots \\
0 & \cdots & 0 & 0 & 0 \\
0 & b_2 & b_1 & -\mathbf{a}_L & -\mathbf{a}_2 & -\mathbf{a}_1
\end{bmatrix}
\]

\[
x(k) = \begin{bmatrix}
\nu(k) \\
x(k-L+1) \\
\vdots \\
x(k)
\end{bmatrix}
\]

The output set of the \( M \) analysis filters at time \( k \) can be represented by the vector \( \mathbf{u}(k) \) in (5). Using (14), \( \mathbf{u}(k) \) can be expressed in terms of the input signal vector \( \mathbf{x}(k) \) by

\[
\mathbf{u}(k) = \begin{bmatrix} <h_1, \mathbf{x}(k)> \cdots <h_M, \mathbf{x}(k)> \end{bmatrix}^T
\]

\[
\triangleq \mathbf{C}\mathbf{x}(k)
\]

where \( \mathbf{C} \) is an \( M \times L \) matrix whose \( i \)-th row is composed of the coefficients of the \( i \)-th analysis filter, i.e.

\[
\mathbf{C} = \begin{bmatrix}
h_1(L-1) & h_1(L-2) & \cdots & h_1(0) \\
h_2(L-1) & h_2(L-2) & \cdots & h_2(0) \\
\vdots & \vdots & \ddots & \vdots \\
h_M(L-1) & h_M(L-2) & \cdots & h_M(0)
\end{bmatrix}
\]

Remark: If the input signal is modeled as an ARMA process, the vector \( \mathbf{h}_i \) and the matrix \( \mathbf{C} \) are of the following forms:

\[
\mathbf{h}_i = \begin{bmatrix} 0 & \cdots & 0 & h_i(L-1) & h_i(L-2) & \cdots & h_i(0) \end{bmatrix}^T
\]

\[
\mathbf{C} = \begin{bmatrix}
0 & \cdots & 0 & h_1(L-1) & h_1(L-2) & \cdots & h_1(0) \\
0 & \cdots & 0 & h_2(L-1) & h_2(L-2) & \cdots & h_2(0) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & h_M(L-1) & h_M(L-2) & \cdots & h_M(0)
\end{bmatrix}
\]

The dimensions of \( \mathbf{h}_i \) and \( \mathbf{C} \) are \((L+q) \times 1 \) and \( M \times (L+q) \), respectively.

Combining (10) and (16), the subband signal vector \( \mathbf{u}(k) \) can be described by the following state-space form:

\[
\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{b}\nu(k+1)
\]

\[
\mathbf{u}(k) = \mathbf{C}\mathbf{x}(k), \text{ for } k = 0, 1, 2, \cdots
\]

The state-space form (19) is called the basic model. In order to include different time scale evolution of (19) after subsampling and upsampling in the filter bank systems, the basic model has to be modified into a block signal model in the next subsection.

B. Block Signal Model

From Fig. 2, we know the received noisy subband sequence \( \{\mathbf{y}(k), k = 0, R, 2R, \cdots\} \) and the desired signal sequence \( \{x(k), k = 0, 1, 2, \cdots\} \) are now evolved in different time scales. In order to combine them into a single system model with the same time scale, we must transform (19) into an equivalent block generation model with time scale \( k = iR \) to match the time scale of the available noisy subband sequence \( \{\mathbf{y}(k), k = 0, R, 2R, \cdots\} \) in (5). From (19), we can get an equivalent block representation of subband signal vector \( \mathbf{u}(k) \) with block length 2 as follows [15]:

\[
\begin{bmatrix}
\mathbf{u}(k) \\
\mathbf{u}(k+1)
\end{bmatrix} = \begin{bmatrix}
\mathbf{A}^2 & \mathbf{A}\mathbf{b} \\
\mathbf{b}\mathbf{A} & \mathbf{b}\mathbf{b}^T
\end{bmatrix}\begin{bmatrix}
\mathbf{x}(k) \\
\mathbf{x}(k+1)
\end{bmatrix} + \begin{bmatrix}
\mathbf{0} \\
\mathbf{0}
\end{bmatrix}\begin{bmatrix}
\nu(k+1) \\
\nu(k+2)
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathbf{u}(k) \\
\mathbf{u}(k+1)
\end{bmatrix} = \begin{bmatrix}
\mathbf{C} & \mathbf{b}
\end{bmatrix}\begin{bmatrix}
\mathbf{x}(k) \\
\mathbf{x}(k+1)
\end{bmatrix} + \begin{bmatrix}
\mathbf{0} \\
\mathbf{0}
\end{bmatrix}\begin{bmatrix}
\nu(k+1) \\
\nu(k+2)
\end{bmatrix}
\]

where \( k = 0, 2, 4, \cdots \). By a similar procedure, we can obtain an equivalent block generation representation of (19) with block length \( R \) as follows:

\[
\begin{bmatrix}
\mathbf{x}(k+R) \\
\mathbf{u}(k)
\end{bmatrix} = \begin{bmatrix}
\mathbf{A}^R & \mathbf{B}_R \\
\mathbf{C}_R & \mathbf{D}_R
\end{bmatrix}\begin{bmatrix}
\mathbf{x}(k) \\
\mathbf{v}(k)
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathbf{x}(k+R) \\
\mathbf{u}(k)
\end{bmatrix} = \begin{bmatrix}
\mathbf{A}^R & \mathbf{B}_R \\
\mathbf{C}_R & \mathbf{D}_R
\end{bmatrix}\begin{bmatrix}
\mathbf{x}(k) \\
\mathbf{v}(k)
\end{bmatrix}
\]
where \( k = 0, R, 2R, \cdots \) and the vector of block subband signal \( u_{R}(k) \), the vector of block driving signal \( v(k) \) and the parameter matrices \( B_{R}, C_{R}, \) and \( D_{R} \), respectively, are as follows:

\[
\begin{align*}
\mathbf{u}_{R}(k) &= \begin{bmatrix}
\mathbf{u}(k) \\
\mathbf{u}(k + 1) \\
\vdots \\
\vdots \\
\mathbf{u}(k + R - 1)
\end{bmatrix}, \\
\mathbf{v}(k) &= \begin{bmatrix}
\mathbf{v}(k + 1) \\
\mathbf{v}(k + 2) \\
\vdots \\
\mathbf{v}(k + R)
\end{bmatrix}, \\
\mathbf{B}_{R} &= \begin{bmatrix}
\mathbf{A}^{R-1} \mathbf{b} \\
\mathbf{A}^{R-2} \mathbf{b} \\
\vdots \\
\mathbf{A} \mathbf{b} \\
\mathbf{b}
\end{bmatrix}, \\
\mathbf{C}_{R} &= \begin{bmatrix}
\mathbf{C} \\
\mathbf{C} \mathbf{A} \\
\vdots \\
\mathbf{C} \mathbf{A}^{R-1}
\end{bmatrix}, \\
\mathbf{D}_{R} &= \begin{bmatrix}
0 \\
\mathbf{C} \mathbf{b} \\
\vdots \\
\mathbf{C} \mathbf{A}^{R-2} \mathbf{b} \\
\mathbf{C} \mathbf{b}
\end{bmatrix}
\end{align*}
\tag{22}
\]

Now, the equivalent block representation of the basic model (19) has been constructed. We know from (5) that output \( u_{R}(k) \) of the system (21) is decimated, corrupted by the additive noise \( n(k) \), and then upsampled to obtain the noisy subband signal \( \hat{y}(k) \). More detailed description is given in the next subsection.

**C. Multirate Signal Model**

The subband signal sequence \( \{u(k), k = 0, 1, 2, \cdots\} \) is decimated by a factor \( R \), corrupted by additive noise sequence \( \{n(k), k = 0, R, 2R, \cdots\} \), and upsampled to produce the available noisy subband sequence \( \{\hat{y}(k), k = 0, R, 2R, \cdots\} \). The corrupting noises \( \{n_{i}(k), k = 0, R, 2R, \cdots\} \) are assumed to be mutually uncorrelated, zero-mean, white Gaussian noises with time-varying covariance \( E[n_{i}(k)n_{j}(j)] = \gamma_{i}(k)\delta(k - j) \) and uncorrelated with driving signal \( v(k) \). After the output \( u_{R}(k) \) of the block system (21) is decimated by a factor \( R \), only the first vector element \( u(k) \) of the vector \( u_{R}(k) \) is reserved, and the other \( R - 1 \) vector elements are dropped, i.e.

\[
\begin{align*}
\mathbf{x}(k + R) &= \mathbf{A}^{R} \mathbf{x}(k) + \mathbf{B}_{R} \mathbf{v}(k) \\
\mathbf{u}(k) &= \mathbf{C} \mathbf{x}(k) \text{for } k = 0, R, 2R, \cdots
\end{align*}
\tag{23}
\]

From (5) and (23), we know that the noisy subband signals \( \hat{y}_{i}(k), i = 1, 2, \cdots, M \) can be modeled as follows:

\[
\begin{align*}
\mathbf{x}(k + R) &= \mathbf{A}^{R} \mathbf{x}(k) + \mathbf{B}_{R} \mathbf{v}(k) \\
\hat{y}(k) &= \mathbf{C} \mathbf{x}(k) + \mathbf{n}(k)
\end{align*}
\tag{24}
\]

where \( k = 0, R, 2R, \cdots \). Now, the multirate state-space signal model, combining different sampling rate signals of an \( M \)-band filter bank system in Fig. 2, has been established in (24). The input signal is embedded in the state vector, and the input signal reconstruction problem of the \( M \)-band filter bank system becomes how to estimate the state vector \( \mathbf{x}(k) \) in (24). Since the time evolution in the multirate system (24) is \( k = 0, R, 2R, \cdots \), the technique of multirate Kalman filter will be employed to obtain the optimal state estimate \( \hat{x}(k) \) for \( k = 0, R, 2R, \cdots \). The synthesis filters in conventional filter bank systems can be completely replaced by the multirate Kalman synthesis filter. Further design procedure is given in the next subsection.

**IV. OPTIMAL SIGNAL RECONSTRUCTION IN NOISY FILTER BANK SYSTEMS**

From the analysis in the previous section, the received signals \( \hat{y}_{i}(k) \) in the \( M \)-band filter bank system with corrupting noises \( n_{i}(k) \) can be modeled by a multirate state-space form in (24) for \( k = 0, R, 2R, \cdots \). The optimal signal reconstruction will be achieved via a multirate Kalman synthesis filter (Fig. 2).

Before the design of multirate Kalman synthesis filter for optimal signal reconstruction in a noisy filter bank system, the statistical characteristics of block driving noise \( v(k) \) and block subband noise \( n(k) \) in (24) must be discussed. It is seen from (22) that the block driving noise \( v(k) \), consisting of \( v(k + 1) \cdots v(k + R) \), is a zero mean, white Gaussian noise...
vector, whose covariance matrix is of the following diagonal form:

$$E[v(k)v^T(i)] = 
\begin{bmatrix}
\sigma^2_v(k+1) & 0 \\
0 & \vdots \\
& \sigma^2_v(k+R)
\end{bmatrix} \delta(k-i)$$

$$\triangleq Q_R(k)\delta(k-i), \text{ for } k, i = 0, R, 2R, \cdots \quad (25)$$

where $\delta(k)$ is the Kronecker delta function. The covariance matrix $Q_R(k)$ is an $R \times R$ positive matrix and will be used in the multirate Kalman filtering algorithm for input signal estimation.

Similarly, the statistical characteristics of the block subband noise $n(k)$ can be obtained from a white Gaussian assumption of subband noise $n_i(k), i = 1, 2, \cdots, M$. The block subband noise $n(k)$ is a zero mean, white Gaussian vector whose covariance matrix is of the following diagonal form:

$$E[n(k)n^T(i)] =
\begin{bmatrix}
r_1(k) & 0 \\
0 & \vdots \\
r_{M}(k)
\end{bmatrix} \delta(k-i)$$

$$\triangleq R_M(k)\delta(k-i) \text{ for } k, i = 0, R, 2R, \cdots \quad (26)$$

where $R_M(k)$ is an $M \times M$ positive matrix. Furthermore, the subband noises $n_i(k)$ are assumed to be uncorrelated with the driving signal $v(k)$. Therefore, the block subband noise $n(k)$ remains uncorrelated with the block driving noise $v(k)$.

Remark: The quantization noises would depend on the input signal and may be correlated for adjacent subbands. In this situation, the covariance matrix in (26) must be modified as follows:

$$E[n(k)n^T(i)] =
\begin{bmatrix}
r_{11}(k) & r_{12}(k) & \cdots & r_{1M}(k) \\
r_{21}(k) & r_{22}(k) & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
r_{M1}(k) & r_{M2}(k) & \cdots & r_{MM}(k)
\end{bmatrix} \delta(k-i)$$

$$\triangleq R_M(k)\delta(k-i) \quad (27)$$

where $r_{ij}(k), i \neq j$ denotes the correlation between the $i$th subband noise and $j$th subband noise.

Since the block subband noise $n(k)$ and the block driving noise $v(k)$ of the multirate state-space signal model (24) are uncorrelated, the multirate Kalman filter can be applied to this multirate system as a minimum-variance estimator of the state vector $x(k)$, given the available noisy subband signal $\hat{y}(k)$ for $k = 0, R, 2R, \cdots$. The multirate Kalman filtering algorithm for state vector estimation in the multirate state-space model (24) of the filter bank system is derived as follows:

$$x(k+R) = [I - K(k+R)C]A_Rx(k) + K(k+R)\hat{y}(k+R)$$

$$\triangleq x(k+R) = [I - K(k+R)C]A_Rx(k) + K(k+R)\hat{y}(k+R)$$

where $I$ is an identity matrix with adequate dimension, and the time scale evolves by $k = 0, R, 2R, \cdots$. The above Kalman filter gain $K(k)$ will be recursively updated as follows:

$$K(k+R) = P(k+R)C^T [CP(k+R)C^T + R_M(k+R)]^{-1}$$

$$P(k+R) = [I - K(k+R)C]P(k+R)$$

$$P(k+R) = [I - K(k+R)C]P(k+R)$$

where $P(k+R) = E[\hat{x}(k+R)\hat{x}(k+R)^T]$ and $P(k+R) = E[\hat{x}(k+R)\hat{x}(k+R)^T]$ are the filtering and prediction state error covariance matrices, respectively. Both of them provide performance measures of the recursive state estimator (28) and can be calculated prior to any processing of real data sequence $\{\hat{y}(k), k = 0, R, 2R, \cdots\}$. After the optimal state estimate $\hat{x}(k)$ is obtained, the optimal signal reconstruction in the filter bank is given by

$$\hat{x}(k+R+1) = [0 \ I_R] \hat{x}(k)$$

$$\hat{x}(k+R+2) = [0 \ I_R] \hat{x}(k)$$

where $I_R$ is an $R \times R$ identity matrix. If a delay of $l$ time unit is permissible, an optimal reconstruction of input signal with delay $l$ is obtained as

$$\hat{x}(k-l+1) = [0 \ I_R \ 0_{R \times l}] \hat{x}(k)$$

$$\hat{x}(k-l+2) = [0 \ I_R \ 0_{R \times l}] \hat{x}(k)$$

$$\hat{x}(k-l) = [0 \ I_R \ 0_{R \times l}] \hat{x}(k)$$

where $l = 0, R, 2R, \cdots$.

As the delay $l$ increases, the performance of reconstruction is improved. In practice, when the delay is equal to two or three times of the dominant time constant of the multirate state-space model (24), a nearly optimal performance is achieved.

The initial setting of the above multirate Kalman filtering algorithm is discussed below. The initial state estimate $\hat{x}(0)$
and the initial state error covariance matrix $P(0|0)$ must be set up by another initial-setting filter as follows:

$$\dot{x}(0) = [I - K(0)C]A\dot{x}(1 - 1) + K(0)\dot{y}(0)$$

$$P(0) = [I - K(0)C]P(0|0)$$

(32)

where the optimal initial guess $\dot{x}(1)$ is taken to be the mean $\mu_0$ of the initial signal vector $x_0$ (in Section III), i.e., $\dot{x}(1) = E[x(1 - 1) \cdots x(1) x(1 - 1)] = \mu_0$, and the optimal initial Kalman gain $K(0)$ takes prior knowledge of the covariance matrix $P_0$ of the initial signal vector $x_0$ as follows:

$$K(0) = P(0|0)^{-1}C^T[CP(0|0)^{-1}C^T + R_M(0)^{-1}]$$

$$P(0|0) = AP(-1|0)A^T + \sigma_x^2(0)b^Tb^T$$

(33)

V. SIMULATION AND DISCUSSION

In this section, two numerical examples of noisy two-band and five-band filter bank systems are given to illustrate the performance of optimal signal reconstruction with the proposed multirate Kalman synthesis filter. The input signal is modeled by an AR or ARMA process, and the driving noise in the signal model is white Gaussian distributed. Three test cases are executed in the computer simulation, each has the length of 10,000 signal points. Some notations are defined before further discussion of numerical simulation. Let us define

$$SNR_{st} = 10\log_{10} \left( \frac{\sum_{k=jR}^j x_k^2(k)}{\sum_{k=jR}^j n_k^2(k)} \right)$$

for $i = 1, 2, \cdots, M$. (34)

It shows the SNR of the $i$th subband signal to the $i$th subband noise. For simplicity, we denote $SNR_s = SNR_{st}$, for $i = 1, 2, \cdots, M$ in this paper because, in our simulation examples, the SNR $SNR_{st}$ is the same in each subband. Another notation $SNR_r$ denotes the SNR of the input signal to the reconstruction error

$$SNR_r = 10\log_{10} \left( \frac{\sum_{k} x_k^2(k)}{\sum_{k} (x(k) - \hat{x}(k))^2} \right).$$

(35)

The performance index $SNR_r$ serves as the improvement measure, and it is an important pointer indicating the degree of signal reconstruction in filter bank systems.

A. Case 1: Six-Pole AR Process

In the following simulation examples, the input signal $x(k)$ is produced by an AR generation model with order $p = 6$. The parameter set $\{a_1, a_2, \cdots, a_p\}$ of AR signal model is chosen to have six poles at $[0.5 \pm j0.3, 0.7 \pm j0.4, 0.8 \pm j0.5]$, respectively. For the impulse responses of the conventional analysis and synthesis filters, the design results in [11] are used. The frequency responses of these filters are shown in Figs. 3–6.

Some simulation results of the two-band filter bank system are listed in Table I. If the filter bank system is free of subband noises, the $SNR_r$ is very high. However, as additive subband noises appear, $SNR_r$ is obviously down. The signal reconstruction performance of a conventional filter bank is seen to be sensitive to the subband noises. When $SNR_s$ becomes small, the performance is even worse. The multirate Kalman synthesis filter (with 30 sampling time delay) is used for the attenuation of subband noises in the two-band filter bank example. The improvement of signal reconstruction $SNR_r$ (or the noise attenuation ability) of the multirate Kalman synthesis filter is remarkable in all test cases. In the $SNR_s = 0$ dB case, the $SNR_r$ improvement is about 7 dB. The noise-free case is also simulated. It is observed even in the noise-free case, the performance of the multirate Kalman synthesis filter is better than the conventional filter bank. The reason is only an approximation solution is obtained for perfect reconstruction (3) in conventional filter bank designs. Our approach, however, is an optimal solution taking into consideration the statistical properties of the input signal.

From the results of Table I, the proposed multirate Kalman synthesis filter for five-band filter bank system is also seen to achieve an improvement in signal reconstruction $SNR_r$. In this case, the multirate Kalman synthesis filter (with 50 sampling time delay) employed is employed in the five-band system to attenuate the effect of subband noises. If subband noises are added, the $SNR_r$ is obviously down. In the $SNR_s = 0$ dB case, the $SNR_r$ improvement is about 9 dB. The performance improvement is also seen as obvious in both noise-corrupted and noise-free cases.

B. Case 2: Ten-Pole AR Process

The input in this case is generated by an AR model with poles at

$$-0.9, -0.5 \pm j0.3, -0.7 \pm j0.4,$$

$$-0.5, -0.6, -0.7, -0.8, -0.9.$$
TABLE III  
Performance Comparison $SNR_e$ of Noisy Two-Band Filter Bank Systems for an AR(10) Input Signal

<table>
<thead>
<tr>
<th>$SNR_e$</th>
<th>Kalman synthesis filter</th>
<th>conventional filter bank</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>11.25</td>
<td>0.27</td>
</tr>
<tr>
<td>5</td>
<td>15.30</td>
<td>5.12</td>
</tr>
<tr>
<td>10</td>
<td>19.21</td>
<td>10.36</td>
</tr>
<tr>
<td>15</td>
<td>23.38</td>
<td>15.19</td>
</tr>
<tr>
<td>20</td>
<td>27.47</td>
<td>20.19</td>
</tr>
<tr>
<td>25</td>
<td>32.23</td>
<td>25.38</td>
</tr>
<tr>
<td>$\infty$ (noise-free)</td>
<td>155.77</td>
<td>83.35</td>
</tr>
</tbody>
</table>

TABLE IV  
Performance Comparison $SNR_e$ of Noisy Five-Band Filter Bank Systems for an AR(10) Input Signal

<table>
<thead>
<tr>
<th>$SNR_e$</th>
<th>Kalman synthesis filter</th>
<th>conventional filter bank</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>12.45</td>
<td>-1.11</td>
</tr>
<tr>
<td>5</td>
<td>16.36</td>
<td>4.21</td>
</tr>
<tr>
<td>10</td>
<td>20.72</td>
<td>9.15</td>
</tr>
<tr>
<td>15</td>
<td>25.69</td>
<td>14.27</td>
</tr>
<tr>
<td>20</td>
<td>30.37</td>
<td>19.55</td>
</tr>
<tr>
<td>25</td>
<td>35.17</td>
<td>24.29</td>
</tr>
<tr>
<td>$\infty$ (noise-free)</td>
<td>169.13</td>
<td>91.60</td>
</tr>
</tbody>
</table>

The conventional analysis and synthesis filters adopted in this case are the same as those in Case 1.

The time delay of the multirate Kalman synthesis filter in the two-band filter bank is set to 30. The simulation results are listed in Table III. The time delay in the five-band filter bank in our design is set to 50, and the simulation results are listed in Table IV. It can be seen that the performance improvements in this case are even more significant than those in Case 1. Our explanation of this phenomenon is in the ten-pole case, the spectrum of the input signal is more narrowbanded, and the autocorrelation is stronger than the six-pole case. As a result, the estimation of the Kalman filter is more precise. If the spectrum of input signal is more widebanded, the performance improvement of Kalman synthesis filter is not as significant.

Tables I-IV illustrate the performance of signal reconstruction for each of the two- and five-band filter banks equipped with the proposed multirate Kalman synthesis filter. All of them show that the improvement degree of signal reconstruction by the proposed method is attractive in all test cases.

Remarks:

1) The multirate Kalman synthesis filter reconstructs the input signal from the viewpoint of optimal state estimation, in contrast to the viewpoint of perfect reconstruction in conventional filter bank systems. It is capable of optimally reconstructing the input signal provided the analysis filter bank covers the whole frequency band so that no information is lost in transmission. Even in the case that the analysis filters are not well bandlimited, the performance is still satisfactory. A simulation experiment was performed to demonstrate this aspect. Simulation result is described below. Consider a two-band analysis filter bank with a lowpass filter $(1 + z^{-1})/2$ and a highpass filter $(1 - z^{-1})/2$. Their frequency responses are plotted in Fig. 7. The reconstruction performance of the multirate Kalman synthesis filter for the AR(6) input signal in Case 1 is listed in Table V. Comparing this result with that in Table I, we see the reconstruction performance is still very satisfactory. This observation indicates that the reconstruction performance is not influenced by the frequency characteristics of the analysis filter bank. However, in many application areas (such as subband coding), analysis filter banks with good frequency characteristics are required; therefore, the subband signals are well bandlimited.

2) Since perfect reconstruction is not important for our proposed algorithm, the analysis filter bank can be any orthogonal transformation or any linear transformation. For example, in a wavelet filter bank case, the analysis filter bank is based on wavelet transform. The coefficients of the two-band wavelet analysis filter bank adopted in our simulation is derived from [18]. The simulation result by the proposed optimal synthesis filter is listed in Table VI. The reconstruction performance is also very satisfactory.

VI. Conclusion

The filter bank systems with perfect reconstruction property have been widely addressed in the literature under the assump-
tion of noise-free channel/coding subband paths. However, effects of transmission channel and operation of the encoding algorithm cause the subband signals to be noisy [13, 14, 19]; therefore, the output signal is no longer the same as the input signal. If the corrupting noises possess considerable energy compared with the subband signals, the deterioration of the signal reconstruction performance of filter bank systems will be serious. Hence, through the study of multirate reconstruction theory in this paper, we wish to contribute some efforts to upgrade the filter bank design technique under the noisy subband case.

We have treated the optimal signal reconstruction problem in this paper from the multirate space-state modeling viewpoint. A multirate Kalman synthesis filter is equipped in the subband paths to achieve optimal signal reconstruction. Based on the optimal state estimation theory, the proposed multirate Kalman synthesis filter provides the minimum-variance reconstruction of the input signal. From the simulation results, the performance improvement of signal reconstruction in the noisy filter bank systems is found to be remarkable.

Finally, the "optimal signal reconstruction problem" proposed in this paper is adequate to the actual applications such as subband coding systems, which always have noisy channel/coding subband paths. The subject of this paper, therefore, is worthy of study. Furthermore, the potential of the proposed multirate Kalman synthesis filter to specific applications (e.g., speech or image subband coding system with some kind of waveform coding algorithm) shows the the proposed multirate space-state modeling theory will be attractive in the future.

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REFERENCES


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