Robust decision-feedback equalizer design method under parameter perturbation in a transmission channel

Chin-Wei Lin\textsuperscript{a}, Bor-Sen Chen\textsuperscript{a,*}, Tsang-Yi Yang\textsuperscript{b}

\textsuperscript{a}Control and Signal Processing Lab., Department of Electrical Engineering, National Tsing-Hua University, Hsinchu, Taiwan, 30043, ROC
\textsuperscript{b}Southern Information Systems Inc., Hsinchu, Taiwan, ROC

Received 9 September 1994, revised 11 January 1995

Abstract

An optimal decision-feedback equalizer (DFE) design method is proposed in this paper for the signal transmission systems with small parameter perturbation. The perturbative parameters of the dispersive channel and noise model are of probabilistic structure. The proposed DFE minimizes not only the distortion from the channel and noise, but also the effect due to the parameter perturbation of transmission channel and noise model. Hence, it is more robust than the conventional DFE. The calculus of variation technique and spectral factorization method will be employed to treat this problem. It is also discussed for extending the method to the case with unknown but bounded second-order statistics of parameter perturbation. Finally, a numerical example is given to illustrate the simulation result of the proposed method.

Zusammenfassung


Résumé

Nous proposons dans cet article une méthode de conception d'égaliseur à décision par boucle de rétroaction (DFE) optimal pour des systèmes de transmission de signaux ayant une petite perturbation des paramètres. Les paramètres perturbants du canal dispersif et le modèle de bruit sont de structure probabiliste. Le DFE proposé minimise non

\textsuperscript{*} Corresponding author. Tel. 886-35-731155. Fax: 886-35-715971. E-mail: bscchen@athena.ee.nthu.edu.tw

0165-1684/95/$8.50 © 1995 Elsevier Science B.V. All rights reserved
SSDI 0165-1684(95)00014-3
1. Introduction

The recovery of the transmitted signals in the presence of noise and intersymbol interference (ISI) is an important research topic in digital communications. The ISI is caused by the nonideal channel characteristics and limits the achievable transmission rate. The task of an equalizer is to reconstruct the transmitted sequence.

Linear equalizers (LE) [5, 13, 22] have been developed for many years. Although an LE has simple structure, the performance is often unsatisfactory. Especially, if the channel has spectral nulls, an LE exhibits excessive noise enhancement at the frequencies most attenuated by the channel, and the output of the LE can be very noisy. Maximum-likelihood sequence estimator (MLSE) [8] provides another approach to data estimation. The MLSE Viterbi algorithm has the best performance in the bit-error rate. However, if the channel is of long impulse response, the algorithm becomes very complicated unless suboptimal measurements, such as state truncation or prefiltering, are employed [4, 15, 17].

The decision-feedback equalizer (DFE) [7, 11, 14, 23, 24] has a similar computational complexity as the linear equalizer, but it attains almost the same performance as the Viterbi equalizer for many channels [6]. The DFE includes two filters, i.e., a forward filter and a feedback filter, and a detector. It equalizes the channel to produce a sampled causal response and subtract the part of the ISI at the sample time due to past data decisions. It can attain channel inversion without amplifying the noise simultaneously. But the DFE has a drawback, i.e., it propagates decision error [20]. However, due to its advantages, signal reconstruction by a DFE is often considered.

In general, the optimal DFE can be obtained while the transfer function of the channel is precisely known and the statistical properties of the input signal and corrupting noise are also exactly given [23]. However, the transfer function may change with time due to component drift, aging, or temperature variation. Besides, the transmission channel may be inaccurately modeled or identified possibly intentionally, for example, model reduction or linearization, and so on. The characteristics of corrupting noise may also change due to the variation of environment, for example, the influence of weather on communication systems. Hence, we need a design technique for reconstructing the input signal under such variations.

One approach for the equalization problem under parameter perturbation is to use an adaptive scheme [9, 10, 12, 16]. However, the adaptive equalization scheme may be unpractical because of cost and complexity. It is desired to find a fixed robust equalizer which gives an acceptable performance over the range of parameter perturbation. Several recent works for designing linear equalizers have concerned with the noise uncertainty from the viewpoint of the worst case of the second-order statistics. Moustakids and Kassam [18] and Peng and Chen [21] used the minimax approach to design a robust Wiener filter while the frequency response of the transmission channel changes within a given range. Chen and Peng solved the robust deconvolution by channel sensitivity method in [3]. In [2], a robust deconvolution filter was proposed with consideration of the covariances of parameter variation and noise uncertainty. Chen and Chen [1] designed a realizable robust filter from the viewpoint of time domain. The transmission system is described in a state-space model. A robust Kalman filter is obtained based on minimax approach.
Since the DFE has some advantages over the LF, the purpose of this paper is to design a robust DFE for signal reconstruction. An optimal DFE design method is proposed for solving the equalization problem under parameter perturbation in the dispersive channel and corrupting noise. The perturbative parameters are of probabilistic structure. The forward and feedback filters are represented in infinite impulse response (IIR) forms. The proposed DFE minimizes not only the distortion from the channel and noise, but also the effect due to parameter perturbation. Some methods such as calculus of variation technique and spectral factorization will be employed to derive a realizable DFE. It is found that both minimum-phase and nonminimum-phase perturbative transmission systems can be handled. An expression for the minimum mean square error is also derived. Besides, an extension of the design method to the unknown but bounded second-order statistics of the parameter perturbation is also discussed. Since the design method is proposed with consideration of the parameter perturbation, it has the robustness property. Finally, a numerical example is given for illustrating the simulation result of the proposed DFE.

The organization of this paper is as follows. In the next section, the problem formulation and some notations are stated. In Section 3, we represent the mean square estimation error under parameter perturbation as a linear quadratic optimization problem in the frequency domain. Then, the optimal solution of the DFE is derived based on calculus of variation technique and spectral factorization. An extension of the design method to the unknown but bounded second-order statistics of the parameter perturbation is also discussed. A numerical example is presented in Section 4 to illustrate the performance of the proposed method. Finally, some conclusions are given in Section 5.

2. System description and problem formulation

Fig. 1 illustrates the discrete-time channel model of the DFE system under investigation. The transmitted signal \( u(k) \) is assumed to be zero-mean and white with variance \( \sigma^2_u \). The model for the dispersive channel is represented by a stable rational function \( q^{-d}H(q^{-1}, \alpha + \delta \alpha) \) in which \( q^{-1} \) represents the backward shift operator \( (q^{-1}u(k) = u(k - 1)) \). The \( N \)-dimensional column vector \( \alpha \) (the superscript \( T \) denotes the transpose).

\[
\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_N]^T, \tag{1}
\]

represents the mean or expected values of the channel parameters, and any deviation from the mean is denoted by a column vector \( \delta \alpha \). Thus

\[
E\{\delta \alpha_i\} = 0, \quad i = 1, 2, \ldots, N, \tag{2}
\]

where \( E\{\cdot\} \) denotes the expected value and \( \delta \alpha_i \) is the \( i \)th element of \( \delta \alpha \). It is assumed that the covariance matrix

\[
\Sigma_\alpha = E\{\delta \alpha(\delta \alpha)^T\} = \begin{bmatrix} \sigma_{\alpha_1}^2 & \cdots & \sigma_{\alpha_1\alpha_N}^2 \\ \vdots & \ddots & \vdots \\ \sigma_{\alpha_N\alpha_1}^2 & \cdots & \sigma_{\alpha_N}^2 \end{bmatrix}, \quad \sigma_{\alpha_i}^2 = \sigma_{\alpha_i}^2 = E\{\delta \alpha_i \delta \alpha_j\} \tag{3}
\]

\[
\Sigma_\alpha = E\{\delta \alpha(\delta \alpha)^T\} = \begin{bmatrix} \sigma_{\alpha_1}^2 & \cdots & \sigma_{\alpha_1\alpha_N}^2 \\ \vdots & \ddots & \vdots \\ \sigma_{\alpha_N\alpha_1}^2 & \cdots & \sigma_{\alpha_N}^2 \end{bmatrix}, \quad \sigma_{\alpha_i}^2 = \sigma_{\alpha_i}^2 = E\{\delta \alpha_i \delta \alpha_j\} \tag{3}
\]
is known a priori, and that the variations $\delta x_i$ are small and independent of the signals in the system. $q^{-d}$ represents a transmission delay of $d$ samples. The received signal $y(k)$ is corrupted by a colored noise $w(k)$,

$$y(k) = q^{-d}H(q^{-1}, \alpha + \delta \alpha)u(k) + w(k),$$

(4)

where $w(k)$ is described as

$$w(k) = D(q^{-1}, \beta + \delta \beta)n(k).$$

(5)

In (5), $n(k)$ is a zero-mean white noise with variance $\sigma_n^2$ and is independent of $u(k)$. Also, the $M$-dimensional column vector $\beta$,

$$\beta = [\beta_1, \beta_2, \ldots, \beta_M]^T,$$

(6)

represents the expected values of the parameters in the noise model, and any deviation from the mean is denoted by $\delta \beta$. Thus

$$E\{\delta \beta_i\} = 0, \quad i = 1, 2, \ldots, M.$$  

(7)

It is assumed that the covariance matrix

$$\Sigma_\beta = E\{\delta \beta \delta \beta^T\} = \{\sigma_{ij}^\beta\}$$

(8)

is also known a priori, and that the variations $\delta \beta_i$ are small and independent of $n(k)$.

A DFE makes the estimate

$$\hat{u}(k-l|k) = F(q^{-1})y(k) - q^{-1}B(q^{-1})\hat{u}(k-l)$$

(9)

about $u(k-l)$ and then converts it to a final decision $\hat{u}(k-l)$ with a nonlinear memoryless circuit (e.g., the circuit is an $m$-output quantizer if the input signals $\{u(k)\}$ are $m$-ary). Here, the smoothing lag $l$ is supposed to satisfy the relationship $l \geq d$.

Given the received signal $y(k)$, the problem lies in finding a realizable DFE $(F(q^{-1}), B(q^{-1}))$ such that the mean square error $J$ is minimized, where

$$J = E\{e^2(k-l)\}$$

$$= E\{(k-l) - \hat{u}(k-l|k))^2\}. \quad (10)$$

Same as the most previous discussions of the DFE problem, we suppose that the past decisions are correct (i.e., $\{\hat{u}(k-l-1)\} = \{u(k-l-1)\}$). In the next section, we derive the optimal DFE $(F(q^{-1}), B(q^{-1}))$ such that $E\{e^2(k-l)\}$ is minimized. Some notations are first introduced here.

**Notations**

- $H = H(q^{-1})$ rational function in the complex variable $z^{-1}$, substituting for $q^{-1}$.
- $H^* = H^T(z)$ rational function which is the complex conjugate and transpose of $H(z^{-1})$.
- $H(\alpha + \delta \alpha)$ rational function which is equal to $H(z^{-1}, \alpha + \delta \alpha)$.
- $\{H\}_+ / \{H\}_-$ analytic part outside/inside the unit circle on the $z$-plane. That is, $H = \{H\}_+ + \{H\}_-$.

3. The optimal decision-feedback equalizer

3.1. Analysis of the performance index

The transmission system is described in (4), (5), and (9). Following those relationships, the error signal can be expressed as

$$e(k-l) = u(k-l) - \hat{u}(k-l|k)$$

$$= (q^{-l} - q^{-d}F(q^{-1})H(q^{-1}, \alpha + \delta \alpha)$$

$$+ q^{-l}B(q^{-1}))u(k)$$

$$- F(q^{-1})D(q^{-1}, \beta + \delta \beta)n(k).$$

(11)

Since $u(k)$ and $n(k)$ are mutually independent, applying Parseval’s theorem yields

$$J = \frac{1}{2\pi} \int\int \{z^{-l} - z^{-d}FH(\alpha + \delta \alpha) + z^{-l-1}B\}^\ast$$

$$\times \{z^{-l} - z^{-d}FH(\alpha + \delta \alpha) + z^{-l-1}B\} \sigma_u^2$$

$$+ E\{D^*(\beta + \delta \beta)F^*FD(\beta + \delta \beta)\} \sigma_n^2 \frac{dz^2}{z^2},$$

(12)

where \(\int\int\) denotes integration around the unit circle.

Let $\delta H(\alpha)$ and $\delta D(\beta)$ be defined as follows:

$$\delta H(\alpha) \triangleq H(z + \delta \alpha) - H(z),$$

(13)

$$\delta D(\beta) \triangleq D(\beta + \delta \beta) - D(\beta).$$
Now, expanding $H(x + \delta x)$ in a Taylor series about $x$, we get
\[
H(x + \delta x) = H(x) + \sum_{i=1}^{\infty} \left[ \frac{\partial H(x)}{\partial x_i} \right] \delta x_i + \text{higher-order terms.} \tag{14}
\]

Since $\delta x_i$ are so small that the higher-order terms can be neglected, $\delta H(x)$ is approximated as
\[
\delta H(x) \approx \sum_{i=1}^{N} \left[ \frac{\partial H(x)}{\partial x_i} \right] \delta x_i = (\delta x)^T \frac{\partial H(x)}{\partial x}, \tag{15}
\]
and the first- and second-order statistics of $\delta H(x)$ are respectively
\[
E\{\delta H(x)\} = \sum_{i=1}^{N} \left[ \frac{\partial H(x)}{\partial x_i} \right] E\{\delta x_i\},
\]
\[
E\{\delta H^*(x)\delta H(x)\} = \left( \frac{\partial H(x)}{\partial x} \right)^* \Sigma_x \frac{\partial H(x)}{\partial x} = \Sigma_{\delta H}. \tag{16}
\]

Similarly, $\delta D(\beta)$ is approximated as
\[
\delta D(\beta) \approx \sum_{i=1}^{M} \frac{\partial D(\beta)}{\partial \beta_i} \delta \beta_i = (\delta \beta)^T \frac{\partial D(\beta)}{\partial \beta}, \tag{17}
\]
and the first- and second-order statistics of $\delta D(\beta)$ are respectively
\[
E\{\delta D(\beta)\} = \sum_{i=1}^{M} \left[ \frac{\partial D(\beta)}{\partial \beta_i} \right] E\{\delta \beta_i\},
\]
\[
E\{\delta D^*(\beta)\delta D(\beta)\} = \left( \frac{\partial D(\beta)}{\partial \beta} \right)^* \Sigma_{\delta \beta} \frac{\partial D(\beta)}{\partial \beta} = \Sigma_{\delta D}. \tag{18}
\]

Expanding the above equation yields
\[
J = \frac{1}{2\pi} \int E\{(z^{-1} - z^{-d} F^* H^*(x) + z^{l+1} B^*) \}
\times (z^{-1} - z^{-d} F H(x) + z^{l+1} B) \sigma_n^2
+ E\{[D(\beta) + \delta D(\beta)] F^* [D(\beta) \delta D(\beta)] \} \sigma_n^2 \frac{dz}{z}. \tag{19}
\]

Since the parameter perturbations $\delta x$ and $\delta \beta$ are independent of the signals, the performance index can be reduced to
\[
J = \frac{1}{2\pi} \int (z^{-1} - z^{-d} F^* H^*(x) + z^{l+1} B^*)
\times (z^{-1} - z^{-d} F H(x) + z^{l+1} B) \sigma_n^2
+ F^* \Sigma_{\delta H} F \sigma_n^2 + D^* F^* D(\beta) \sigma_n^2
\frac{dz}{z} \tag{20}
\]
and the problem is equivalent to finding the optimal DFE $(F(z^{-1}), B(z^{-1}))$ such that (23) is minimized.

**Remark.** The second and fourth terms in (23) represent the error effects due to the parameter perturbation from the channel $H(x)$ and noise model $D(\beta)$, respectively. Hence, the proposed DFE minimizes not only the distortion from the channel and noise, but also the effect due to the parameter perturbation.
3.2. The design method

Let
\[
F(z^{-1}) = F_o(z^{-1}) + \varepsilon_1 \eta_1(z^{-1}),
\]
\[
B(z^{-1}) = B_o(z^{-1}) + \varepsilon_2 \eta_2(z^{-1}),
\]
where \((F_o(z^{-1}), B_o(z^{-1}))\) is the optimal DFE, \(\eta_1(z^{-1})\) and \(\eta_2(z^{-1})\) are any realizable rational functions with all poles in \(|z| < 1\), and \(\varepsilon_1, \varepsilon_2\) is a small real vector. Substituting (24) into (23) yields
\[
J = \frac{1}{2\pi j} \oint (z^{-1} - z^{-d}(F_o + \varepsilon_1 \eta_1)^* H^*(x)
+ z^{-1} (B_o + \varepsilon_2 \eta_2)^*)
\times (z^{-1} - z^{-d}(F_o + \varepsilon_1 \eta_1) H(x)
+ z^{-1} (B_o + \varepsilon_2 \eta_2)) \sigma_u^2 + \sigma_u^2
\]
\[
+ (F_o + \varepsilon_1 \eta_1)^* \Sigma_{dH}(F_o + \varepsilon_1 \eta_1) \sigma_u^2
+ D^*(\beta)(F_o + \varepsilon_1 \eta_1)^* (F_o + \varepsilon_1 \eta_1) D(\beta) \sigma_u^2
+ (F_o + \varepsilon_1 \eta_1)^* \Sigma_{dH}(F_o + \varepsilon_1 \eta_1) \sigma_u^2 \frac{dz}{z}.
\]
Meanwhile, we introduce the spectral factorization as follows:
\[
A^* A = \Sigma_{dH} \sigma_u^2 + D^*(\beta) D(\beta) \sigma_u^2 + \Sigma_{dO} \sigma_u^2,
\]
where \(A\) is free of poles and zeros in \(|z| > 1\). The necessary conditions for a minimum of \(J\) are
\[
\frac{\partial J}{\partial \varepsilon_1} \bigg|_{\varepsilon_1 = 0} = 0 \quad \text{and} \quad \frac{\partial J}{\partial \varepsilon_2} \bigg|_{\varepsilon_2 = 0} = 0.
\]

From (25) and (27), we get the following two equations:
\[
\frac{1}{\pi j} \oint (1 + z^{-1} B_o) z^{-1+d} H^*(x) \sigma_u^2 - F_o H^*(x) H(x) \sigma_u^2 + F_o A^* A ) \eta_1 \frac{dz}{z} = 0
\]
and
\[
\frac{1}{\pi j} \oint (1 - z^d F_o H(x) + z^{-1} B_o) z \eta_2 \frac{dz}{z} = 0.
\]
where \(A\) was defined in (26). By Cauchy’s residue theorem, conditions (28) and (29) are satisfied only if both \(P^*\) and \(X^*\) are analytic inside the unit circle on the \(z\)-plane [19], where
\[
P^* = 1 - z^{-d} F_o H(x) + z^{-1} B_o
\]
and
\[
X^* = (1 + z^{-1} B_o) z^{-1+d} H^*(x) \sigma_u^2 + F_o H^*(x) H(x) \sigma_u^2 + F_o A^* A
\]
From (30), the only choice of \(P^*\) is a polynomial in \(z\) with degree \(l - d\), i.e.,
\[
P^* = p_0 + p_1 z + \cdots + p_{l-d} z^{l-d},
\]
since \(H(x), F_o,\) and \(B_o\) must be stable (analytic outside the unit circle).

Substitution of (30) into (31) results in
\[
z X^* = - z^{-1+d} H^*(x) P^* \sigma_u^2 + F_o A^* A
\]
Multiplying both sides of (33) by \(A^{-1}\) yields
\[
z X^* A^{-1} = - z^{-1-d} H^*(x) A^{-1} P^* \sigma_u^2 + F_o A
\]
Let \(z^{-1-d} H^*(x) A^{-1} P^* \sigma_u^2\) be decomposed as
\[
z^{-1-d} H^*(x) A^{-1} P^* \sigma_u^2 = - z^{-1-d} H^*(x) A^{-1} P^* \sigma_u^2
\]
\[
+ \{ z^{-1-d} H^*(x) A^{-1} P^* \sigma_u^2 \}
\]
Substituting (35) into (34) and rearranging, we obtain
\[
z X^* A^{-1} = \{ z^{-1-d} H^*(x) A^{-1} P^* \sigma_u^2 \}
\]
\[
= - \{ z^{-1-d} H^*(x) A^{-1} P^* \sigma_u^2 \} + F_o A = 0
\]
Note that the left-hand side of (36) is analytic inside the unit circle, while the right-hand side is analytic outside the unit circle. Hence, the only solution is that both of them are identically zero, i.e.,
\[
z^{-1+d} H^*(x) A^{-1} P^* \sigma_u^2 = 0
\]
or
\[
F_o = \{ z^{-1+d} H^*(x) A^{-1} P^* \sigma_u^2 \}^{-1}
\]
The optimal feedback filter is obtained from (30) as
\[
B_o = z [P^* + z^{-d} F_o H(x) - 1].
\]
perturbation is expressed as
\[
\min J = \frac{1}{2\pi} \int d \beta \int_{-\pi}^{\pi} \left( \mathbf{p}^* \mathbf{p} \sigma_n^2 + \left\{ z^{-1-d} H^*(x) A - \mathbf{p}^* \sigma_n^2 \right\}^2 \right) \, d\beta.
\]  
(40)

**Remark.** (1) If the transmission system is free of parameter perturbation, i.e., \( \delta x = \delta \beta = 0 \), the spectral factorization in (26) is reduced to
\[
\mathbf{A}_1^* A_1 = D^*(\beta) D(\beta) \sigma_n^2, 
\]  
(41)
where \( \beta \) is a constant vector. The corresponding DFE \( (F_{12}, B_{12}) \) minimizing the mean square error has the following forms:
\[
F_{12} = \left\{ z^{-1-d} H^*(x) A_1^* \mathbf{p}^* \sigma_n^2 \right\} A_1^{-1},
\]  
(42a)
\[
B_{12} = \left[ \mathbf{p}_1 + z^{-1-d} F_{12} H(x) - 1 \right],
\]  
(42b)
where \( \mathbf{p}_1 \) is a polynomial in \( z \) with degree \( l - d \) to be determined. This solution is equivalent to the one which neglects the effect of parameter perturbation.

(2) From the relationships between (26) and (41), the optimal DFE design method under parameter perturbation is equivalent to the one without channel parameter perturbation but with the following equivalent noise model:
\[
y(k) = q^{-d} H(q^{-1}, x) u(k) + \Delta(q^{-1}) e(k).
\]  
(43)

In (43), \( e(k) \) is the zero-mean innovations sequence with variance 1 (cf. Fig. 2).

### 3.3. Solution of optimal DFE \( (F_{12}, B_{12}) \) in (38) and (39)

It is observed from the previous subsection that \( \mathbf{p}^* \) is a polynomial in \( z \) with degree \( l - d \), i.e.,
\[
\mathbf{p}^* = p_0 + p_1 z + \cdots + p_{l-d} z^{l-d}.
\]  
(44)

For convenience, we define \( S \) as
\[
S = \left\{ z^{-1-d} H^*(x) A - \mathbf{p}^* \sigma_n^2 \right\}.
\]  
(45)
and represent \( H(x) A^{-1} \) in the following infinite impulse response form:
\[
H(x) A^{-1} = \sum_{i=0}^{\infty} r_i z^{-i}.
\]  
(46)

Thus, (38) and (39) can be rewritten as
\[
F_0 = S A^{-1},
\]  
\(47a\)
\[
B_0 = z \left( \mathbf{p}^* + z^l S H(x) A^{-1} - 1 \right).
\]  
(47b)

From (45), the only choice of \( S \) is a polynomial in \( z^{-1} \) with degree \( l - d \), i.e.,
\[
S = s_0 + s_1 z^{-1} + \cdots + s_{l-d} z^{-l+d},
\]  
(48)
since \( H^*(x) A - \mathbf{p}^* \) is analytic inside the unit circle while \( S \) is analytic outside the unit circle.

Substituting (46) into (45) and comparing the coefficients of orders from \( z^0 \) to \( z^{l-d} \) in (47b), we get
\[
\begin{bmatrix}
s_0 \\
s_1 \\
\vdots \\
s_{l-d}
\end{bmatrix} = \begin{bmatrix}
0 & r_0 & \cdots & r_{l-d} \\
0 & 0 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
p_0 \\
p_1 \\
\vdots \\
p_{l-d}
\end{bmatrix},
\]  
(49)
where
\[
A = \begin{bmatrix}
r_0 & r_1 & \cdots & r_{l-d} \\
r_0 & 0 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
r_0 & 0 & \cdots & 0
\end{bmatrix},
\]  
(50)
Also, comparing the coefficients of orders from \( z^{l-d} \) to \( z^0 \) in (47b), we get
\[
\begin{bmatrix}
p_{l-d} \\
p_{l-d-1} \\
\vdots \\
p_0
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\vdots \\
1
\end{bmatrix} \begin{bmatrix}
r_0 \\
r_1 \\
\vdots \\
r_{l-d}
\end{bmatrix} + \begin{bmatrix}
s_0 \\
s_1 \\
\vdots \\
s_{l-d}
\end{bmatrix},
\]  
(51)
Substituting (49) to (51) yields

\[
\begin{bmatrix}
    p_{l-d} \\
    \vdots \\
    p_1 \\
    p_0
\end{bmatrix}
= \begin{bmatrix}
    0 \\
    \vdots \\
    -\sigma_u^2 A^T A \\
    1
\end{bmatrix}
\begin{bmatrix}
    p_{l-d} \\
    \vdots \\
    p_1 \\
    p_0
\end{bmatrix}.
\]

Thus, the coefficients of \( P^* \) are calculated from

\[
(I + \sigma_u^2 A^T A)
\begin{bmatrix}
    p_{l-d} \\
    \vdots \\
    p_1 \\
    p_0
\end{bmatrix}
= \begin{bmatrix}
    0 \\
    \vdots \\
    0 \\
    1
\end{bmatrix}.
\]

(53)

It is obvious that the matrix \((I + \sigma_u^2 A^T A)\) is always positive definite and invertible. Therefore, \( P^* \) can be obtained from (53).

Substituting \( P^* \) into (49), we get \( S \). The optimal forward filter \( F_o(z^{-1}) \) is then obtained by calculating (47a). Besides, the feedback filter \( B_o(z^{-1}) \) is obtained by calculating (47b).

The complete algorithm is summarized in Table 1.

### Table 1

The DFE optimization algorithm under parameter perturbation

1. Obtain \( \delta H(\alpha)/\delta \alpha \) and \( \delta H(\beta)/\delta \beta \).
2. Compute \( \Sigma_{\alpha} \) and \( \Sigma_{\beta} \) from (17) and (20).
3. Perform the spectral factorization \( A^* A \) in (26).
4. Derive the impulse response of \( H(z) A^{-1} \).
5. Obtain \( P^* \) from (53).
6. Obtain \( S \) from (49).
7. Derive the optimal forward filter \( F_o = SA^{-1} \).
8. Derive the optimal feedback filter \( B_o \) from (47a).

The matrices \( \Sigma_{\alpha} \) and \( \Sigma_{\beta} \) are known as follows:

\[
0 < \Sigma_{\alpha} \leq \Sigma_{x} \leq \bar{\Sigma}_{\alpha},
\]

\[
0 < \Sigma_{\beta} \leq \Sigma_{\beta} \leq \bar{\Sigma}_{\beta}.
\]

(54)

In this situation, the DFE is considered to minimize the worst case of the mean square error under uncertain covariances of perturbative parameters, i.e., to minimize the following criterion:

\[
\max_{\Sigma_{\alpha}, \Sigma_{\beta}} E \{ e^2(k - l) \}.
\]

(55)

It is a typical minimax problem. Similarly, according to Parseval's theorem, the optimal DFE problem for the worst case of mean square error is equivalent to minimizing the following criterion:

\[
\max_{\Sigma_{\alpha}, \Sigma_{\beta}} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{(z^2 - z^d F^* H^*(\alpha) + z^{l+1} F^* B^*) \sigma_u^2}
\]

\[
\times (z^{-2z} + z^d F H(\alpha) + z^{l-1} B) \sigma_u^2 + F^* \Sigma_{\alpha} F \sigma_u^2 + D^*(\beta) F^* \Sigma_{\beta} F \sigma_u^2
\]

\[
+ F^* \Sigma_{\delta} F \sigma_u^2 \frac{dz}{z}.
\]

(56)

3.4. Extension of the design method to the case with unknown but bounded second-order statistics of the parameter perturbation

Suppose that the covariance matrices of parameter perturbations \( \delta x \) and \( \delta \beta \) are not precisely known, but the ranges of the covariances \( \Sigma_{x} \) and \( \Sigma_{\beta} \) are bounded. In this situation, we can extend the design method of the DFE to accommodate the uncertainty in the covariance matrices.
Define
\[ \Sigma_{sh} = \left( \frac{\partial H(x)}{\partial \alpha} \right)^* \Sigma_x \left( \frac{\partial H(x)}{\partial \alpha} \right) \] (57)

and
\[ \Sigma_{sd} = \left( \frac{\partial D(\beta)}{\partial \beta} \right)^* \Sigma_x \left( \frac{\partial D(\beta)}{\partial \beta} \right). \] (58)

It is obvious that the mean square error in the worst case becomes
\[
\max_{x, \beta} J = \frac{1}{2\pi j} \int \left( z^{-1} - z^{-d} F^* H^*(x) + z^{l+1} \beta \right) \times \left( z^{-1} - z^{-d} FH(x) + z^{-l-1} B \right) \sigma_u^2 + F^* \Sigma_{sh} F \sigma_u^2 + D^*(\beta) F^* FD(\beta) \sigma_n^2 \] (59)

By a procedure similar to that in the previous subsection, a corresponding minimax DFE design algorithm is obtained as follows. First, perform the spectral factorization as
\[ A^* A = \Sigma_{sh} \sigma_u^2 + D^*(\beta) D(\beta) \sigma_n^2 + \Sigma_{sd} \sigma_u^2. \] (60)

Then, using a similar technique, the corresponding minimax DFE (F_{02}, B_{02}) is obtained:
\[ F_{02} = \{ z^{-l+d} H^*(x) A_{02}^* P_{02}^* \sigma_u^2 \}^* A_{02}^{-1}, \] (61a)
\[ B_{02} = z [ P_{02}^* + z^{-d} F_{02} H(x) - 1]. \] (61b)

The minimum mean square error in the worst case is
\[
\begin{aligned}
\max_{x, \beta, z, \alpha, \beta} & \left( \max_{x, \beta} J \right) \\
= \frac{1}{2\pi j} \int & P_{02}^* P_{02} \sigma_u^2 + \{ z^{-l+d} H^*(x) A_{02}^* P_{02}^* \sigma_u^2 \}^* \\
\times & \left( z^{-l+d} H^*(x) A_{02}^* P_{02}^* \sigma_u^2 \right) + \frac{dz}{z}. \end{aligned} \] (62)

4. Simulation results

In this section, an application of the proposed design method in Section 3 is considered. The signal-to-noise ration (SNR) is defined as
\[
\text{SNR} = 10 \log \left( \frac{1}{2\pi j} \int E \{ H^*(x + \delta x) H(x + \delta x) \} \sigma_u^2 \frac{dz}{z} \right) \] (63)

Example. The following transmission system is considered:
\[
y(k) = \frac{0.3536 + a q^{-1}}{1 + b q^{-1}} u(k) \] (64)
\[
+ \frac{c}{1 - 0.6 q^{-1} + 0.73 q^{-2}} z n(k),
\]
where \( u(k) \) is a binary equiprobable i.i.d. sequence, i.e., \( P(u(k) = 1) = P(u(k) = -1) = \frac{1}{2} \), and \( n(k) \) is a zero-mean, white Gaussian noise with variance 1. The parameters \( a, b, \) and \( c \) are random variables with expected values \( 0.7071, -0.5, \) and 0.3, respectively. The deviations of the parameters are mutually independent and Gaussian distributed with variances all equal to 0.0033. The corresponding DFE with \( l = 1 \) is obtained from the following steps:

(1) Perform the spectral factorization
\[
A^* A = \left[ \begin{array}{cc} z & (0.3536 + 0.7071 z) z \\ 1 - 0.5 z & (1 - 0.5 z)^2 \end{array} \right] \]
\[
\times \left[ \begin{array}{cc} 0.0033 & 0 \\ 0 & 0.0033 \end{array} \right]
\]
\[
\times \left[ \begin{array}{cc} z^{-1} & (0.3536 + 0.7071 z^{-1}) z^{-1} \\ 1 - 0.5 z^{-1} & (1 - 0.5 z^{-1}) z^{-1} \end{array} \right]
\]

In general, if the covariances of \( \delta x \) and \( \delta \beta \) cannot be correctly measured or if they are time varying, the upper bounds of those covariances can be used in the design of optimal DFE from the minimax approach.
\[
+ \frac{(0.3)^2}{(1 - 0.6z + 0.73z^2)(1 - 0.6z^{-1} + 0.73z^{-2})} \\
+ \frac{0.0033}{1 - 0.6z + 0.73z^2}(1 - 0.6z^{-1} + 0.73z^{-2}).
\]

(65)

The solution is given by
\[
\Lambda = \\
0.3265 - 0.3030z^{-1} + 0.0862z^{-2} - 0.0018z^{-3} \\
1 - 1.6z^{-1} + 1.58z^{-2} - 0.88z^{-3} + 0.1825z^{-4}.
\]

(66)

(2) Employing the technique stated in Section 3.3 yields
\[
P^* = 0.2515 - 0.2481z
\]

(67)

and
\[
S = 0.2291 + 0.2724z^{-1}.
\]

(68)

(3) Evaluate the optimal DFE
\[
F_o = \frac{0.7017 - 0.2886z^{-1} - 0.2260z^{-2} + 0.7005z^{-3} - 0.6060z^{-4} + 0.1522z^{-5}}{1 - 0.9281z^{-1} + 0.2640z^{-2} - 0.0056z^{-3}}
\]

(69)

and
\[
B_o = z[P^* + zF_oH(z) - 1] = \frac{0.6043 - 0.4229z^{-1} + 0.3834z^{-2} - 0.3768z^{-3} + 0.1077z^{-4}}{1 - 1.4281z^{-1} + 0.7280z^{-2} - 0.1376z^{-3} + 0.0028z^{-4}}.
\]

(70)

(4) Evaluate the minimum mean square error using the one-lag DFE
\[
\min J = 0.2517.
\]

(71)

The bit-error rate versus SNR is shown in Fig. 3. For comparison we also give the simulation result for the DFE in [23] (in which the effect of parameter perturbation is neglected). When SNR = 0, the major part of (26) is \(D^*(\beta)D(\beta)\sigma_n^2\), and the performances of the proposed method and [23] are almost the same. However, in the high SNR region, the performance of the proposed method is much better than that of [23].

5. Conclusion

The design method of the optimal DFE is introduced in this paper with consideration of small random parameter perturbation in the dispersive

![Fig. 3. Bit-error rate with smoothing lag \(l = 1\) : the proposed DFE; \(\times\) : the method of [23].](image)
channel and noise model. The past decisions are assumed to be correct. The calculus of variation technique and spectral factorization method are adopted to derive a realizable optimal estimator. An explicit solution of the DFE has been presented. It can be applied to nonminimum-phase channels as well as minimum-phase ones. A simple expression for the minimum mean square estimation error is also obtained.

The simulation result demonstrates that the proposed method yields a significant improvement of performance compared to the one in [23], since our method is a compromise between good estimation and low sensitivity to noise and parameter perturbations, to achieve the minimization of mean square error. In addition, it can be found that a good performance can be achieved by the proposed method with a small value of lag.

References


