Abstract—In general, due to the interactions among subsystems, it is difficult to design an $H_{\infty}$ decentralized controller for nonlinear interconnected systems. In this study, the model reference tracking control problem of nonlinear interconnected systems is studied via $H_{\infty}$ decentralized fuzzy control method. First, the nonlinear interconnected system is represented by an equivalent Takagi–Sugeno type fuzzy model. A state feedback decentralized fuzzy control scheme is developed to override the external disturbances such that the $H_{\infty}$ model reference tracking performance is achieved. Furthermore, the stability of the nonlinear interconnected systems is also guaranteed. If states are not all available, a decentralized fuzzy observer is proposed to estimate the states of each subsystem for decentralized control. Consequently, a fuzzy observer-based state feedback decentralized fuzzy controller is proposed to solve the $H_{\infty}$ tracking control design problem for nonlinear interconnected systems. The problem of $H_{\infty}$ decentralized fuzzy tracking control design for nonlinear interconnected systems is characterized in terms of solving an eigenvalue problem (EVP). The EVP can be solved very efficiently using convex optimization techniques. Finally, simulation examples are given to illustrate the tracking performance of the proposed methods.

Index Terms—$H_{\infty}$ decentralized fuzzy tracking control, linear matrix inequality problem (LMIP) and eigenvalue problem (EVP).

I. INTRODUCTION

THE past three decades have witnessed serious applications of large-scale interconnected system methodologies to urban planning, economic models, spacecraft dynamics, power systems, industrial processes, transportation networks and others. The properties of interconnected systems have been widely studied and many different approaches have been proposed to stabilize the interconnected linear systems [1]–[4]. On the other hand, there are few studies concerning with the stabilization control for the interconnected nonlinear systems [5], [6]. Since linearization technique and linear robust control are used, these results are always conservative and only applicable to some special nonlinear interconnected systems. Due to the physical configuration and high dimensionality of interconnected systems, a centralized control is neither economically feasible nor even necessary [7]. Therefore, decentralized scheme is preferred in control design of the large-scale interconnected systems [8]–[10]. In other words, decentralized control scheme attempts to avoid difficulties in complexity of design, debugging, data gathering, and storage requirements. However, due to the effects of nonlinear interconnection among subsystems, there is still no efficient way to deal with the decentralized control problem of nonlinear interconnected systems, especially for the model reference tracking control case.

In the past few years, there has been rapidly growing interest in fuzzy control of nonlinear systems, and there have been many successful applications. The most important issue for fuzzy control systems is how to get a system design with the guarantee of stability and control performance, and recently there have been significant research efforts on these issues in fuzzy control systems [19]–[21], [23]–[27]. In these studies, a nonlinear plant was approximated by a Takagi–Sugeno fuzzy linear model [15], and then a model-based fuzzy control was developed to stabilize the Takagi–Sugeno fuzzy linear model. Similarly, there are very few studies concerning with the control problems for the nonlinear interconnected systems using Takagi–Sugeno fuzzy model.

The tasks of stabilization and tracking are two typical control problems. In general, tracking problems are more difficult than stabilization problems. Since fuzzy model is a suitable method to approximate a nonlinear system, a Takagi and Sugeno fuzzy model is employed in this study to approximate the nonlinear interconnected systems. In this paper, a state feedback decentralized fuzzy controller with constant control parameters is proposed to tackle the $H_{\infty}$ model reference tracking control design problem for nonlinear interconnected systems. A robust technique is developed to efficiently override the effect of external disturbances and interconnections among subsystem to guarantee the stability of global system. The problem of state feedback decentralized $H_{\infty}$ fuzzy tracking control design is characterized in terms of solving an eigenvalue problem (EVP).

All the results mentioned above are generated with state feedback. In practice, the states of the system are not all available. Several researchers have devoted to the observer design for large-scale systems; e.g., [11], [12]. If a full state observer constructed by the centralized method is used to estimate the state of a large-scale system, it will be difficult to implement and large estimation errors will also arise due to the number of computations involved [13], [14]. In this study, a decentralized fuzzy observer is proposed to estimate the states of each subsystem for decentralized control. Then, a fuzzy observer-based state feedback decentralized fuzzy controller is proposed to solve the $H_{\infty}$ model reference tracking control design problem for nonlinear interconnected systems. The problem of output feedback $H_{\infty}$ decentralized fuzzy tracking control design for nonlinear interconnected systems is also characterized in terms...
of solving an eigenvalue problem (EVP). The EVP can be solved very efficiently using convex optimization techniques [31], [32].

The paper is organized as follows. The problem formulation is presented in Section II. In Section III, the design problems of $H_{\infty}$ decentralized fuzzy tracking control for the nonlinear interconnected systems are introduced, while a fuzzy observer-based $H_{\infty}$ decentralized tracking control design for the nonlinear interconnected systems is considered in Section IV. In Section V, simulation examples are provided to demonstrate the design procedures. Finally, concluding remarks are made in Section VI.

II. PROBLEM FORMULATION

Consider a class of nonlinear interconnected system $S$ which is composed of $N$ subsystems $S_i$ ($i = 1, \ldots, N$) as follows:

$$\dot{x}_i(t) = f_i(x_i(t)) + g_i(x_i(t))u_i(t) + \sum_{j=1, j \neq i}^{N} f_{ij}(x_j(t)) + w_i(t)$$

where

- $x_i(t) \in \mathbb{R}^{n_i}$ state vector;
- $u_i(t) \in \mathbb{R}^{m_i}$ control signal;
- $w_i(t)$ external disturbance of the $i$th subsystem;
- $f_i(x_i(t)), g_i(x_i(t))$ smooth functions, where $f_{ij}(x_j(t))$ denotes the interconnection between the $i$th subsystem and the $j$th subsystem.

A fuzzy dynamic model has been proposed by Takagi and Sugeno [15] to represent locally linear input/output relations for nonlinear systems. This fuzzy dynamic model is described by fuzzy If-Then rules and will be employed here to deal with the control design problem of a nonlinear interconnected system. The $k$th rule of this fuzzy model for the nonlinear interconnected subsystem $S_i$ is proposed as the following form:

**Plant Rule $k$:**

**If** $z_{i_k}(t)$ is $F_{k_1}$ and ... and $z_{i_{a_k}}(t)$ is $F_{k_a};$

**Then**

$$\dot{x}_i(t) = A_{ik}x_i(t) + B_{ik}u_i(t) + \sum_{j=1, j \neq i}^{N} A_{ijk}x_j(t) + w_i(t)$$

for $k = 1, 2, \ldots, L$, where $x_i(t) \in \mathbb{R}^{n_i}$ is the state and $u_i(t) \in \mathbb{R}^{m_i}$ is the control signal of the $i$th subsystem, $F_{k_j}$ is the fuzzy set, $L$ is the number of If-Then rules, the matrices $A_{ik}, B_{ik}$ and $A_{ijk}$ are of appropriate dimensions, and $z_{i_1}(t), z_{i_2}(t), \ldots, z_{i_{a_k}}(t)$ are the premise variables, for $i, j = 1, 2, \ldots, N$.

The overall fuzzy model of subsystem $S_i$ can be rearranged as the following form [19], [20], [22], [23]:

$$\dot{x}_i(t) = \sum_{k=1}^{L} h_k(z_i(t)) \left[A_{ik}x_i(t) + B_{ik}u_i(t) + \sum_{j=1, j \neq i}^{N} A_{ijk}x_j(t) + w_i(t)\right]$$

where

$$h_k(z_i(t)) = \prod_{j=1}^{a_k} F_{k_j}(z_j(t))$$

$$\mu_k(z_i(t)) = \mu_k(z_i(t)) = \sum_{k=1}^{L} \mu_k(z_i(t))$$

$$z_i(t) = [z_{i_1}(t), z_{i_2}(t), \ldots, z_{i_{a_k}}(t)]$$

(4)

where $F_{k_j}(z_{i_j}(t))$ is the grade of membership of $z_{i_j}(t)$ in $F_{k_j}$ [16]–[18]. We assume

$$\mu_k(z_i(t)) \geq 0$$

and

$$\sum_{k=1}^{L} \mu_k(z_i(t)) > 0 \quad \text{for } k = 1, 2, \ldots, L$$

for all $t$. Therefore, we get

$$h_k(z_i(t)) \geq 0 \quad \text{for } k = 1, 2, \ldots, L$$

and

$$\sum_{k=1}^{L} h_k(z_i(t)) = 1.$$  

Consider a reference model for the $i$th subsystem as follows [29], [30]:

$$\dot{x}_r(t) = A_{ri}x_r(t) + r_i(t)$$

where $x_r(t)$ denotes reference state; $A_{ri}$ denotes a specific asymptotically stable matrix; $r_i(t)$ denotes bounded reference input. It is assumed that $r_i(t)$, for all $t \geq 0$, represents a desired trajectory for $x_i(t)$ to follow.

Suppose the following fuzzy decentralized controller is employed to deal with the stabilization problem of the above interconnected subsystem $S_i$.

**Control Rule $k$:**

**If** $z_{i_k}(t)$ is $F_{k_1}$ and ... and $z_{i_{a_k}}(t)$ is $F_{k_a};$

**Then**

$$u_i(t) = K_{ik}(x_i(t) - x_r(t)).$$

(8)

Hence, the fuzzy decentralized controller is given by

$$u_i(t) = \sum_{k=1}^{L} h_k(z_i(t))[K_{ik}(x_i(t) - x_r(t))].$$

(9)

Substituting (9) into (3) yields the closed-loop decentralized control of the subsystem $S_i$ as the following:

$$\dot{x}_i(t) = \sum_{k=1}^{L} h_k(z_i(t))h_{ri}(z_i(t)) \left[(A_{ik} + B_{ik}K_{im})x_i(t) - B_{ik}K_{im}x_r(t) + \sum_{j=1, j \neq i}^{N} A_{ijk}x_j(t) + w_i(t)\right]$$

(10)
Let us consider the $H_\infty$ tracking performance related to the tracking error $x_i(t) - x_\pi(t)$ as follows \cite{28,23}:

$$\int_0^{t_f} \{ (x_i(t) - x_\pi(t))^T Q_i (x_i(t) - x_\pi(t)) \} dt \leq \rho^2$$ \tag{11}

or

$$\int_0^{t_f} \{ (x_i(t) - x_\pi(t))^T Q_i (x_i(t) - x_\pi(t)) \} dt \leq \rho^2 \int_0^{t_f} \bar{w}_i(t)^T \bar{w}_i(t) dt$$ \tag{12}

where $\bar{w}_i(t) = [w_i(t), r_i(t)]^T$ for all reference input $r_i(t)$ and external disturbance $w_i(t); t_f$ is terminal time of control, $Q_i$ is symmetric positive definite weighting matrix, $\rho$ is a prescribed attenuation level. The physical meaning of (11) or (12) is that the effect of any $\bar{w}_i(t)$ on tracking error $x_i(t) - x_\pi(t)$ must be attenuated below a desired level $\rho$ from the viewpoint of energy, no matter what $\bar{w}_i(t)$ is, i.e., the $L_2$ gain from $\bar{w}_i(t)$ to $x_i(t) - x_\pi(t)$ must be equal to or less than a prescribed value $\rho^2$. The $H_\infty$ tracking performance with a prescribed attenuation level is useful for a robust tracking design without knowledge of $w_i(t)$ and $r_i(t)$.

After manipulation, the augmented system can be expressed as the following form:

$$\begin{bmatrix}
\dot{x}_i(t) \\
\dot{x}_\pi(t)
\end{bmatrix} = \sum_{j=1, j\neq i}^N \sum_{k=1}^L \sum_{m=1}^L h_{jk}(z_i(t)) h_m(z_i(t))$$

$$\times \begin{bmatrix}
A_{ik} + B_{ik} K_{im} & -B_{ik} K_{im} \\
0 & A_{ir}
\end{bmatrix}
\begin{bmatrix}
x_i(t) \\
x_\pi(t)
\end{bmatrix}
$$

$$+ \begin{bmatrix}
A_{ijk} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_j(t) \\
x_\pi(t)
\end{bmatrix} + \begin{bmatrix}
w_i(t) \\
r_i(t)
\end{bmatrix},$$ \tag{13}

Let us denote

$$\bar{A}_{ikm} = A_{ik} + B_{ik} K_{im} - B_{ik} K_{im}$$

$$\bar{B}_{ijk} = \begin{bmatrix}
A_{ijk} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
w_i(t) \\
r_i(t)
\end{bmatrix}.$$ \tag{14}

Therefore, the augmented system defined in (13) can be expressed as the following form:

$$\begin{bmatrix}
\dot{x}_i(t) \\
\dot{x}_\pi(t)
\end{bmatrix} = \sum_{j=1, j\neq i}^N \sum_{k=1}^L \sum_{m=1}^L h_{jk}(z_i(t)) h_m(z_i(t))$$

$$\times (\bar{A}_{ikm} x_i(t) + \bar{B}_{ijk} x_\pi(t)) + \bar{w}_i(t).$$ \tag{15}

Hence, the $H_\infty$ tracking performance can be rewritten as follows

$$\int_0^{t_f} \{ (x_i(t) - x_\pi(t))^T Q_i (x_i(t) - x_\pi(t)) \} dt$$

$$= \int_0^{t_f} \bar{x}_i(t)^T Q_i \bar{x}_i(t) dt$$

$$\leq \rho^2 \int_0^{t_f} \bar{w}_i(t)^T \bar{w}_i(t) dt$$ \tag{16}

where

$$\bar{Q}_i = \begin{bmatrix}
Q_i & -Q_i \\
-Q_i & Q_i
\end{bmatrix}.$$ \tag{17}

If the initial condition is also considered, the inequality (16) can be modified as

$$\int_0^{t_f} \bar{x}_i(t)^T Q_i \bar{x}_i(t) dt$$

$$\leq \rho^2 \int_0^{t_f} \bar{w}_i(t)^T \bar{w}_i(t) dt$$ \tag{18}

where $\bar{P}_i$ is a symmetric positive definite weighting matrix.

The purpose of this study is to determine a decentralized fuzzy controller in (9) for the augmented system in (15) with the guaranteed $H_\infty$ tracking performance in (18) for all $\bar{w}_i(t)$. Thereafter, the attenuation level $\rho^2$ can also be minimized so that the $H_\infty$ tracking performance in (18) is reduced as small as possible. Furthermore, the closed-loop systems for whole interconnected systems

$$\begin{bmatrix}
\dot{x}_i(t) \\
\dot{x}_\pi(t)
\end{bmatrix} = \sum_{j=1, j\neq i}^N \sum_{k=1}^L \sum_{m=1}^L h_{jk}(z_i(t)) h_m(z_i(t))$$

$$\times (\bar{A}_{ikm} x_i(t) + \bar{B}_{ijk} x_\pi(t)) + \bar{B}_{ijk} x_\pi(t).$$ \tag{19}

Therefore, stability is the most important issue in the control system. Obviously, it is appealing for control engineers to specify the decentralized control parameters $K_{ik}$ in the fuzzy controller (9) such that the stability for the whole interconnected nonlinear system $S$ can be guaranteed. To prove the augmented closed-loop system in (19) to be stable, let us define a Lyapunov function for the system of (19) as

$$V(t) = \sum_{i=1}^N \psi_i(t) = \sum_{i=1}^N \bar{x}_i(t)^T \bar{P}_i \bar{x}_i(t).$$ \tag{21}

The time derivative of $V(t)$ is

$$\dot{V}(t) = \sum_{i=1}^N \left[ \bar{x}_i(t)^T \bar{P}_i \bar{x}_i(t) + \bar{x}_i(t)^T \bar{P}_i \bar{x}_i(t) \right].$$ \tag{22}

By substituting (19) into (22), we get

$$\dot{V}(t) = \sum_{i=1}^N \left[ \bar{x}_i(t)^T \bar{P}_i \bar{x}_i(t) + \bar{x}_i(t)^T \bar{P}_i \bar{x}_i(t) \right]$$

$$= \sum_{i=1}^N \sum_{j=1, j\neq i}^N \sum_{k=1}^L \sum_{m=1}^L h_{jk}(z_i(t)) h_m(z_i(t))$$

$$\times \left[ (\bar{A}_{ikm} \bar{x}_i(t) + \bar{B}_{ijk} \bar{x}_i(t))^T \bar{P}_i \bar{x}_i(t)$$

$$+ \bar{x}_i(t)^T \bar{P}_i (\bar{A}_{ikm} \bar{x}_i(t) + \bar{B}_{ijk} \bar{x}_i(t)) \right]$$

$$= \sum_{i=1}^N \sum_{j=1, j\neq i}^N \sum_{k=1}^L \sum_{m=1}^L h_{jk}(z_i(t)) h_m(z_i(t))$$

$$\times \left[ \begin{bmatrix}
\bar{x}_i(t) \\
\bar{x}_i(t)
\end{bmatrix}^T \begin{bmatrix}
\bar{A}_{ikm} \bar{P}_i + \bar{P}_i \bar{A}_{ikm} & \bar{P}_i \bar{B}_{ijk} \\
\bar{B}_{ijk} \bar{P}_i & 0
\end{bmatrix} \right.$$

$$\times \begin{bmatrix}
\bar{x}_i(t) \\
\bar{x}_i(t)
\end{bmatrix}.$$ \tag{23}
Then, we get the following result:

**Theorem 1:** In the augmented nonlinear interconnected system (19), if \( P_t = P_t^T > 0 \) is the common solution of the following matrix inequalities:

\[
\begin{bmatrix}
\bar{A}_{ikm} P_t + \bar{P}_t \bar{A}_{ikm} & \bar{P}_t P_{ijk} \\
\bar{B}_{ijk} P_t & 0
\end{bmatrix} \leq 0
\]  

for \( i,j = 1,2,\ldots,N \ (j \neq i) \) and \( k,m = 1,2,\ldots,L \), then the whole interconnected nonlinear system \( S \) is stable in the sense of Lyapunov.

**Proof:** From (23), we get

\[
\hat{V}(t) = \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \sum_{k=1}^{L} \sum_{m=1}^{L} h_k(z_i(t))h_m(z_i(t))
\]

\[
\times \begin{bmatrix}
\bar{x}_i(t) \\
\bar{x}_j(t)
\end{bmatrix}^T \begin{bmatrix}
\bar{A}_{ikm} P_t + \bar{P}_t \bar{A}_{ikm} & \bar{P}_t P_{ijk} \\
\bar{B}_{ijk} P_t & 0
\end{bmatrix} \begin{bmatrix}
\bar{x}_i(t) \\
\bar{x}_j(t)
\end{bmatrix}.
\]  

(24)

From (24), we obtain \( \hat{V}(t) \leq 0 \). Then, the whole interconnected nonlinear system \( S \) is stable in the sense of Lyapunov. This completes the proof. \( \square \)

Next, the design purpose in this study is to specify the decentralized fuzzy control in (9) to achieve \( H_\infty \) tracking control performance in (18).

Then, we obtain the following main result.

**Theorem 2:** In the nonlinear system (15), if \( P_t = P_t^T > 0 \) is the common solution of the following matrix inequalities:

\[
\begin{bmatrix}
\bar{A}_{ikm} P_t + \bar{P}_t \bar{A}_{ikm} + \bar{Q}_i & \bar{P}_t P_{ijk} & \bar{P}_t \\
\bar{B}_{ijk} P_t & 0 & -\rho^2 I
\end{bmatrix} \leq 0
\]  

for \( i,j = 1,2,\ldots,N \ (j \neq i) \) and \( k,m = 1,2,\ldots,L \). Then, the whole interconnected nonlinear system \( S \) is stable in the sense of Lyapunov if \( \bar{\sigma}_i(t) = 0 \) and the \( H_\infty \) tracking control performance in (18) is guaranteed for a prescribed \( \rho^2 \).

**Proof:** Note that (26) implies

\[
\begin{bmatrix}
\bar{A}_{ikm} P_t + \bar{P}_t \bar{A}_{ikm} \\
\bar{B}_{ijk} P_t
\end{bmatrix} \leq 0.
\]  

According to Theorem 1, the stability of the whole interconnected nonlinear system is immediately followed. From (16), we obtain

\[
\int_0^{t_f} \{ (x_i(t) - x_{vi}(t))^T Q_i (x_i(t) - x_{vi}(t)) \} dt
\]

\[
= \int_0^{t_f} \bar{x}_i(t)^T \bar{Q}_i \bar{x}_i(t) dt
\]

\[
= \bar{x}_i(t_0)^T \bar{Q}_i \bar{x}_i(t_0) - \bar{x}_i(t_f)^T \bar{P}_t \bar{x}_i(t_f)
\]

\[
+ \int_0^{t_f} \left\{ \bar{x}_i(t)^T \bar{Q}_i \bar{x}_i(t) + \frac{d}{dt} (\bar{x}_i(t)^T \bar{P}_t \bar{x}_i(t)) \right\} dt
\]

\[
\leq \bar{x}_i(t_0)^T \bar{Q}_i \bar{x}_i(t_0) + \int_0^{t_f} \left\{ \bar{x}_i(t)^T \bar{Q}_i \bar{x}_i(t) + \bar{x}_i(t)^T \bar{P}_t \bar{x}_i(t) + \bar{x}_i(t)^T P_{ijk} \bar{x}_i(t) \right\} dt
\]

\[
= \bar{x}_i(t_0)^T \bar{Q}_i \bar{x}_i(t_0) + \bar{x}_i(t_f)^T \bar{P}_t \bar{x}_i(t_f) + \int_0^{t_f} \left\{ \bar{x}_i(t)^T \bar{Q}_i \bar{x}_i(t) + \bar{x}_i(t)^T \bar{P}_t \bar{x}_i(t) + \bar{x}_i(t)^T P_{ijk} \bar{x}_i(t) \right\} dt
\]

\[
= \bar{x}_i(t_0)^T \bar{Q}_i \bar{x}_i(t_0) + \int_0^{t_f} \left\{ \bar{x}_i(t)^T \bar{Q}_i \bar{x}_i(t) + \bar{x}_i(t)^T \bar{P}_t \bar{x}_i(t) + \bar{x}_i(t)^T P_{ijk} \bar{x}_i(t) \right\} dt
\]

(25)

By (26), we obtain

\[
\int_0^{t_f} \bar{x}_i(t)^T \bar{Q}_i \bar{x}_i(t) dt \leq \bar{x}_i(t_0)^T \bar{Q}_i \bar{x}_i(t_0) + \rho^2 \int_0^{t_f} \bar{\sigma}_i(t)^T \bar{\sigma}_i(t) dt.
\]  

(28)

Therefore, the \( H_\infty \) tracking control performance is achieved with a prescribed \( \rho^2 \). This completes the proof. \( \square \)

To obtain a better robust tracking performance, the robust tracking control problem can be formulated as the following minimization problem so that the \( H_\infty \) tracking performance in (18) is reduced as small as possible

\[
\min \rho^2
\]

subject to \( P_t = P_t^T > 0 \) and (26).

(29)

Note that the matrix inequalities in (26) can be transformed into certain forms of linear matrix inequalities (LMIs). There-
fore, the minimization problem in (29) can be formulated as a minimization problem subject to some LMIs, which is also called an eigenvalue problem (EVP).

By introducing a new matrix

\[
W_i = \begin{bmatrix} \bar{W}_i & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} \bar{P}_i^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}
\]

where \( \bar{W}_i = \bar{P}_i^{-1} \) and multiplying it into (26), we obtain

\[
\begin{aligned}
&\left[ \bar{X}_{ikm} \bar{P}_i + \bar{P}_i \bar{X}_{km} + \bar{Q}_i \bar{P}_i \bar{B}_{ijk} \bar{P}_i \right] W_i \\
&= \left[ \bar{W}_i \bar{X}_{ikm} + \bar{X}_{km} \bar{W}_i + \bar{W}_i \bar{Q}_i \bar{W}_i \bar{B}_{ijk} + I \right] \\
&\leq 0.
\end{aligned}
\]

(30)

By the Schur complements [31], (30) is equivalent to

\[
\begin{aligned}
&\begin{bmatrix} \bar{W}_i \bar{A}_{ikm} + \bar{A}_{km} \bar{W}_i & \bar{W}_i \left( \bar{Q}_i^{1/2} \right)^T & -I & 0 & 0 \\
& \left( \bar{Q}_i^{1/2} \right)^T \bar{W}_i & -I & 0 & 0 \\
& \bar{B}_{ijk} & 0 & 0 & -\rho^2 I \end{bmatrix} \leq 0.
\end{aligned}
\]

(31)

Therefore, the minimization problem in (29) is equivalent to

\[
\min_{W_i} \rho^2
\]

subject to \( W_i = W_i^T > 0 \) and (31).

For the convenience of design, let

\[
\bar{P}_i = \begin{bmatrix} P_{i11} & 0 \\ 0 & P_{i11} \end{bmatrix}
\]

(33)

where \( P_{i11} = P_{i11}^T > 0 \) and thus

\[
\bar{W}_i = \begin{bmatrix} W_{i11} & 0 \\ 0 & W_{i11} \end{bmatrix}
\]

(34)

where \( W_{i11} = P_{i11}^{-1} \).

With \( Y_{im} = K_{im} W_{i11} \), (31) are equivalent to the following linear matrix inequalities (LMIs) (see (35) at the bottom of the page) where

\[
\begin{bmatrix} Q_{i11} & Q_{i12} \\ Q_{i21} & Q_{i22} \end{bmatrix} = \bar{Q}_i^{1/2}.
\]

Finally, the minimization problem in (32) is equivalent to the following eigenvalue problem (EVP).

\[
\min_{\{W_{i11}, Y_{im}\}} \rho^2
\]

subject to \( W_{i11} = W_{i11}^T > 0 \) and (35) (36)

Based on the analysis above, the design procedures for the \( H_{\infty} \) fuzzy decentralized tracking control of interconnected systems are summarized as follows:

**Design Procedure 1:**

Step 1) Construct the fuzzy plant rules (2).

Step 2) Solve the EVP in (36) to obtain \( W_{i11} \) and \( Y_{im} \) (thus \( K_{im} = Y_{im} W_{i11}^{-1} \) can also be obtained).

Step 3) Obtain fuzzy decentralized control rule in (9).

IV. DECENTRALIZED OBSERVER SYNTHESIS FOR NONLINEAR INTERCONNECTED SYSTEMS

In the previous sections, we assumed that all the state variables are available. In practice, this assumption often does not hold. In this situation, we need to estimate state vector \( x_i(t) \) from output \( y_k(t) \) of each subsystem for state feedback control. The \( k \)th rule of this fuzzy model for the nonlinear interconnected subsystem \( S_k \) is proposed as the following form:

**Plant Rule k:**

\[
\begin{array}{l}
\text{If } z_{i_k}(t) = F_k \ldots \text{ and } z_{i_1}(t) = F_k y_k \\
\text{Then } \dot{x}_i(t) = A_{ik} x_i(t) + B_{ik} u_i(t) \\
y_k(t) = C_{ik} x_i(t) + v_i(t)
\end{array}
\]

(37)

where \( v_i(t) \) is measurement noise of subsystem \( S_i \).

Therefore, the overall output of subsystem \( S_k \) can be arranged as the following form:

\[
y_k(t) = \sum_{k=1}^{L} h_k(z_i(t)) C_{ik} x_i(t) + v_i(t).
\]

(38)

\[
\begin{bmatrix}
W_{i11} A_{ik} + Y_{im} B_{ik} & -B_{ik} Y_{im} & W_{i11} Q_{i11} & W_{i11} Q_{i12} & A_{ijk} & I & 0 \\
W_{i11} A_{ik} + A_{ir} W_{i11} & W_{i11} Q_{i11} + W_{i11} Q_{i21} & 0 & 0 & 0 & I & 0 \\
W_{i11} A_{ik} + A_{ir} W_{i11} & W_{i11} Q_{i11} + W_{i11} Q_{i21} & 0 & 0 & 0 & I & 0 \\
0 & 0 & -\rho^2 I & -\rho^2 I & 0 & 0 & 0 \\
\end{bmatrix} \leq 0
\]

(35)
Suppose the following fuzzy decentralized observer is proposed to deal with the state estimation of the nonlinear interconnected system

\[ x_k(t) = \sum_{i=1, i \neq k}^{N} A_{ijk} \hat{x}_j(t) + B_{ik}u_i(t), \]

\[ + \sum_{i=1, i \neq k}^{N} A_{ijk} \hat{x}_j(t) + L_{ik}(y_k(t) - \hat{y}_k(t)) \]  \hspace{1cm} (39)

where \( L_{ik} \) are the fuzzy observer gains for the \( k \)th observer rule for the \( i \)th interconnected subsystem \( S_i \) and \( \hat{y}_k(t) = \sum_{k=1}^{K} h_k(z(t))C_{ik} \hat{x}_k(t) \).

The overall fuzzy decentralized observer for the \( i \)th subsystem is represented as follows:

\[ \dot{\hat{x}}_i(t) = \sum_{k=1}^{L} h_k(z_i(t)) \left[ A_{ik} \hat{x}_i(t) + B_{ik}u_i(t) \right] + \sum_{j=1, j \neq i}^{N} A_{ijk} \hat{x}_j(t) + L_{ik}(y_i(t) - \hat{y}_i(t)) \]  \hspace{1cm} (40)

The fuzzy observer-based decentralized controller is modified as

\[ u_i(t) = \sum_{m=1}^{L} h_m(z_i(t)) \left[ K_{im} (\hat{x}_i(t) - x_{ri}(t)) \right] \]  \hspace{1cm} (41)

for \( i = 1, 2, \ldots, N \), where \( \hat{x}_i(t) \) is obtained from the fuzzy observer in (40).

**Remark 1:** The premise variables \( z_i(t) \) can be measurable state variables, outputs or combination of measurable state variables. For Takagi–Sugeno type fuzzy model, using state variables as premise variables are common, but not always [19], [21]–[23]. The limitation of this approach is that some state variables must be measurable to construct the fuzzy observer and fuzzy controller. This is a common limitation for control system design of Takagi–Sugeno fuzzy approach [22], [23]. If the premise variables of the fuzzy observer depend on the estimated state variables, i.e., \( \dot{\hat{x}}(t) \) instead of \( z_i(t) \) in the fuzzy observer, the situation becomes more complicated. In this case, it is difficult to directly find decentralized control gains \( K_{im} \) and observer gains \( L_{ik} \).

Let us denote the estimation error as

\[ e_i(t) = x_i(t) - \hat{x}_i(t). \]  \hspace{1cm} (42)

By differentiating (42), we get error dynamic of fuzzy decentralized observer as follows:

\[ \dot{e}_i(t) = \dot{x}_i(t) - \dot{\hat{x}}_i(t) \]

\[ = \sum_{k=1}^{L} h_k(z_i(t)) \left[ A_{ik} x_i(t) + B_{ik}u_i(t) \right] + \sum_{j=1, j \neq i}^{N} A_{ijk} x_j(t) + w_i(t) \]

\[ \left\{ \begin{array}{l}
A_{ik} \hat{x}_i(t) + B_{ik}u_i(t) \\
\sum_{j=1, j \neq i}^{N} A_{ijk} \hat{x}_j(t) + L_{ik}(y_i(t) - \hat{y}_i(t)) 
\end{array} \right\} 
= \sum_{j=1, j \neq i}^{N} \sum_{k=1}^{L} h_k(z_i(t)) \sum_{m=1}^{L} h_m(z_i(t)) \times \left[ (A_{ik} - L_{ik}C_{im})x_i(t) + A_{ijk} \hat{x}_j(t) + w_i(t) \right]
\]  \hspace{1cm} (43)

Then, the augmented system is equivalent to the following form:

\[ \begin{bmatrix}
\dot{x}_i(t) \\
\dot{x}_{ri}(t) \\
\dot{e}_i(t)
\end{bmatrix} = \begin{bmatrix}
\sum_{j=1, j \neq i}^{N} \sum_{k=1}^{L} h_k(z_i(t)) \sum_{m=1}^{L} h_m(z_i(t)) \\
\sum_{j=1, j \neq i}^{N} \sum_{k=1}^{L} h_k(z_i(t)) \sum_{m=1}^{L} h_m(z_i(t)) \\
\sum_{j=1, j \neq i}^{N} \sum_{k=1}^{L} h_k(z_i(t)) \sum_{m=1}^{L} h_m(z_i(t))
\end{bmatrix} \times \begin{bmatrix}
A_{ik} \hat{x}_i(t) + B_{ik}u_i(t) \\
\sum_{j=1, j \neq i}^{N} A_{ijk} \hat{x}_j(t) + L_{ik}(y_i(t) - \hat{y}_i(t)) \\
\left[ (A_{ik} - L_{ik}C_{im})x_i(t) + A_{ijk} \hat{x}_j(t) + w_i(t) \right]
\end{bmatrix}. \]  \hspace{1cm} (44)

After manipulation, (44) can be expressed as the following form:

\[ \begin{bmatrix}
\dot{x}_i(t) \\
\dot{x}_{ri}(t) \\
\dot{e}_i(t)
\end{bmatrix} = \sum_{j=1, j \neq i}^{N} \sum_{k=1}^{L} h_k(z_i(t)) \sum_{m=1}^{L} h_m(z_i(t)) \times \begin{bmatrix}
A_{ik} + B_{ik}K_{im} & -B_{ik}K_{im} & -B_{ik}K_{im} \\
0 & A_{ri} & 0 \\
0 & 0 & A_{ik} - L_{ik}C_{im}
\end{bmatrix}
\times \begin{bmatrix}
x_i(t) \\
x_{ri}(t) \\
e_i(t)
\end{bmatrix} + \begin{bmatrix}
A_{ijk} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & A_{ijk}
\end{bmatrix} \begin{bmatrix}
x_{ri}(t) \\
e_i(t) \\
e_i(t)
\end{bmatrix} + \begin{bmatrix}
I & 0 & 0 \\
0 & I & 0 \\
0 & I & -L_{ik}
\end{bmatrix} \begin{bmatrix}
w_i(t) \\
r_i(t) \\
v_i(t)
\end{bmatrix}. \]  \hspace{1cm} (45)

Let us denote

\[ \dot{A}_{ikm} = \begin{bmatrix}
A_{ik} + B_{ik}K_{im} & -B_{ik}K_{im} & -B_{ik}K_{im} \\
0 & A_{ri} & 0 \\
0 & 0 & A_{ik} - L_{ik}C_{im}
\end{bmatrix}, \]

\[ \dot{z}_i(t) = \begin{bmatrix}
x_i(t) \\
x_{ri}(t) \\
e_i(t)
\end{bmatrix}, \]

\[ \dot{E}_{ik} = \begin{bmatrix}
I & 0 & 0 \\
0 & I & 0 \\
0 & I & -L_{ik}
\end{bmatrix}, \]

\[ \ddot{w}_i(t) = \begin{bmatrix}
w_i(t) \\
r_i(t) \\
v_i(t)
\end{bmatrix}. \]
Therefore, the augmented system of the $i$th subsystem defined in (45) can be expressed as the following form:

$$\dot{x}_i(t) = \sum_{j=1}^{N} \sum_{l=1}^{L} h_k(z_l(t)) \sum_{m=1}^{L} h_m(z_l(t)) \times [\dot{x}_{ikm} \dot{x}_i(t) + \dot{B}_{ijk} \dot{x}_j(t) + \dot{E}_{ik} \dot{w}_i(t)].$$

Hence, the $H_\infty$ tracking performance in (18) can be modified as follows:

$$\int_{0}^{t_f} \{ (x_i(t) - x_{ri}(t))^T Q_i (x_i(t) - x_{ri}(t)) \} dt$$

$$= \int_{0}^{t_f} \dot{x}_i(t) Q_i \dot{x}_i(t) dt$$

$$\leq \dot{x}_i^T(0) \hat{P}_i \dot{x}_i(0) + \rho^2 \int_{0}^{t_f} \dot{w}_i^T(0) \dot{w}_i(t) dt$$

(47)

where $\hat{P}_i$ is a symmetric positive definite weighting matrix and

$$\hat{Q}_i = \begin{bmatrix} Q_i & -Q_i & 0 \\ -Q_i & Q_i & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  

(48)

The purpose of this study is to determine a fuzzy controller in (43) for the augmented system in (46) with the guaranteed $H_\infty$ tracking performance in (47) for all $\dot{w}_i(t)$. Thereafter, the attenuation level $\rho^2$ can also be minimized so that the $H_\infty$ tracking performance in (47) is reduced as small as possible. Furthermore, the closed-loop systems for whole interconnected systems

$$\dot{x}_i(t) = \sum_{j=1}^{N} \sum_{l=1}^{L} h_k(z_l(t)) \sum_{m=1}^{L} h_m(z_l(t)) \times [\dot{x}_{ikm} \dot{x}_i(t) + \dot{B}_{ijk} \dot{x}_j(t)],$$

(49)

$i = 1, \ldots, N$ are stable.

Let us choose a Lyapunov function for each subsystem (49) as follows:

$$\hat{V}_i(t) = \dot{x}_i^T(t) \hat{P}_i \dot{x}_i(t), \quad \text{for } i = 1, 2, \ldots, N$$

(50)

where the weighting matrix $\hat{P}_i = \hat{P}_i^T > 0$.

Thus, the Lyapunov function $\hat{V}(t)$ for the overall interconnected system $S$ is

$$\hat{V}(t) = \sum_{i=1}^{N} \hat{V}_i(t) = \sum_{i=1}^{N} \dot{x}_i^T(t) \hat{P}_i \dot{x}_i(t).$$

(51)

By differentiating (51), we get

$$\dot{\hat{V}}(t) = \sum_{i=1}^{N} \left\{ \dot{x}_i^T(t) \hat{P}_i \dot{x}_i(t) + \dot{x}_i^T(t) \hat{P}_i \dot{x}_i(t) \right\}$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{l=1}^{L} h_k(z_l(t)) \sum_{m=1}^{L} h_m(z_l(t)) \times [\dot{x}_{ikm} \dot{x}_i(t) + \dot{B}_{ijk} \dot{x}_j(t)]$$

$$\times \left[ \dot{\hat{Q}}_i \dot{x}_i(t) + \dot{\hat{B}}_{ijk} \dot{x}_j(t) \right],$$

(52)

Then, we get the following results:

**Theorem 3:** In the augmented nonlinear interconnected system (49), if $\hat{P}_i = \hat{P}_i^T > 0$ is the common solution of the following matrix inequalities

$$\begin{bmatrix} \hat{A}_{ikm}^T \hat{P}_i + \hat{B}_{ijk} \dot{x}_j(t) + \hat{E}_{ik} \dot{w}_i(t) \\ \hat{B}_{ijk}^T \hat{P}_i \hat{P}_i + \hat{E}_{ij} \end{bmatrix} \leq 0$$

for $i, j = 1, 2, \ldots, N$ (j $\neq$ i) and $k, m = 1, \ldots, L$, then the whole interconnected nonlinear system $S$ is stable in the sense of Lyapunov.

**Proof:** From (52) and (53), we obtain $\dot{\hat{V}}(t) \leq 0$. Then, the whole interconnected nonlinear system $S$ is stable in the sense of Lyapunov. This completes the proof.

**Theorem 4:** In the nonlinear augmented system (46), if $\hat{P}_i = \hat{P}_i^T > 0$ is the common solution of the following matrix inequalities

$$\begin{bmatrix} \hat{A}_{ikm}^T \hat{P}_i + \hat{B}_{ijk} \dot{x}_j(t) + \hat{E}_{ik} \dot{w}_i(t) \\ \hat{B}_{ijk}^T \hat{P}_i \hat{P}_i + \hat{E}_{ij} \end{bmatrix} \leq 0$$

for $i, j = 1, 2, \ldots, N$ (j $\neq$ i) and $k, m = 1, \ldots, L$. Then, the whole interconnected nonlinear system $S$ is stable in the sense of Lyapunov if $\dot{w}_i(t) = 0$ and the $H_\infty$ tracking control performance in (47) is guaranteed for a prescribed $\rho^2$.

**Proof:** According to Theorem 3 and by the same argument as that in Theorem 2, the stability of the whole interconnected nonlinear system is obtained. From (47), we obtain

$$\int_{0}^{t_f} \{ (x_i(t) - x_{ri}(t))^T Q_i (x_i(t) - x_{ri}(t)) \} dt$$

$$= \int_{0}^{t_f} \dot{x}_i^T(t) Q_i \dot{x}_i(t) dt$$

$$= \dot{x}_i^T(0) \hat{P}_i \dot{x}_i(0) - \dot{x}_i^T(t_f) \hat{P}_i \dot{x}_i(t_f)$$

$$+ \int_{0}^{t_f} \dot{x}_i^T(t) Q_i \dot{x}_i(t) + \frac{d}{dt} (\dot{x}_i^T(t) \hat{P}_i \dot{x}_i(t)) dt$$

$$\leq \dot{x}_i^T(0) \hat{P}_i \dot{x}_i(0) + \int_{0}^{t_f} \dot{x}_i^T(t) Q_i \dot{x}_i(t)$$

$$+ \dot{x}_i^T(t) \hat{P}_i \dot{x}_i(t) + \dot{x}_i^T(t) \hat{P}_i \dot{x}_i(t) dt$$

$$= \dot{x}_i^T(0) \hat{P}_i \dot{x}_i(0) + \int_{0}^{t_f} \dot{x}_i^T(t) Q_i \dot{x}_i(t)$$

$$+ \left( \sum_{j=1}^{N} \sum_{l=1}^{L} h_k(z_l(t)) \sum_{m=1}^{L} h_m(z_l(t)) \times (\dot{x}_{ikm} \dot{x}_i(t) + \dot{B}_{ijk} \dot{x}_j(t) + \dot{E}_{ik} \dot{w}_i(t))^T \hat{P}_i \dot{x}_i(t) \right) dt$$

$$+ \dot{x}_i^T(t) \hat{P}_i \left( \sum_{j=1}^{N} \sum_{l=1}^{L} h_k(z_l(t)) \sum_{m=1}^{L} h_m(z_l(t)) \times (\dot{x}_{ikm} \dot{x}_i(t) + \dot{B}_{ijk} \dot{x}_j(t) + \dot{E}_{ik} \dot{w}_i(t)) \right) dt.$$
Therefore, the $H_{\infty}$ tracking control performance is achieved with a prescribed $\rho^2$. This completes the proof. \hfill \Box

To obtain a better robust tracking performance, the robust tracking control problem can be formulated as the following minimization problem so that the $H_{\infty}$ tracking performance in (47) is reduced as small as possible.

$$\min_{\bar{P}_i} \rho^2$$
subject to $\bar{P}_i = \bar{P}_i^T > 0$ and (54).

From the analysis above, the most important work of the $H_{\infty}$ fuzzy decentralized observer-based tracking control problem is how to solve the common solution $\bar{P}_i = \bar{P}_i^T > 0$ from the minimization problem (57). In general, it is not easy to analytically determine common solution $\bar{P}_i = \bar{P}_i^T > 0$ for (57). Similarly, (57) can be transferred into a minimization problem subject to some linear matrix inequalities (LMIs) called eigenvalue problem (EVP) [31]. The EVP can be solved in a computationally efficient manner using a convex optimization technique such as the interior point method [31].

The following decoupling technique is employed to simplify the design problem. For the convenience of design, we assume

$$\bar{P}_i = \begin{bmatrix} \bar{P}_{i11} & 0 & 0 \\ 0 & \bar{P}_{i22} & 0 \\ 0 & 0 & \bar{P}_{i33} \end{bmatrix}$$

where $\bar{P}_{i11} = \bar{P}_{i11}^T > 0$, $\bar{P}_{i22} = \bar{P}_{i22}^T > 0$, $\bar{P}_{i33} = \bar{P}_{i33}^T > 0$.

This choice is suitable for the separate design of the fuzzy controller and fuzzy observer. By substituting (58) into (54), we obtain (see (59) at the bottom of the page) where $M_{11} = (A_{ik} + B_{ik}C_{im})^T \bar{P}_{i11} + \bar{P}_{i11}(A_{ik} + B_{ik}K_{im}) + Q_i$. 

$$\begin{bmatrix} M_{11} \\ -(B_{ik}K_{im})^T \bar{P}_{i11} - Q_i \\ -(B_{ik}K_{im})^T \bar{P}_{i22} + \bar{P}_{i22}A_{in} + Q_i \\ A_{ik}^T \bar{P}_{i11} \\ 0 \\ 0 \\ \bar{P}_{i11} \\ 0 \\ 0 \\ 0 \\ 0 \\ -L_{ik} \bar{P}_{i33} \end{bmatrix} 
\begin{bmatrix} \bar{P}_{i11}A_{jk} \\ 0 \\ \bar{P}_{i11} \\ 0 \\ 0 \\ \bar{P}_{i22} \\ 0 \\ \bar{P}_{i33} \end{bmatrix} 
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\rho^2I \\ 0 \\ 0 \end{bmatrix} \leq 0 \quad (59)$$
By introducing a new matrix

\[
\bar{W}_i = \begin{bmatrix}
\bar{W}_{i11} & 0 & \cdots & 0 \\
0 & I & \cdots & \cdots \\
\vdots & 0 & \ddots & \cdots \\
\vdots & \vdots & \ddots & I \\
0 & \cdots & \cdots & 0 & I \\
0 & \cdots & \cdots & 0 & 0
\end{bmatrix}
\]

where \( \bar{W}_{i11} = \tilde{P}_{i11}^{-1} \) and multiplying it into (59), we obtain

\[ W_i \times \{ \text{Left hand side of (59)} \} \times W_i \leq 0, \quad (60) \]

With \( Z_{ik} = \tilde{P}_{i33} L_{ik} \) and \( Y_{im} = K_{im} \tilde{W}_{i11} \), the inequalities in (60) are equivalent to following matrix inequalities (see (61) at the bottom of the page) where \( M_{22} = (A_{ik} \tilde{W}_{i11} + B_{ik} Y_{im})^T + (A_{ik} \tilde{W}_{i11} + B_{ik} Y_{im}) + \tilde{W}_{i11} Q_i \tilde{W}_{i11} \).

Therefore, the \( H_\infty \) fuzzy decentralized observer-based tracking control problem can be reformulated as the following optimization problem:

\[
\begin{aligned}
\min_{\{\tilde{W}_{i11}, \tilde{P}_{i22}, \tilde{P}_{i33}\}} & \quad \rho^2 \\
\text{subject to} & \quad \tilde{W}_{i11} = \tilde{W}_{i11}^T > 0, \\
& \quad \tilde{P}_{i22} = \tilde{P}_{i22}^T > 0, \\
& \quad \tilde{P}_{i33} = \tilde{P}_{i33}^T > 0 \quad \text{and} \quad (61). \quad (62)
\end{aligned}
\]

The analysis above shows that when dealing with the stabilization design problem of the fuzzy observer-based decentralized control system, the most important task is to solve common solutions \( \tilde{W}_{i11} = \tilde{W}_{i11}^T > 0, \tilde{P}_{i22} = \tilde{P}_{i22}^T > 0 \) and \( \tilde{P}_{i33} = \tilde{P}_{i33}^T > 0 \) from (62). Since the variables \( K_{im} \) and \( Y_{im} \) are cross-coupled, there are no effective algorithms for solving these matrix inequalities till now. By the choice of (58), the control problem and observer problem can be decoupled and can be solved separately by the following two-stage procedures which solve the fuzzy control parameters first and then solve the fuzzy observer parameters. These will be discussed in detail in the following.

We first solve \( \tilde{W}_{i11} \) and \( Y_{im} \) from the following matrix inequalities

\[
(A_{ik} \tilde{W}_{i11} + B_{ik} Y_{im})^T + (A_{ik} \tilde{W}_{i11} + B_{ik} Y_{im}) + \tilde{W}_{i11} Q_i \tilde{W}_{i11} \leq 0, \quad (63)
\]

Note that the Schur complements, (63) is equivalent to (see (64) at the bottom of the page).

Note that solving \( \tilde{W}_{i11} \) and \( Y_{im} \) from (64) is a convex linear matrix inequality problem (LMIP). After solving the LMIP in (64) to obtain \( \tilde{W}_{i11} \) and \( Y_{im} \) (thus \( K_{im} = Y_{im} \tilde{W}_{i11}^{-1} \)) and by substituting \( \tilde{W}_{i11} \) and \( Y_{im} \) and \( K_{im} \) into (61), (61) becomes a convex linear matrix inequalities (LMIs).

Second, we can solve the following eigenvalue problem (EVP)

\[
\begin{aligned}
\min_{\{\tilde{P}_{i22}, \tilde{P}_{i33}, Z_{ik}\}} & \quad \rho^2 \\
\text{subject to} & \quad \tilde{P}_{i22} = \tilde{P}_{i22}^T > 0, \\
& \quad \tilde{P}_{i33} = \tilde{P}_{i33}^T > 0 \quad \text{and} \quad (61) \quad (65)
\end{aligned}
\]

to obtain \( \tilde{P}_{i22}, \tilde{P}_{i33} \) and \( Z_{ik} \) (thus \( L_{ik} = \tilde{P}_{i33}^{-1} Z_{ik} \)).

Therefore, the \( H_\infty \) fuzzy decentralized observer-based tracking control problem in (62) is equivalent to solving

\[
\begin{aligned}
\min_{\{\tilde{W}_{i11}, \tilde{P}_{i22}, \tilde{P}_{i33}\}} & \quad \rho^2 \\
\text{subject to} & \quad (64) \quad \text{and} \quad (65).
\end{aligned}
\]

\[
\begin{bmatrix}
-M_{22} & -(B_{ik} K_{im})^T - \tilde{W}_{i11} Q_i & 0 \\
-(B_{ik} K_{im})^T & A_{ik}^T \tilde{P}_{i22} + \tilde{P}_{i22} A_{ik} + Q_i & 0 \\
0 & 0 & I \\
0 & 0 & 0 \\
0 & 0 & \tilde{P}_{i22} \\
0 & 0 & 0
\end{bmatrix} \leq 0 \quad (61)
\]
The design procedures for the $H_{\infty}$ fuzzy observer-based decentralized tracking control of interconnected systems are summarized as follows:

**Design Procedure 2:**

1. Select fuzzy plant rules (37).
2. Solve the LMIP in (64) to obtain $\hat{W}_{i11}$ and $Y_{im}$ (thus $K_{im} = Y_{im} \hat{W}_{i11}$) can also be obtained.
3. Substituting $\hat{W}_{i11}, Y_{im}$, and $K_{im}$ into (61) and then solve the EVP in (65) to obtain $P_{222}, P_{233}$ and $Z_{ik}$ (thus $L_{ik} = P_{235}^{-1} Z_{ik}$ can also be obtained).
4. Construct the fuzzy observer in (40).
5. Obtain fuzzy decentralized control rule in (41).

V. SIMULATION EXAMPLES

We consider a two-machine interconnected system which is composed of two-machine subsystems $S_i$ as follows [33]

$$
S_i : \dot{x}_{i1}(t) = x_{i2}(t)
$$

$$
\dot{x}_{i2}(t) = -\frac{D_i}{M_i} x_{i2}(t) + \frac{1}{M_i} u_i(t)
$$

$$
+ \sum_{j=1, j \neq i}^{2} \frac{E_i E_j Y_{ij}}{M_i} [\cos (\delta_{ij}^0 - \theta_{ij})
- \cos(x_{i1}(t) - x_{j1}(t) + \delta_{ij}^0 - \theta_{ij})]
+ w_{i2}(t)
$$

$$
y_i(t) = x_{i1}(t) + v_i(t) \tag{67}
$$

where

- $x_{i1}(t), x_{i2}(t)$ absolute rotor angle and angular velocity of the $i$th machine, respectively;
- $M_i$ inertia coefficient;
- $D_i$ damping coefficient;
- $E_i$ internal voltage;
- $Y_{ij}$ modulus of the transfer admittance between the $i$th and $j$th machines;
- $\theta_{ij}$ phase angle of the transfer admittance between the $i$th and $j$th machines;
- $w_{i2}(t)$ external disturbance which is assumed to be sinusoidal with amplitude 1 and period $\pi$;
- $v_i(t)$ measurement noise which is assumed to be zero mean white noise with variance 0.1 for $i, j = 1, 2 (i \neq j)$.

We assume the two-machine interconnected systems’ parameters as follows:

- $E_1 = 1.017$  $E_2 = 1.005$
- $M_1 = 1.03$  $M_2 = 1.25$
- $D_1 = 0.8$  $D_2 = 1.2$
- $Y_{12} = Y_{21} = 1.98$
- $\theta_{12} = -\theta_{21} = 1.5$  $\delta_{12}^0 = -\delta_{21}^0 = 1.2$

**Example 1:** Suppose the state variables are available. Now, following the **Design Procedure 1** in the above section, a design procedure of fuzzy decentralized state feedback controller for model reference tracking of the interconnected system is given as follows:

1. To use the fuzzy decentralized control approach, we represent the interconnected system (67) by a fuzzy model. The problem of constructing Takagi–Sugeno fuzzy model for nonlinear systems can be found in [35]–[37]. To minimize the design effort and complexity, we try to use as few rules as possible.

Rule 1

If $x_{11}(t)$ is about $-\frac{\pi}{2}$ and $x_{21}(t)$ is about $-\frac{\pi}{2}$,

Then $\dot{x}_i(t) = A_{i1} x_i(t) + B_{i1} u_i(t)$

$$
+ \sum_{j=1, j \neq i}^{2} A_{ij} x_j(t) + w_i(t),
$$

Rule 2

If $x_{11}(t)$ is about $-\frac{\pi}{2}$ and $x_{21}(t)$ is about 0,

Then $\dot{x}_i(t) = A_{i1} x_i(t) + B_{i1} u_i(t)$

$$
+ \sum_{j=1, j \neq i}^{2} A_{ij} x_j(t) + w_i(t),
$$

Rule 3

If $x_{11}(t)$ is about $-\frac{\pi}{2}$ and $x_{21}(t)$ is about $\frac{\pi}{2}$,

Then $\dot{x}_i(t) = A_{i1} x_i(t) + B_{i1} u_i(t)$

$$
+ \sum_{j=1, j \neq i}^{2} A_{ij} x_j(t) + w_i(t),
$$

Rule 4

If $x_{11}(t)$ is about 0 and $x_{21}(t)$ is about $-\frac{\pi}{2}$,

Then $\dot{x}_i(t) = A_{i1} x_i(t) + B_{i1} u_i(t)$

$$
+ \sum_{j=1, j \neq i}^{2} A_{ij} x_j(t) + w_i(t),
$$

Rule 5

If $x_{11}(t)$ is about 0 and $x_{21}(t)$ is about 0,

Then $\dot{x}_i(t) = A_{i1} x_i(t) + B_{i1} u_i(t)$

$$
+ \sum_{j=1, j \neq i}^{2} A_{ij} x_j(t) + w_i(t).$$
Rule 6

If \( x_{11}(t) \) is about \( \frac{\pi}{2} \) and \( x_{21}(t) \) is about \( \frac{\pi}{2} \),
Then
\[
\dot{x}_i(t) = A_{i1}x_i(t) + B_{i1}u_i(t) + \sum_{j=1,j\neq i}^{2} A_{ij}x_j(t) + w_i(t).
\]

Rule 7

If \( x_{11}(t) \) is about \( \frac{\pi}{2} \) and \( x_{21}(t) \) is about \( -\frac{\pi}{2} \),
Then
\[
\dot{x}_i(t) = A_{i2}x_i(t) + B_{i2}u_i(t) + \sum_{j=1,j\neq i}^{2} A_{ij}x_j(t) + w_i(t).
\]

Rule 8

If \( x_{11}(t) \) is about \( \frac{\pi}{2} \) and \( x_{21}(t) \) is about 0,
Then
\[
\dot{x}_i(t) = A_{i3}x_i(t) + B_{i3}u_i(t) + \sum_{j=1,j\neq i}^{2} A_{ij}x_j(t) + w_i(t).
\]

Rule 9

If \( x_{11}(t) \) is about \( \frac{\pi}{2} \) and \( x_{21}(t) \) is about \( \frac{\pi}{2} \),
Then
\[
\dot{x}_i(t) = A_{i4}x_i(t) + B_{i4}u_i(t) + \sum_{j=1,j\neq i}^{2} A_{ij}x_j(t) + w_i(t).
\]

where \( A_{ik}, B_{ik}, A_{ijk} \) are listed in Appendix A, \( w_i(t) = [0, w_{2i}(t)]^T \) and \( C_i = [1, 0] \) for \( i = 1, 2 \) and \( k = 1, \ldots, 9 \).

The reference models are given as

\[
A_{r1} = \begin{bmatrix} 0 & 1 \\ -100 & -101 \end{bmatrix}
\]

and

\[
r_i(t) = \begin{bmatrix} 0 \\ 100\cos(0.5t) \\ 100\sin(0.5t) \end{bmatrix} \quad \text{for } i = 1, 2.
\]

For the convenience of design, triangle type membership functions are chosen for Rule 1–Rule 9.

Step 2) Solve EVP using the LMI optimization toolbox in Matlab [34]. In this case

\[
W_{11} = \begin{bmatrix} 0.0092 & -0.0376 \\ -0.0376 & 2.6558 \end{bmatrix} \times 10^5
\]

and

\[
W_{21} = \begin{bmatrix} 0.0098 & -0.0471 \\ -0.0471 & 3.4882 \end{bmatrix} \times 10^5.
\]

Step 3) The control parameters can be found in Appendix B. Figs. 1–4 present the simulation results. Initial condition is assumed to be \((x_{11}(0), x_{12}(0), x_{21}(0), x_{22}(0))^T = (1, 0, 1, 0)^T\) in the simulations. The trajectories of the state variable \( x_{11}(t) \) including reference state variable \( x_{r1}(t) \) are shown in Fig. 1.
Example 2: Suppose the state variables are not all available. Then, following the Design Procedure 2 in the above section, a design procedure of fuzzy observer-based decentralized controller for the model reference tracking of the interconnected system is given as follows. We assume the state variables \( x_{11}(t) \) and \( x_{21}(t) \) can be measured to construct the fuzzy model. In other words, the state variables \( x_{12}(t) \) and \( x_{22}(t) \) should be estimated.

Step 1) The same as Example 1.
Step 2) Solve LMIP and EVP using the LMI optimization toolbox in Matlab. In this case
\[
\hat{W}_{111} = \begin{bmatrix}
0.0077 & -0.0493 \\
-0.0493 & 1.7908
\end{bmatrix} \times 10^7
\]
\[
\hat{W}_{211} = \begin{bmatrix}
0.0079 & -0.0488 \\
-0.0488 & 1.7743
\end{bmatrix} \times 10^7
\]
\[
\hat{P}_{122} = \begin{bmatrix}
0.5506 & 0.0070 \\
0.0070 & 0.0025
\end{bmatrix}
\]
\[
\hat{P}_{222} = \begin{bmatrix}
33.8545 & 33.5434 \\
33.5434 & 33.8907
\end{bmatrix}
\]
\[
\hat{P}_{133} = \begin{bmatrix}
0.0686 & -0.0067 \\
-0.0067 & 0.0017
\end{bmatrix}
\]
\[
\hat{P}_{233} = \begin{bmatrix}
0.0924 & -0.0088 \\
-0.0088 & 0.0020
\end{bmatrix}
\]

Step 3) The observer parameters can be found in Appendix C.
Step 4) The control parameters can be found in Appendix D.

Figs. 5–10 present the simulation results. The trajectories of the state variable \( x_{11}(t) \) including reference state variable \( x_{r11}(t) \) is shown in Fig. 5. The trajectories of the state variable \( x_{12}(t) \) including reference state variable \( x_{r12}(t) \) is shown in Fig. 6. The trajectories of the state variable \( x_{12}(t) \) including
estimated state variable $\hat{x}_{21}(t)$ is shown in Fig. 7. The trajectories of the state variable $x_{21}(t)$ including reference state variable $x_{r2}(t)$ is shown in Fig. 8. The trajectories of the state variable $x_{22}(t)$ including reference state variable $x_{r2}(t)$ is shown in Fig. 9. The trajectories of the state variable $x_{23}(t)$ including estimated state variable $\hat{x}_{23}(t)$ is shown in Fig. 10. From the simulation results, the proposed decentralized $H_\infty$ control scheme can solve the tracking problem for nonlinear interconnected systems effectively and systematically with the aid of LMI toolbox in Matlab.

VI. CONCLUSION

In this paper, a Takagi and Sugeno fuzzy model is proposed to study the model reference tracking control problems for nonlinear interconnected systems using fuzzy decentralized control. A fuzzy decentralized controller is proposed to overcome the effect of external disturbances such that the $H_\infty$ model reference tracking performance is achieved. Furthermore, the stability of the nonlinear interconnected systems is also guaranteed. If states are not all available, a decentralized fuzzy observer is also proposed to estimate the states of each subsystem for decentralized control. Consequently, a fuzzy observer-based state feedback decentralized fuzzy controller is proposed to solve the $H_\infty$ tracking control design problem for nonlinear interconnected systems. The problem of $H_\infty$ decentralized fuzzy tracking control design for nonlinear interconnected systems is characterized in terms of solving an eigenvalue problem (EVP). The EVP can be efficiently solved using convex optimization techniques. The proposed fuzzy observer-based decentralized control scheme is simple without complex control algorithms. Therefore, it is suitable for practical applications. Simulation example is given to illustrate the design procedure and tracking performance of the proposed method.

APPENDIX A

$$A_{11} = \begin{bmatrix} 0 & 1 \\ -0.7046 & -0.7767 \end{bmatrix}$$
$$B_{11} = \begin{bmatrix} 0 \\ 0.9709 \end{bmatrix}$$
$$A_{12} = \begin{bmatrix} 0 & 1 \\ -1.4809 & -0.7767 \end{bmatrix}$$
$$B_{12} = \begin{bmatrix} 0 \\ 0.9709 \end{bmatrix}$$
$$A_{13} = \begin{bmatrix} 0 & 1 \\ -1.4536 & -0.7767 \end{bmatrix}$$
$$B_{13} = \begin{bmatrix} 0 \\ 0.9709 \end{bmatrix}$$
$$A_{14} = \begin{bmatrix} 0 & 1 \\ 1.0472 & -0.7767 \end{bmatrix}$$
$$B_{14} = \begin{bmatrix} 0 \\ 0.9709 \end{bmatrix}$$
$$A_{15} = \begin{bmatrix} 0 & 1 \\ -0.5139 & -0.7767 \end{bmatrix}$$
$$B_{15} = \begin{bmatrix} 0 \\ 0.9709 \end{bmatrix}$$

Fig. 8. The trajectories of the state variable $x_{21}(t)$ (solid line) and reference state variable $x_{r2}(t)$ (dashdot line).

Fig. 9. The trajectories of the state variable $x_{22}(t)$ (solid line) and reference state variable $x_{r2}(t)$ (dashdot line).

Fig. 10. The trajectories of the state variable $x_{22}(t)$ (solid line) and estimated state variable $\hat{x}_{r2}(t)$ (dashdot line).
Appendix B

\[ A_{1a} = \begin{bmatrix} 0 & 1 \\ -1.5480 & -0.7767 \end{bmatrix} \]
\[ B_{1a} = \begin{bmatrix} 0 & 0 \\ 0.9709 \end{bmatrix} \]
\[ A_{1b} = \begin{bmatrix} 1 \\ 1.1261 & -0.7767 \end{bmatrix} \]
\[ B_{1b} = \begin{bmatrix} 0 & 0 \\ 0.9709 \end{bmatrix} \]
\[ A_{1c} = \begin{bmatrix} 1 \\ 0.7086 & -0.7767 \end{bmatrix} \]
\[ B_{1c} = \begin{bmatrix} 0 & 0 \\ 0.9709 \end{bmatrix} \]
\[ A_{1d} = \begin{bmatrix} 1 \\ -0.5066 & -0.7767 \end{bmatrix} \]
\[ B_{1d} = \begin{bmatrix} 0 & 0 \\ 0.9709 \end{bmatrix} \]
\[ A_{2a} = \begin{bmatrix} 1 \\ 0.2776 & -0.96 \end{bmatrix} \]
\[ B_{2a} = \begin{bmatrix} 0 & 0 \\ 0.800 \end{bmatrix} \]
\[ A_{2b} = \begin{bmatrix} 1 \\ -0.6478 & -0.96 \end{bmatrix} \]
\[ B_{2b} = \begin{bmatrix} 0 & 0 \\ 0.800 \end{bmatrix} \]
\[ A_{2c} = \begin{bmatrix} 1 \\ -0.9528 & -0.96 \end{bmatrix} \]
\[ B_{2c} = \begin{bmatrix} 0 & 0 \\ 0.800 \end{bmatrix} \]
\[ A_{2d} = \begin{bmatrix} 1 \\ 1.2532 & -0.96 \end{bmatrix} \]
\[ B_{2d} = \begin{bmatrix} 0 & 0 \\ 0.800 \end{bmatrix} \]
\[ A_{2e} = \begin{bmatrix} 1 \\ 0.3200 & -0.96 \end{bmatrix} \]
\[ B_{2e} = \begin{bmatrix} 0 & 0 \\ 0.800 \end{bmatrix} \]
\[ A_{2f} = \begin{bmatrix} 1 \\ -0.1293 & -0.96 \end{bmatrix} \]
\[ B_{2f} = \begin{bmatrix} 0 & 0 \\ 0.800 \end{bmatrix} \]
\[ A_{2g} = \begin{bmatrix} 1 \\ 1.2135 & -0.96 \end{bmatrix} \]
\[ B_{2g} = \begin{bmatrix} 0 & 0 \\ 0.800 \end{bmatrix} \]
\[ A_{2h} = \begin{bmatrix} 1 \\ 1.2206 & -0.96 \end{bmatrix} \]
\[ B_{2h} = \begin{bmatrix} 0 & 0 \\ 0.800 \end{bmatrix} \]
\[ A_{2i} = \begin{bmatrix} 1 \\ 0.5133 & -0.96 \end{bmatrix} \]
\[ B_{2i} = \begin{bmatrix} 0 & 0 \\ 0.800 \end{bmatrix} \]

Appendix C

\[ L_{11} = \begin{bmatrix} 15.2449 \\ 58.9788 \end{bmatrix} \]
\[ L_{12} = \begin{bmatrix} 15.2449 \\ 58.9788 \end{bmatrix} \]
\[ L_{13} = \begin{bmatrix} 15.2449 \\ 58.9788 \end{bmatrix} \]
\[ L_{14} = \begin{bmatrix} 15.2449 \\ 58.9788 \end{bmatrix} \]
\[ L_{15} = \begin{bmatrix} 15.2449 \\ 58.9788 \end{bmatrix} \]
\[ L_{16} = \begin{bmatrix} 15.2449 \\ 58.9788 \end{bmatrix} \]
\[ L_{17} = \begin{bmatrix} 15.2449 \\ 58.9788 \end{bmatrix} \]
\[ L_{18} = \begin{bmatrix} 15.2449 \\ 58.9788 \end{bmatrix} \]
\[ L_{19} = \begin{bmatrix} 15.2449 \\ 58.9788 \end{bmatrix} \]
\[ L_{20} = \begin{bmatrix} 15.2449 \\ 58.9788 \end{bmatrix} \]
\[ L_{21} = \begin{bmatrix} 15.2449 \\ 58.9788 \end{bmatrix} \]
\[ L_{22} = \begin{bmatrix} 15.2449 \\ 58.9788 \end{bmatrix} \]
\[ L_{23} = \begin{bmatrix} 15.2449 \\ 58.9788 \end{bmatrix} \]
\[ L_{24} = \begin{bmatrix} 15.2449 \\ 58.9788 \end{bmatrix} \]
\[ L_{25} = \begin{bmatrix} 15.2449 \\ 58.9788 \end{bmatrix} \]
\[ L_{26} = \begin{bmatrix} 15.2449 \\ 58.9788 \end{bmatrix} \]
\[ L_{27} = \begin{bmatrix} 15.2449 \\ 58.9788 \end{bmatrix} \]
\[ L_{28} = \begin{bmatrix} 15.2449 \\ 58.9788 \end{bmatrix} \]

Appendix D

\[ K_{11} = \begin{bmatrix} -225.3436 \\ 23.8345 \end{bmatrix} \]
\[ K_{12} = \begin{bmatrix} -224.4896 \\ 23.8259 \end{bmatrix} \]
\[ K_{13} = \begin{bmatrix} -224.5195 \\ 23.8262 \end{bmatrix} \]
\[ K_{14} = \begin{bmatrix} -227.2713 \\ 23.8540 \end{bmatrix} \]
\[ K_{15} = \begin{bmatrix} -225.5534 \\ 23.8366 \end{bmatrix} \]
\[ K_{16} = \begin{bmatrix} -224.4156 \\ 23.8251 \end{bmatrix} \]
\[
\]

REFERENCES


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