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Adaptive Attitude Control of Spacecraft: Mixed $H_2/H_\infty$ Approach

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An adaptive mixed $H_2/H_\infty$ attitude control of nonlinear spacecraft systems with unknown or uncertain inertia matrix and external disturbances is presented. The design objective is to specify a controller with a parameter adaptive law such that the adaptive $H_2$ optimal control performance can be achieved under a desired adaptive $H_\infty$ disturbance attenuation constraint. It can be derived that the spacecraft model satisfies the properties of linear parameterization and skew symmetry. An explicit solution to the adaptive mixed $H_2/H_\infty$ attitude control problem can be obtained by combining nonlinear minimax theory and linear quadratic optimal control techniques. Moreover, by virtue of the skew symmetric property of the spacecraft system and adequate choice of state variable transformation, this adaptive control problem can be reduced to solving two algebraic equations instead of a pair of coupled time-varying differential equations. Furthermore, with some simplification, a closed-form solution to these two algebraic equations can lead to a very simple adaptive controller that can be viewed as a mixed $H_2/H_\infty$ proportional and derivative controller with adaptive law. Finally, experimental simulation results based on the Republic of China Satellite-1 spacecraft system are presented to demonstrate the effectiveness of the designed methods.

Nomenclature

$\{h_1, h_2, h_3\} = \text{body fixed reference frame}$

$\{\epsilon_1, \epsilon_2, \epsilon_3\} = \text{inertial fixed reference coordinates}$

$h = \text{total spacecraft angular momentum in body axes,}$

$[h_1, h_2, h_3]^T, J \omega$

$J = \text{inertia matrix}$

$\{\alpha_1, \alpha_2, \alpha_3\} = \text{orbital coordinate reference}$

$P = \text{weighting matrix of attitude state with respect}$

$\text{to the initial time}$

$Q = \text{weighting matrix of attitude state}$

$Q_f = \text{weighting matrix of attitude state with respect}$

$\text{to the final time}$

$Q_1 = \text{weighting matrix of attitude state for the } H_\infty$

$\text{performance}$

$Q_{1f} = \text{weighting matrix of attitude state with respect}$

$\text{to the final time for the } H_\infty$

$\text{performance}$

$Q_2 = \text{weighting matrix of attitude state for the } H_2$

$\text{performance}$

$Q_{2f} = \text{weighting matrix of attitude state with respect}$

$\text{to the final time for the } H_2$

$r = \text{radius of link of attitude}$

$a_{th} = \text{saturation value of the } i\text{th actuator output torque}$

$S_t = \text{weighting matrix of parameter estimation error}$

$\text{for the } H_2\text{ performance}$

$S_\omega = \text{weighting matrix of parameter estimation error}$

$\text{for the } H_\infty\text{ performance}$

$S = \text{state-space transformation matrix}$

$u = \text{control torque vector}$

$u = \text{mixed } H_2/H_\infty\text{ control torque vector}$

$W = \text{weighting matrix of control torque}$

$W_1 = \text{weighting matrix of control torque}$

$\text{for the } H_\infty\text{ performance}$

$W_2 = \text{weighting matrix of control torque}$

$\text{for the } H_2\text{ performance}$

$x = \text{attitude state vector}$

$\gamma = \text{desired disturbance attenuation level}$

$\delta = \text{attitude Euler angle, } \{\delta_1, \delta_2, \delta_3\}\text{ }$

\[ [v \times] = \text{cross product matrix associated with the vector} \]

\[ v = [v_1, v_2, v_3]^T, \]

\[ v_1 0 -v_1 \]

\[ -v_1 v_1 0 \]

$\tau_a = \text{torque vector due to actuators, } [\tau_{a1}, \tau_{a2}, \tau_{a3}]^T$

$\tau_{aee} = \text{aerodynamic disturbance torque vector}$

$\tau_d = \text{external disturbance torque vector}$

$\tau_g = \text{gravity gradient torque vector}$

$\tau_w = \text{solar radiation pressure disturbance torque vector}$

$\Omega = \text{bounded region, } \{x \in \mathbb{R}^3 \mid -\pi/2 < \theta < \pi/2\}$

$\omega = \text{general angular velocity vector in body axes,}$

$[\omega_1, \omega_2, \omega_3]^T$

$\omega_0 = \text{orbital rate}$

Introduction

The attitude control of spacecraft has received extensive attention in recent decades, and several methods of the spacecraft attitude control have been developed to treat this problem. Based on linearization, coordinate transformation and nonlinear feedback, a controller for attitude control has been derived. The feasibility of applying the feedback linearization technique to spacecraft attitude control and the momentum management problem has also been discussed. A sliding manifold approach has also been used for spacecraft attitude control. More relevant to this study, the optimal control theory has been used for attitude control of spacecraft systems. The approach of $H_2$ control has been applied to the space station attitude and momentum control problem while taking into consideration the linearized equations of motion. Recently, Chen and Wu developed a nonlinear $H_\infty$ control design to treat the spacecraft attitude control problem under parameter perturbation and external noise.

Over the past 10 years, mixed $H_2/H_\infty$ optimal control has been studied for linear systems. The main purpose of this type of control is to design an $H_\infty$ optimal control for the worst-case external disturbance whose effects on system output must be attenuated below a desired value, that is, to design an $H_2$ optimal control under the $H_\infty$ disturbance attenuation constraint. This control design is very suitable for the optimal control problem of systems under uncertain external disturbance. Along this line, based on the nonlinear two-player Nash differential game theorem, Wu et al. also developed a unified design for $H_2$, $H_\infty$, and mixed $H_2/H_\infty$ attitude control of spacecraft.

However, in a practical situation, system parameter variations such as those generated by payload motion, vehicle docking, the
rotation of solar arrays, and so on, are inevitable. One can develop an adaptive scheme that will continuously adjust controller gains to compensate for parameter uncertainties and environmental effects to ensure stable and robust performance. For this problem, adaptive attitude control has been presented for spacecraft systems.\textsuperscript{17–22}

In this paper, we design an attitude controller with a parameter adaptive law so that the adaptive $H_2$ optimal control performance can be achieved under a desired adaptive $H_\infty$ disturbance attenuation constraint when the inertia matrix of spacecraft is uncertain (or unknown) and an external disturbance exists. A pair of coupled time-varying differential equations with an adaptive constraint must first be solved to design the controller. By an adequate choice of a state transformation and use of the skew-symmetric property of spacecraft systems, two time-varying coupled differential equations can be transformed into two corresponding algebraic coupled equations, and the adaptive constraint holds. Then, through Cholesky factorization (see Ref. 22), these two coupled algebraic equations can be solved easily. As a result, the structure of the controller becomes very simple and can be viewed as a mixed $H_2/H_\infty$ adaptive proportional and derivative (PD) controller with the control gains depending on the disturbance attenuation level $\gamma$, which is assigned according to mission requirements.

Mathematical Model and Problem Formulation

Spacecraft Model

Consider a spacecraft moving in a circular orbit. The coordinate systems used in the attitude control are shown in Fig. 1. The inertial fixed reference coordinates, $e_1$, $e_2$, and $e_3$, with their origin at the center of the Earth are used to determine the orbital position of the spacecraft. The orbital coordinate reference, $e_1$, $e_2$, and $e_3$, is rotating about the $\omega_2$ axis with respect to the fixed inertial coordinate system $e_1$, $e_2$, and $e_3$ at the orbital rate $\omega_2$. The axes of this reference frame are chosen such that the roll axis $\phi_1$ is in the line of sight, the pitch axis $\theta_2$ is perpendicular to the orbital plane, and the yaw axis $\psi_3$ points toward the Earth. The last reference system used is the body-fixed reference frame, $b_1$, $b_2$, and $b_3$. The orientation of the spacecraft with respect to the reference frame, $e_1$, $e_2$, and $e_3$, is obtained, in this work, by a yaw-pitch-roll ($\phi_1-\theta_2-\psi_3$) sequence of rotations. The origins of both orbit coordinates and body-fixed coordinates are at the center of mass of the spacecraft.

The nonlinear equations of motion, in terms of components along the body-fixed control axes, are given by the attitude kinematics and the spacecraft dynamics and can be written as\textsuperscript{18,24}

$$\omega = R(\theta)\dot{\theta} - \omega(\theta)$$

(1)

for attitude kinematics and

$$J\dot{\omega} = [h \times]\omega + \tau_e + \tau_s + \tau_d$$

(2)

for spacecraft dynamics where

$$R(\theta) = \begin{bmatrix} 1 & 0 & -\sin \theta_2 \\ 0 & \cos \theta_2 & \sin \theta_2 \cos \theta_3 \\ 0 & -\sin \theta_2 \cos \theta_3 & \cos \theta_2 \end{bmatrix}$$

$$\alpha_\theta(\theta) = \alpha_0 + \alpha_1 \begin{bmatrix} \cos \theta_2 \sin \theta_1 \\ \cos \theta_2 \cos \theta_3 \sin \theta_1 + \sin \theta_2 \sin \theta_3 \\ -\sin \theta_2 \cos \theta_3 \cos \theta_1 \sin \theta_3 \end{bmatrix}$$

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}$$

$$\tau_e = 3\omega_0^2(c \times J_c)$$

with

$$c = \begin{bmatrix} -\sin \theta_2 \\ \sin \theta_2 \cos \theta_3 \\ \cos \theta_2 \cos \theta_3 \end{bmatrix}$$

and

$$\tau_s = \tau_{\text{act}} + \tau_{\text{dc}}.$$

Remark 1: If the control torque is limited by the saturation of actuator, the spacecraft dynamics (2) is of the following form:

$$J\dot{\omega} = [h \times]\omega + \tau_e + \tau_{\text{sat}} + \tau_{\text{act}} + \tau_{\text{dc}}$$

(3)

where $\tau_{\text{sat}} = \min(\max(\tau_{\text{act}} - \tau_{\text{sat}}, 0), \tau_{\text{sat}})$ and

$$\text{sat}(\tau_{\text{sat}}) = \begin{cases} \tau_{\text{sat}} & \tau_{\text{sat}} \geq \tau_{\text{sat}} \\ \tau_{\text{sat}} - |\tau_{\text{sat}}| & |\tau_{\text{sat}}| < \tau_{\text{sat}} \\ -\tau_{\text{sat}} & \tau_{\text{sat}} \leq -\tau_{\text{sat}} \end{cases}, \text{ for } i = 1, 2, 3.$$

To apply the proposed design method, the dynamic equation (3) can be put into the form of Eq. (2) with $\tau_e = \tau_{\text{sat}} + \tau_{\text{act}} + \tau_{\text{dc}}$. In this situation, the deviation $\tau_{\text{sat}} + \tau_{\text{act}}$ can be considered as a part of the external disturbance.

Remark 2: This definition of Eq. (1) is defined for all $\theta_1$, $\theta_2$, and $\theta_3$ except $\theta_2 = \pm(2n + 1)(\pi/2)\text{ for any integer } n$. The singularity [i.e., the determinant of matrix $R(\theta)$ becomes zero at $\theta_2 = \pm(2n + 1)(\pi/2)$] arises owing to the choice of the set of rotations that define the orientation of the spacecraft relative to the orbital frame. However, when the orientation corresponding to the singularity at $\theta_2 = \pm(2n + 1)(\pi/2)$ lies in the control region of attitude angles, another set of rotations can be defined to eliminate this singularity.\textsuperscript{28}

In this paper, we are interested in the attitudes in the bounded region $\Omega$.

Differentiating Eq. (1) gives

$$\dot{\omega} = R(\theta)\dot{\theta} - \frac{d}{dt}R(\theta)\dot{\theta} - \frac{d}{dt}\alpha_\theta(\theta)$$

(4)

Substituting $\omega$ and $\dot{\omega}$ from Eqs. (1) and (4) into Eq. (2) and premultiplying Eq. (4) by the matrix $R^T(\theta)$, we obtain

$$M(\theta)\dot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + G(\theta, \dot{\theta}) = u + d$$

(5)

where

$$M(\theta) = R^T(\theta)JR(\theta)$$

$$C(\theta, \dot{\theta}) = R^T(\theta)J\frac{d}{dt}R(\theta) - R^T(\theta)[h \times]R(\theta)$$

$$G(\theta, \dot{\theta}) = -R^T(\theta)J\frac{d}{dt}\alpha_\theta(\theta) + R^T(\theta)[h \times]\omega(\theta) - 3\omega_0^2R^T(\theta)(c \times J_c)$$

$$u = R^T(\theta)\tau_e,$$

$$d = R^T(\theta)\tau_d$$

Fig. 1 ROCSAT-1 spacecraft and coordinate systems used in the spacecraft attitude control.
In this paper, we consider the nonlinear spacecraft model in Eq. (5) under the following assumption.

Assumption: The nominal inertia matrix \( J \) is an unknown constant and is symmetric positive definite. Moreover, the external disturbance \( \tau_d \) is bounded but unknown.

Properties: Under the preceding assumption, the spacecraft system (5) has the following properties.

1) The matrix \( M(\theta) \) is symmetric positive definite.
2) The matrix \( \frac{1}{2} \{ (\partial C/\partial \theta) \} M(\theta) - C(\theta, \dot{\theta}) \) is skew symmetric,

\[
x^T \begin{bmatrix} 
\frac{1}{2} \frac{d}{dt} M(\theta) - C(\theta, \dot{\theta}) \end{bmatrix} x = 0, \quad \forall x \in R^3 \tag{6}
\]

3) The spacecraft parameter matrices \( M, C, \) and \( G \) form the following linear parameterization property

\[
M(\theta)\dot{y} + C(\theta, \dot{\theta})y + G(\theta, \dot{\theta}) = \Phi(\theta, \dot{\theta}, y, y)\xi \tag{7}
\]

where \( \xi = [J_1, J_2, J_3, J_4, J_5, J_6]^T \) denotes the unknown constant parameter vector and the known regression matrix \( \Phi(\theta, \dot{\theta}, y, y) \) is described in the Appendix.

Problem Formulation

In general, precise knowledge about system parameters \( M, C, \) and \( G \) is required while calculating an appropriate applied torque \( \tau_d \) in the mixed \( H_2/H_\infty \) attitude control problem of the given spacecraft systems. 24 In this paper, we deal with the adaptive mixed \( H_2/H_\infty \) attitude control problem of the given spacecraft systems under unknown inertia matrix \( J \) and external disturbance \( \tau_d \). In this case, the control law must be identified and adapted to the operation conditions.

If we define the controlled state as follows

\[
x = \begin{bmatrix} \dot{\theta} \end{bmatrix} \tag{8}
\]

and introduce a filtered link of state \( r \) and a state-space transformation matrix \( T \) as

\[
r = \lambda_1 \theta + \lambda_2 \dot{\theta} \tag{9a}
\]

\[
T = \begin{bmatrix} I_3 \\ \lambda_1 I_3 \times \lambda_2 I_3 \times I_3 \\ I_3 \times \lambda_3 \\ 0_3 \times 3 \end{bmatrix} \tag{9b}
\]

for some positive constants \( \lambda_1 \) and \( \lambda_2 \), which should be adequately determined later, then we have

\[
r = -M^{-1}(\theta)C(\theta, \dot{\theta})r + \lambda_2 M^{-1}(\theta)\Phi(\theta, \dot{\theta}, \lambda_1/\lambda_2, \theta).
\]

\[
(\lambda_1/\lambda_2)\dot{\xi} + u + d \tag{10}
\]

where

\[
\Phi(x, t, \xi) = \Phi(\theta, \dot{\theta}, \lambda_1/\lambda_2, \theta, \lambda_1/\lambda_2, \dot{\theta}) \xi = (\lambda_1/\lambda_2)M(\theta)\dot{\xi} + (\lambda_1/\lambda_2)C(\theta, \dot{\theta})\xi - G(\theta, \dot{\theta})
\]

Thus, by Eqs. (8–10), the dynamic equation of the spacecraft attitude control system can be obtained as

\[
\dot{x}(t) = T^{-1} \begin{bmatrix} r(t) \\ \dot{\theta}(t) \end{bmatrix} = A_T(x, t) x + B_T(x, t) \Phi(x, t, \xi) + u + d \tag{11}
\]

where

\[
A_T(x, t) = \begin{bmatrix} 0_3 \times 3 & I_3 \times 3 \\ -\lambda_1/\lambda_2M^{-1}(\theta)C(\theta, \dot{\theta}) & -M^{-1}(\theta)C(\theta, \dot{\theta}) - (\lambda_1/\lambda_2)I_3 \times 3 \end{bmatrix}
\]

\[
B_T(x, t) = \begin{bmatrix} 0_3 \times 3 \\ M^{-1}(\theta) \end{bmatrix}
\]

If the following applied torque is selected,

\[
u = u_s - \Phi(x, t, \xi) \tag{12}
\]

then the dynamic equation driven by the control torque \( u_s \) becomes

\[
x = A_T(x, t)x + B_T(x, t)\Phi(x, t, \xi) + u_s + d \tag{13}
\]

where \( \xi = \xi - \dot{\xi} \) denotes the estimation error.

Formally, the adaptive attitude control problem that is considered in this paper for the dynamic equation of spacecraft system in Eq. (13) can be stated as follows.

Adaptive Mixed \( H_2/H_\infty \) Attitude Control Problem

Consider the nonlinear spacecraft dynamic system (13). Given a desired disturbance attenuation level \( \gamma > 0 \) and weighting matrices \( Q_1, Q_2, W_1, \) and \( W_2, \) the adaptive mixed \( H_2/H_\infty \) attitude control problem is said to be solved if there exists a control law \( u_s(t) \) and an adaptive law of \( \xi(t) \) such that the following \( H_2 \) (quadratic) optimal attitude control performance

\[
\min_{u_s(t) \in L_2[0, \tau_f]} \left[ \int_0^{\tau_f} \left( x^T(t)Q_2x(t) + \dot{\xi}^T(t)S_1\dot{\xi}(t) + \int_0^{\tau_f} x^T(t)Q_1x(t) + u_s^T(t)W_1u_s(t) \right) dt \right] \tag{14}
\]

can be achieved for all \( t_f \in [0, \infty) \) and for some positive-definite matrices \( Q_2 > 0, S_1 > 0 \) satisfying the disturbance attenuation constraints:

\[
x^T(t)Q_2x(t) + \dot{\xi}^T(t)S_1\dot{\xi}(t) + \int_0^{\tau_f} x^T(t)Q_1x(t) + u_s^T(t)W_1u_s(t) \tag{15}
\]

for some positive-definite matrices \( Q_1 > 0, S_1 > 0. \) That is, not only the \( H_2 \) optimal attitude control performance is achieved, but also the effect of disturbance \( d(t) \) on the state \( x(t) \) and control \( u_s(t) \) must be attenuated below a desired level \( \gamma^2 \) from the energy point of view.

Remark 3: Let

\[
J_1(u_s, d) = \int_0^{\tau_f} [x^T(t)Q_2x(t) + \dot{\xi}^T(t)S_1\dot{\xi}(t) + \int_0^t x^T(s)Q_1x(s) + u_s^T(t)W_1u_s(t)] dt \tag{16a}
\]

\[
J_2(u_s, d) = \int_0^{\tau_f} x^T(t)Q_2x(t) + \dot{\xi}^T(t)S_1\dot{\xi}(t) + \int_0^t x^T(s)Q_1x(s) + u_s^T(t)W_1u_s(t) \tag{16b}
\]

Then, the adaptive mixed \( H_2/H_\infty \) attitude control with both performances (14) and (15) is equivalent to finding the control law \( u_s(t) \), the adaptive law of \( \xi(t) \), and the worst-case disturbance \( d^*(t) \) such that

\[
J_1[u_s(t), d^*(t)] \geq J_1[u_s^*(t), d^*(t)] \tag{17a}
\]

\[
J_2[u_s^*(t), d^*(t)] \leq J_2[u_s, d^*(t)] \tag{17b}
\]

This mixed performance with parameter estimation can be viewed as an adaptive attitude control design approach that minimizes an \( H_2 \) cost function under the \( H_\infty \) disturbance attenuation constraint on the spacecraft attitude control system associated with the unknown parameter and external disturbance.

By use of a state transformation and the property of symmetry, the given adaptive mixed \( H_2/H_\infty \) attitude control problem is solved as follows.
Substituting the dynamic equation (13) into Eq. (21) leads to
\[ J_2(u_c, d) = x^T(0)P_2[x(0), 0]x(0) + \xi^T(0)S\xi(0) \]
\[ + \int_0^T \left[ x^T(t)\left[ \dot{P}_2(x,t) + P_2(x,t)A_27(x,t) + A_7^T(x,t)P_2(x,t) \right]x(t) \right. \]
\[ + Q_2\left[x(t) + u^*(t)Wu_c(t) + \xi^T(t)B_7^T(x,t)P_2(x,t)\right]x(t) \]
\[ \left. + x^T(t)P_2(x,t)B_7(x,t)u(t) + x^T(t)B_7^T(x,t)P_2(x,t)x(t) \right]dt \]
\[ + x^T(t)B_7(x,t)u(t) + \xi^T(t)S\xi(t) + \xi^T(t)S\xi(t) \]
\[ + \xi^T(t)\Phi(x,t)B_7^T(x,t)P_2(x,t)x(t) \]
\[ + x^T(t)P_2(x,t)B_7(x,t)\Phi(x,t)\xi(t) \]}
\[ dt \]
(22)

Thus, by the worst-case disturbance \( d^*(x,t) \) in Eq. (20), the adaptive law \( \xi(x,t) \) in Eq. (18c) given that \( \dot{\xi}(x,t) = -\xi(x,t) \) and the differential equation (19b), we have
\[ J_2(u_c, d^*(x,t)) = x^T(0)P_2[x(0), 0]x(0) + \xi^T(0)S\xi(0) \]
\[ + \int_0^T \left\{ [x(t) + W^{-1}B_7^T(x,t)P_2(x,t)x(t)]^T \right. \]
\[ \times W[u_c(t) + W^{-1}B_7^T(x,t)P_2(x,t)x(t)] \left. \int_0^t dt \right\} \]
(23)

which, by the control law in Eq. (18b), results in
\[ J_2(u^*_c(x,t), d^*(x,t)) = x^T(0)P_2[x(0), 0]x(0) + \xi^T(0)S\xi(0) \]

Then, we have
\[ J_2[u^*_c(x,t), d^*(x,t)] \leq J_2[u_c, d^*(x,t)] \quad \forall u_c(t) \in L_2[0, t_f] \]
(25)

Similarly, the cost function \( J_1(u_c, d) \) in Eq. (16a) can be rewritten as
\[ J_1(u_c, d) = x^T(0)P_1[x(0), 0]x(0) + \xi^T(0)S\xi(0) \]
\[ + \int_0^T \left[ x^T(t)[P_1(x,t) + P_1(x,t)A_27(x,t) + A_7^T(x,t)P_1(x,t) \right]x(t) \]
\[ + Q_1\left[x(t) + u^*_c(t)Wu_c(t) + \xi^T(t)B_7^T(x,t)P_1(x,t)x(t) \right] \]
\[ + x^T(t)P_1(x,t)B_7(x,t)u(t) + x^T(t)B_7^T(x,t)P_1(x,t)x(t) \]
\[ + x^T(t)B_7(x,t)u(t) + \xi^T(t)S\xi(t) + \xi^T(t)S\xi(t) \]
\[ + \xi^T(t)\Phi(x,t)B_7^T(x,t)P_1(x,t)x(t) \]
\[ + x^T(t)P_1(x,t)B_7(x,t)\Phi(x,t)\xi(t) \]}
\[ dt \]
(26)

It can be deduced from the differential equation (19a), the optimal control \( u^*_c(x,t) \) in Eq. (18b), the adaptive law \( \xi(x,t) \) in Eq. (18c), and the relation in Eq. (19c) that
\[ J_1[u^*_c(x,t), d] = x^T(0)P_1[x(0), 0]x(0) + \xi^T(0)S\xi(0) \]
\[ + \int_0^T \left[ x^T(t)[P_1(x,t) + P_1(x,t)A_27(x,t) + A_7^T(x,t)P_1(x,t) \right]x(t) \]
\[ + Q_1\left[x(t) + u^*_c(t)Wu_c(t) + \xi^T(t)B_7^T(x,t)P_1(x,t)x(t) \right] \]
\[ + x^T(t)P_1(x,t)B_7(x,t)u(t) + x^T(t)B_7^T(x,t)P_1(x,t)x(t) \]
\[ + x^T(t)B_7(x,t)u(t) + \xi^T(t)S\xi(t) + \xi^T(t)S\xi(t) \]
\[ + \xi^T(t)\Phi(x,t)B_7^T(x,t)P_1(x,t)x(t) \]
\[ + x^T(t)P_1(x,t)B_7(x,t)\Phi(x,t)\xi(t) \]}
\[ dt \]
(27)

Then, we can conclude that
\[ J_1[u^*_c(x,t), d^*(x,t)] \geq J_1[u^*_c(x,t), d] \quad \forall d(t) \in L_2[0, t_f] \]
(29)

QED
Remark 4: Part 1 of Remark 4 is as follows. If only the adaptive $H_2$ optimal attitude control design is considered, the desired disturbance attenuation constraint is negligible, so $\gamma = \infty$. In this situation, the adaptive $H_2$ control law is given as

$$ u^* (x, t) = u_0^* (x, t) - \Phi (x, t) \hat{\xi} (x, t) \quad (30a) $$

where

$$ u_0^* (x, t) = - W^{-1} B T (x, t) P (x, t) x (t) \quad (30b) $$

$$ \hat{\xi} (x, t) = S^{-1} \Phi^T (x, t) B T^T (x, t) P (x, t) x (t) \quad (30c) $$

and $P (x, t) = P^T (x, t) \geq 0$ is the solution of the following time-varying differential equation:

$$ \dot{P} (x, t) + P (x, t) A_T (x, t) + A_T^T (x, t) P (x, t) + Q = 0 \quad (31a) $$

$$ P (x (t_f), t_f) = Q_f \quad (31b) $$

This result is the same as the time-varying differential equation in Eq. (19a) or Eq. (19b) with

$$ P (x, t) = P_1 (x, t) = P_2 (x, t) \quad Q = Q_1 = Q_2 $$

$$ Q_f = Q_{1f} = Q_{2f} $$

Part 2 of Remark 4 is as follows. If only the adaptive $H_\infty$ attitude control design is considered, the criteria (17a) and (17b) are combined into the following dynamic game problem:

$$ J \left[ u^* (x, t, d) \right] \leq J \left[ u_0^* (x, t, d) \right] \leq J \left[ u_0, d^* (x, t) \right] \quad (32a) $$

with

$$ J (u_0, d) = \int L (x, t) Q (x, t) + \int \left[ W \tilde{u}^T (t) d (t) \right] dt \quad (32b) $$

In this situation, the solution of the adaptive $H_\infty$ attitude control of the spacecraft system is given as

$$ u^* (x, t) = u_0^* (x, t) - \Phi (x, t) \hat{\xi} (x, t) \quad (33a) $$

where

$$ u_0^* (x, t) = - W^{-1} B T (x, t) P (x, t) x (t) \quad (33b) $$

$$ \hat{\xi} (x, t) = S^{-1} \Phi^T (x, t) B T^T (x, t) P (x, t) x (t) \quad (33c) $$

and $P (x, t) = P^T (x, t) \geq 0$ is the solution of the following time-varying differential equation

$$ \dot{P} (x, t) + P (x, t) A_T (x, t) + A_T^T (x, t) P (x, t) + Q = 0 \quad (34a) $$

$$ P (x (t_f), t_f) = Q_f \quad (34b) $$

This result is the same as the time-varying differential equation in Eq. (19a) with $P (x, t) = P_1 (x, t) = P_2 (x, t), Q = Q_1$, and $Q_f = Q_{1f}$.

Solution of the Time-Varying Differential Equations

In general, however, it is difficult to solve $P_1 (x, t)$ and $P_2 (x, t)$ in the coupled time-varying differential equations (19a) and (19b), especially satisfying the adaptive constraint in Eq. (19c) simultaneously. Fortunately, in the spacecraft system, the differential equations (19a) and (19b) can be further simplified to algebraic matrix equations by adequately selecting the nonlinear function matrices $P_1 (x, t)$ and $P_2 (x, t)$ and by using the skew-symmetric property in Eq. (6). Moreover, with the selected matrices $P_1 (x, t)$ and $P_2 (x, t), Q_1$ can be satisfied.

Because state transformation (9b) has been involved in the process of design, without loss of generality, we suggest that the solutions $P_1 (x, t)$ and $P_2 (x, t)$ of the coupled differential equations (19a) and (19b) can be put in more explicit forms as follows:

$$ P_1 (x, t) = T^T B M (x, t) B^T T + \begin{bmatrix} K_1 & 0_{3 \times 3} \\ 0_{3 \times 3} & 0_{3 \times 3} \end{bmatrix} \quad (35a) $$

$$ P_2 (x, t) = T^T B M (x, t) B^T T + \begin{bmatrix} K_2 & 0_{3 \times 3} \\ 0_{3 \times 3} & 0_{3 \times 3} \end{bmatrix} \quad (35b) $$

where $K_1$ and $K_2$ are some positive-definite symmetric constant matrices and $B = \{ I_{3 \times 3}, 0_{3 \times 3} \}^T$. In the following paragraphs, it is demonstrated that, under some conditions, these suggested matrices $P_1 (x, t)$ and $P_2 (x, t)$ are the solutions of differential equations (19a) and (19b). Furthermore, the constant matrices $T, K_1,$ and $K_2$ can be solved from a pair of coupled algebraic equations.

Consider the second and third terms on the left-hand side of differential equations (19a) and (19b). Using the skew-symmetric property in Eq. (6) and the selected relations in Eqs. (35a) and (35b), we get

$$ \dot{P}_1 (x, t) + P_1 (x, t) A_T (x, t) + A_T^T (x, t) P_1 (x, t) = \begin{bmatrix} 0_{3 \times 3} & K_1 \\ K_1 & 0_{3 \times 3} \end{bmatrix} \quad (36a) $$

$$ \dot{P}_2 (x, t) + P_2 (x, t) A_T (x, t) + A_T^T (x, t) P_2 (x, t) = \begin{bmatrix} 0_{3 \times 3} & K_2 \\ K_2 & 0_{3 \times 3} \end{bmatrix} \quad (36b) $$

It can also be easily checked that

$$ B_T (x, t) P_1 (x, t) = \lambda_2 B^T \quad (37a) $$

$$ B_T (x, t) P_2 (x, t) = \lambda_2 B^T \quad (37b) $$

which satisfy the condition in Eq. (19c).

By the use of the results of Eqs. (36a–37b), coupled differential equations (19a) and (19b) can be reduced to the following coupled algebraic equations:

$$ \begin{bmatrix} 0_{3 \times 3} & K_1 \\ K_1 & 0_{3 \times 3} \end{bmatrix} + Q_1 - \lambda_1^2 T^T B \left( W^{-1} - \frac{1}{\gamma^2} I_{3 \times 3} \right) B^T T = 0 \quad (38a) $$

$$ \begin{bmatrix} 0_{3 \times 3} & K_2 \\ K_2 & 0_{3 \times 3} \end{bmatrix} + Q_2 - \lambda_2^2 T^T B \left( W^{-1} - \frac{2}{\gamma^2} I_{3 \times 3} \right) B^T T = 0 \quad (38b) $$

In addition, the optimal control law, adaptive law, and the worst-case disturbance can be rewritten as

$$ u^* (x, t) = - \lambda_1 W^{-1} r (t) \quad (39a) $$

$$ \dot{\xi} (x, t) = \lambda_2 S^{-1} \Phi^T (x, t) r (t) \quad (39b) $$

$$ d^* (x, t) = \left( \lambda_2 / \gamma^2 \right) r (t) \quad (39c) $$

where $r (t)$ is in Eq. (9a). It is obvious that Eqs. (38a–39c) are all based on known matrices or variables and are applicable. From the preceding analysis, matrices $P_1 (x, t)$ and $P_2 (x, t)$ in Eqs. (35a) and (35b) are the solutions of coupled differential equations (19a) and (19b) if matrices $K_1, K_2,$ and $T$ satisfy coupled algebraic equations (38a) and (38b) simultaneously. Furthermore, the positive-definite symmetric property of $K_1$ and $K_2$ must be satisfied. To guarantee
the solvability, further assumptions and constraints on the weighting matrices \( Q_1, Q_2 \), and \( W \) are required.

For the simplicity of design, let

\[
W = a^2 l_{3X3}, \quad S = b l_{6X6}, \quad Q_1 = \alpha Q_1 \tag{40}
\]

where \( a > 0, b > 0, \) and \( \alpha > 0 \) and the positive-definite symmetric matrix \( Q_1 \) can be factorized by the Cholesky factorization (see Ref. 21) as

\[
Q_1 = \begin{bmatrix}
q_{11}^2 l_{3X3} & Q_{12} \\
Q_{12}^T & q_{22}^2 l_{3X3}
\end{bmatrix}
\tag{41}
\]

With the definitions of \( T \) and \( B \) in Eqs. (9b) and (11) and the forms in Eqs. (40) and (41), the coupled algebraic equations in Eqs. (38a) and (38b) can be solved by the following equalities:

\[
\lambda_1 = \frac{\sqrt{a y q_{11}}}{2 \sqrt{q_{22}^2 (y^2 - a^2)}} \tag{42a}
\]

\[
\lambda_2 = \frac{q_{22}^2}{q_{11}} \lambda_1 \tag{42b}
\]

\[
\alpha = \frac{y^2 - 2a^2}{y^2 - a^2} \tag{42c}
\]

\[
K_1 = q_{11} q_{22} l_{3X3} - Q_{12} \tag{42d}
\]

\[
K_2 = \alpha K_1 \tag{42e}
\]

with \( 0 < a < y / \sqrt{2} \) and \( Q_{12} < q_{11} q_{22} l_{3X3} \).

From this analysis, the solution to the adaptive mixed \( H_2/H_{\infty} \) attitude control problem is concluded in the following corollary.

**Corollary:** For the adaptive mixed \( H_2/H_{\infty} \) attitude control, given a desired disturbance attenuation level \( \gamma > 0 \), let the weighting matrix \( W, S, Q_1, \) and \( Q_2 \) be taken as in Eqs. (40) and (41) with \( \alpha \) satisfying the requirement in Eq. (42c) and \( Q_{12} < q_{11} q_{22} l_{3X3} \). If the constant \( a \) in the weighting matrix \( W \) satisfies

\[
0 < a < y / \sqrt{2} \tag{43}
\]

then the following adaptive mixed \( H_2/H_{\infty} \) control law solves the adaptive mixed \( H_2/H_{\infty} \) attitude control problem

\[
u^*(x, t) = u^*_1(x, t) - \Phi(x, t) \xi(x, t) \tag{44a}
\]

where

\[
u^*_1(x, t) = \frac{y}{a \sqrt{y^2 - a^2}} (q_{11} \theta + q_{22} \theta) \tag{44b}
\]

\[
\xi(x, t) = \frac{y}{b \sqrt{y^2 - a^2}} \Phi^T(x, t) (q_{11} \theta + q_{22} \theta) \tag{44c}
\]

Also, the worst-case disturbance is

\[
d^*(x, t) = \left( \frac{y}{a \sqrt{y^2 - a^2}} (q_{11} \theta + q_{22} \theta) \right) \tag{44d}
\]

**Remark 5:** The designed optimal control law in Eq. (44b) can be viewed as a PD control law

\[
u^*_1(x, t) = K_1 \theta + K_2 \theta \tag{45}
\]

with the control gains \( K_1 = -y q_{11} / (a \sqrt{y^2 - a^2}) \) and \( K_2 = -y q_{22} / (a \sqrt{y^2 - a^2}) \), both of which are adjusted by the attenuation level \( \gamma \) according to the mission desired.

**Design Algorithm**

Based on the preceding discussion, the proposed spacecraft adaptive attitude control design can be outlined as the following design algorithm.

The adaptive mixed \( H_2/H_{\infty} \) attitude control design algorithm consists of three steps:

1) Choose a desired level of disturbance attenuation, \( \gamma > 0. \)

2) Select the weighting matrices, \( W = a^2 l_{3X3} \) and \( S = b l_{6X6} \), such that \( 0 < \alpha < y / \sqrt{2} \) and \( b > 0 \)

\[
Q_1 = \begin{bmatrix}
q_{11}^2 l_{3X3} & Q_{12} \\
Q_{12} & q_{22}^2 l_{3X3}
\end{bmatrix}
\]

with \( Q_{12} < q_{11} q_{22} l_{3X3} \) and \( Q_2 = [(y^2 - 2a^2)/(y^2 - a^2)] Q_1. \)

3) Obtain the corresponding adaptive mixed \( H_2/H_{\infty} \) control law associated with the update law for the spacecraft system in Eqs. (1) and (2)

\[
\tau = R^{-T}(\theta) \left[ \begin{array}{c}
\frac{y}{a \sqrt{y^2 - a^2}} (q_{11} \theta + q_{22} \theta) - \Phi(x, t) \xi \\
\frac{y}{b \sqrt{y^2 - a^2}} \Phi^T(x, t) (q_{11} \theta + q_{22} \theta)
\end{array} \right] \tag{46a}
\]

\[
\dot{\xi} = \frac{y}{b \sqrt{y^2 - a^2}} \Phi^T(x, t) (q_{11} \theta + q_{22} \theta) \tag{46b}
\]

where the regression matrix \( \Phi(x, t) \) is defined in the Appendix.

**Remark 6:** By following Remark 4 and using the same techniques as in the preceding paragraph, we have the following results.

1) For the adaptive \( H_2 \) attitude control design case, the applied torque and the update law for the spacecraft system in Eqs. (1) and (2) are

\[
\tau = R^{-T}(\theta) (1/\alpha) (q_{11} \theta + q_{22} \theta) - \Phi(x, t) \xi \tag{47a}
\]

\[
\dot{\xi} = (\alpha/b) \Phi^T(x, t) (q_{11} \theta + q_{22} \theta) \tag{47b}
\]

with \( a > 0 \) and \( b > 0. \)

2) For the adaptive \( H_{\infty} \) attitude control design case, the applied torque and the update law for the spacecraft system in Eqs. (1) and (2) are

\[
\tau = R^{-T}(\theta) \left[ \begin{array}{c}
\frac{y}{a \sqrt{y^2 - a^2}} (q_{11} \theta + q_{22} \theta) - \Phi(x, t) \xi \\
\frac{y}{b \sqrt{y^2 - a^2}} \Phi^T(x, t) (q_{11} \theta + q_{22} \theta)
\end{array} \right] \tag{48a}
\]

\[
\dot{\xi} = \frac{y}{b \sqrt{y^2 - a^2}} \Phi^T(x, t) (q_{11} \theta + q_{22} \theta) \tag{48b}
\]

with \( 0 < a < \gamma \) and \( a > 0. \)

**Simulation Results**

The experimental simulations of the Republic of China Satellite-1 (ROCSAT-1) spacecraft (Fig. 1) have been made with the assistance of the National Space Program Office in Taiwan, Republic of China. To substantiate the performance of the adaptive mixed \( H_2/H_{\infty} \) attitude controller design, we consider the unknown inertia matrix of the form

\[
J = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\
J_{12} & J_{22} & J_{23} \\
J_{13} & J_{23} & J_{33}
\end{bmatrix}
\]

and the unknown parameter vector to be estimated by \( \xi = [J_{11}, J_{12}, J_{13}, J_{22}, J_{23}, J_{33}]^T \) with the initial value \( \xi(0) = [126.98, -1.87, 3.38, 116.63, -2.40, 209.36]^T \). By Eq. (7), we can calculate the regression matrix \( \Phi(\theta, \theta, (\lambda_{1,k} / \lambda_{2,k}) \theta, (\lambda_{1,k} / \lambda_{2,k}) \theta) \) of dimension \( 3 \times 6 \) (see the Appendix). The orbital rate for the ROCSAT-1 spacecraft at its nominal 600 km circular orbit environment is \( \omega_{0} = 0.0011 \text{ rad/s} \). The external disturbance \( \tau_{exo} \) and \( \tau_{eleb} \) in the body frame are shown in Fig. 2. We consider that the attitude state \( x = [\theta] \text{T} \) of spacecraft, with initial values at \( \theta(0) = (\pi/45, \pi/45, \pi/45) \) and \( \dot{\theta}(0) = (0, 0, 0) \), is required to decay to zero. Moreover, for the ROCSAT-1 spacecraft, the output torque vector of the reaction wheel is limited in amplitude due to saturation. Thus, we set the saturation values \( S_{sat} = 0.015 \text{ N.m} \), \( i = 1, 2, 3 \) in Eq. (3).

**Simulation 1**

To verify the ability for disturbance attenuation of the proposed method, three cases of control designs are considered: adaptive \( H_2 \) attitude control design, adaptive \( H_{\infty} \) attitude control design, and the adaptive mixed \( H_2/H_{\infty} \) attitude control design. In all cases, we
choose the weighting matrix $S$ to be the identity matrix $I_{6 \times 6}$, that is, $b = 1$. Other simulation parameters are selected as follows:

1) For case 1, adaptive $H_2$ optimal attitude control, select $\alpha = 0.6$ and $Q = I_{6 \times 6}$.

2) For case 2, adaptive $H_\infty$ attitude control, select $\gamma = 0.6$, $\alpha = 0.42$, and $Q = I_{6 \times 6}$.

3) For case 3, adaptive mixed $H_2/H_\infty$ attitude control, select $\gamma = 0.6$, $Q_{1} = I_{6 \times 6}$, and $\alpha = 0.3464$. Then, from Eq. (42c), we select $\alpha = 0.5$.

For the adaptive $H_2$ optimal attitude control case, the applied torque and update law in Eqs. (47a) and (47b) are used. For the adaptive $H_\infty$ attitude control case, the applied torque and update law in Eqs. (48a) and (48b) are used. For the adaptive mixed $H_2/H_\infty$ attitude control case, the applied torque and update law in Eqs. (46a) and (46b) are used. The simulation results are shown in Figs. 3–7. The attitude angles $\theta_1$, $\theta_2$, and $\theta_3$ are represented in Fig. 3, and the attitude angle rates $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$ are depicted in Fig. 4 for the preceding three cases. As the results of these three proposed methods...
Fig. 4 Attitude angles rates: ——, for adaptive $H_2 (\alpha = 0.5)$; ——, for adaptive $H_\infty (\gamma = 0.6, \alpha = 0.42)$; and ———, for adaptive mixed $H_2/H_\infty (\gamma = 0.6, \alpha = 0.5, \alpha = 0.3464)$.

Fig. 5 Steady-state response of attitude angles: ——, for adaptive $H_2 (\alpha = 0.5)$; ———, for adaptive $H_\infty (\gamma = 0.6, \alpha = 0.42)$; and ———, for adaptive mixed $H_2/H_\infty (\gamma = 0.6, \alpha = 0.5, \alpha = 0.3464)$. 
Fig. 6  Attitude tracking performances: ———, for adaptive $H_2$ ($\alpha = 0.6$); ———, for adaptive $H_\infty$ ($\gamma = 0.6, \sigma = 0.42$); and ———, for adaptive mixed $H_2/H_\infty$ ($\gamma = 0.6, \alpha = 0.5, \sigma = 0.3464$).

Fig. 7  Applied torques: ———, for adaptive $H_2$ ($\alpha = 0.6$); ———, for adaptive $H_\infty$ ($\gamma = 0.6, \sigma = 0.42$); and ———, for adaptive mixed $H_2/H_\infty$ ($\gamma = 0.6, \alpha = 0.5, \sigma = 0.3464$).
Fig. 8  Attitude angles: ---, for adaptive mixed $H_2$/$H_\infty$ ($\gamma = 0.8$, $\alpha = 0.5$); --, for adaptive mixed $H_2$/$H_\infty$ ($\gamma = 0.7$, $\alpha = 0.5$); and ----, for adaptive mixed $H_2$/$H_\infty$ ($\gamma = 0.6$, $\alpha = 0.5$).

Fig. 9  Attitude tracking performances: ---, for adaptive mixed $H_2$/$H_\infty$ ($\gamma = 0.8$, $\alpha = 0.5$); --, for adaptive mixed $H_2$/$H_\infty$ ($\gamma = 0.7$, $\alpha = 0.5$); and ----, for adaptive mixed $H_2$/$H_\infty$ ($\gamma = 0.6$, $\alpha = 0.5$).
reveal, the adaptive mixed \( H_2/H_{\infty} \) attitude control causes quicker decay responses and has superior ability to diminish the effect of an external disturbance under unknown parameters. These results can be expected because the adaptive mixed \( H_2/H_{\infty} \) attitude controller is designed to achieve the adaptive \( H_2 \) attitude control under an adaptive \( H_{\infty} \) disturbance attenuation constraint. Furthermore, the adaptive \( H_2 \) attitude controller causes the fewest disturbance attenuation abilities (with respect to slower decay responses) among the three methods. This is reasonable because the adaptive \( H_2 \) attitude control is designed without consideration of the external disturbance and, therefore, does not, in general, guarantee any robust performance in the face of disturbance. The simulation result in Fig. 5 shows the steady-state response of attitude angles \( \theta_1, \theta_2, \) and \( \theta_3 \) over \( t = 6000 s \). It can be found that the effects of external disturbances to the attitude state are attenuated, even when the external disturbances are at a maximum at \( t = 3000 s \). The attitude control performance

\[
\int_0^{t_f} x^T(t)Q x(t) + u^T_1(t) W u_1(t) \, dt
\]

is plotted in Fig. 5 for the three cases. The applied torques \( \tau_1, \tau_2, \) and \( \tau_3 \) are represented in Fig. 7. It can be seen that the controller that produces better performance (as the adaptive mixed \( H_2/H_{\infty} \) attitude controller) takes larger control torque in amplitude. This is a tradeoff between the attitude control performance and the amplitude of control torque. These simulation results show that the performance of the adaptive \( H_{\infty} \) attitude controller is better than the performance of adaptive \( H_2 \) attitude control and worse than the performance of adaptive mixed \( H_2/H_{\infty} \) attitude control.

**Simulation 2**

To illustrate the robust capability of disturbance attenuation of the proposed adaptive mixed \( H_2/H_{\infty} \) attitude control design, the adaptive mixed \( H_2/H_{\infty} \) attitude control torque and update law in Eqs. (46a) and (46b) are designed to possess the following three different disturbance attenuation levels:

1. For case 1, select \( \gamma = 0.8, Q_2 = I_{6 \times 6}, \) and \( a = 0.462. \) Then \( \gamma = 0.5. \)
2. For case 2, select \( \gamma = 0.7, Q_2 = I_{6 \times 6}, \) and \( a = 0.404. \) Then \( \gamma = 0.5. \)
3. For case 3, select \( \gamma = 0.6, Q_2 = I_{6 \times 6}, \) and \( a = 0.346. \) Then \( \gamma = 0.5. \)

The simulation results are shown in Figs. 8 and 9 for the attitude angles and the performance

\[
\int_0^{t_f} x^T(t)Q x(t) + u^T_1(t) W u_1(t) \, dt
\]

respectively. It is obvious that a smaller \( \gamma \) may yield a better performance (lower cost function) \( J(u, d) \) in attenuating the effect of external disturbances \( d(t) \).

**Conclusions**

This paper presents an adaptive mixed \( H_2/H_{\infty} \) attitude control design of the nonlinear spacecraft system with an external disturbance and uncertain inertia matrix. In practical situations, the inertia matrix is uncertain and external disturbance is inevitable. By the adaptive control method, the uncertain parameters are estimated. Then, by the mixed \( H_2 \) and \( H_{\infty} \) controller design, the effect of an external disturbance to the spacecraft attitude can be restrained and the attitude state, as well as consumed energy of the controller, is minimized. Unlike the conventional nonlinear mixed \( H_2/H_{\infty} \) control design, which is based on solving two coupled differential equations, a general solution can be obtained by the method proposed in this paper via skew-symmetric property and state transformation techniques. The structure of the controller is very simple and can be viewed as an adaptive PD controller with the controller gains depending on the disturbance attenuation level \( \gamma \), which is assigned according to mission requirement. According to the simulation results, the adaptive mixed \( H_2/H_{\infty} \) attitude controller has greater ability to diminish the effects of an external disturbance to achieve better performance than the adaptive \( H_2 \) attitude controller or the adaptive \( H_{\infty} \) attitude controller. Moreover, from the control laws in Eqs. (46a) and (46b), we can see that these two equations are the same with the exception that they differ in the constraints on \( \gamma \), that is, \( H_{\infty} \) control with \( \gamma > 0 \) but mixed \( H_2/H_{\infty} \) control with \( \gamma > \sqrt{2}/2 \). We can conclude that the solutions of the mixed \( H_2/H_{\infty} \) control are contained in the solutions of the \( H_{\infty} \) control. Furthermore, the adaptive mixed \( H_2/H_{\infty} \) attitude controller achieves robust performance for the spacecraft attitude control systems. From the experimental simulation results on the ROCSAT-1 spacecraft system, the proposed design algorithm exhibits significant advantages for the attitude control of the spacecraft system under unknown parameters and a large external disturbance.

**Appendix: Determining the Regression Matrix**

By Eq. (i), the regression matrix \( \Phi(x, t) = [\phi_i] \) for \( i = 1, 2, 3 \) and \( j = 1, 2, \ldots, 6 \) can be determined as

\[
\begin{align*}
\phi_1 &= y_1 - s_1 y_3 - c_2 s_2 y_2 + B_1 \\
\phi_2 &= c_1 y_2 + s_1 c_3 y_3 + s_1 (A_1 - \hat{\theta}_1) y_4 + (\hat{\theta}_1 c_1 c_2 - \hat{\theta}_2 s_2) y_1 \\
&\quad - c_1 c_4 A_1 y_2 + B_2 - a_0 c(s_3 c_3 - c_3 s_3) A_1 \\
\phi_3 &= -s_1 y_2 + c_1 c_3 y_3 - c_1 (\hat{\theta}_1 - A_1) y_2 - s_1 c_2 (\hat{\theta}_1 - A_1) y_1 \\
&\quad + c_2 c_4 A_1 y_1 + B_3 - a_0 (s_1 c_1 + c_1 s_3) A_1 \\
\phi_4 &= s_1 A_2 y_2 + c_1 c_2 A_2 y_3 - a_0 (s_1 c_3 + c_3 s_3) A_2 \\
\phi_5 &= (c_1 A_2 + s_1 A_2) y_3 - (c_1 A_2 - s_1 A_2) y_2 + a_0 (s_1 c_3 + c_3 s_3) A_3 \\
&\quad + s_1 A_2 A_3 y_1 + B_4 + a_0 s_1 s_3 A_2 \\
\phi_6 &= c_1 A_1 y_2 + s_1 c_3 y_3 - a_0 (c_1 c_3 + 
\end{align*}
\]

**Notes**


+\sin c_1 (c_2 s_1 - s_1 s_2) y_5 + B_3 s_1 c_2 + a_0 (s_1 s_2 c_1 - c_1 s_2) A_2 \\
\phi_{16} = -c_2 s_1^2 y_5 + 2sc_1 c_2 y_1 - c_1 c_2 A_2 s_1 y_5 - [c_2 (c_1^2 - s_1^2) s_2 - 2s_1 c_1 s_2 s_3 y_5] \\
+ B_4 c_2 + B_3 s_1 c_2 + a_0 (s_1 c_2 s_3 - c_1 s_2 A_3 \\
+ (c_1 c_2 s_2 + s_1 s_2) A_2) \\
\phi_{16} = -c_1 c_2 c_1 y_5 + c_1 c_2 y_1 - s_1 c_2 A_2 y_3 - (c_1^2 c_2 s_1 + c_1 s_2 A_3) y_2 \\
- c_1 c_2 (c_1 c_2 s_3 + c_1 s_2 s_3) y_5 + B_4 c_2 + a_0 (s_1 s_2 c_3 + s_1 s_3) A_3 \\
Here, we use the short-hand notations \\
c_1 = \cos \theta_1, \quad c_2 = \cos \theta_2, \quad c_3 = \cos \theta_3, \quad s_1 = \sin \theta_1 \\
s_2 = \sin \theta_2, \quad s_3 = \sin \theta_3, \quad A_1 = \theta_1 - \theta_2 - \theta_3 - c_2 s_3 A_3 \\
A_2 = c_1 c_2 s_3 - s_1 c_2 s_3 + a_0 (s_1 c_2 - s_1 s_2 A_3) \\
A_3 = -s_1 c_1 c_2 c_3 + a_0 (c_2 s_3 - s_1 c_2 - c_3 c_2) y_2 \\
B_1 = a_0 (s_1 c_2 s_3 - c_1 s_2 c_3) \\
B_2 = a_0 (s_1 c_1 c_2 s_3 - c_1 c_2 s_3 - c_2 s_3 c_3 + s_1 c_2 - c_3 c_2) \\
B_3 = a_0 (s_1 c_1 c_2 s_2 s_3 - c_1 c_2 s_2 s_3 - c_2 s_3 c_3 + s_1 c_2 - c_3 c_2) \\
y = y_1 y_2 y_3^2 = (\lambda_1 / \lambda_3)^{\theta_3} \\

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