I. NOMENCLATURE

\[ (b_1, b_2, b_3) \]  Body fixed reference frame
\[ e \]  Tracking error vector
\[ (e_1, e_2, e_3) \]  Inertial fixed reference coordinates
\[ h \]  Total spacecraft angular momentum in body axes, \( [h_1 \ h_2 \ h_3]^T; \ h = J \omega \)
\[ J \]  Inertia matrix
\[ (a_1, a_2, a_3) \]  Orbital coordinate reference
\[ P \]  Weighting matrix of tracking error with respect to initial time
\[ Q \]  Weighting matrix of tracking error
\[ Q_f \]  Weighting matrix of tracking error with respect to final time
\[ Q_i \]  Weighting matrix of tracking error for \( H_\infty \) performance
\[ Q_{1f} \]  Weighting matrix of tracking error with respect to final time for \( H_\infty \) performance
\[ Q_2 \]  Weighting matrix of tracking error for \( H_2 \) performance
\[ Q_{2f} \]  Weighting matrix of tracking error with respect to final time for \( H_2 \) performance
\[ r \]  Filtered link of tracking error
\[ S_1 \]  Weighting matrix of fuzzy approximation error for \( H_\infty \) performance
\[ S_2 \]  Weighting matrix of fuzzy approximation error for \( H_2 \) performance
\[ S_a \]  Saturation value of \( i \)th actuator output torque
\[ T \]  State-space transformation matrix
\[ u \]  Control torque vector
\[ u_c \]  Mixed \( H_2/H_\infty \) control torque vector
\[ u_f \]  Adaptive fuzzy control torque vector
\[ W \]  Weighting matrix of control torque
\[ W_i \]  Weighting matrix of control torque for \( H_\infty \) performance
\[ W_2 \]  Weighting matrix of control torque for \( H_2 \) performance
\[ \gamma \]  Desired disturbance attenuation level
\[ \theta \]  Attitude Euler angle, \( [\theta_1 \ \theta_2 \ \theta_3]^T \)
\[ \hat{\theta} \]  Attitude tracking error
\[ \hat{\theta}_r \]  Reference trajectory of attitude Euler angle, \( [\hat{\theta}_1 \ \hat{\theta}_2 \ \hat{\theta}_3]^T \)
\[ [\nu \times] \]  cross product matrix associated with vector \( \nu = [\nu_1 \ \nu_2 \ \nu_3]^T \):
\[ [\nu \times] = \begin{bmatrix} 0 & -\nu_3 & \nu_2 \\ \nu_3 & 0 & -\nu_1 \\ -\nu_2 & \nu_1 & 0 \end{bmatrix} \]
\[ \tau_a \]  Torque vector due to actuator,
\[ [\tau_a \ \tau_d \ \tau_s]^T \]  Aerodynamic disturbance torque vector
\[ \tau_d \]  External disturbance torque vector
\[ \tau_g \]  Gravity gradient torque vector
\[ \tau_{solar} \]  Solar radiation pressure disturbance torque vector

This paper presents an adaptive fuzzy mixed \( H_2/H_\infty \) attitude control of nonlinear spacecraft systems with unknown or uncertain inertia matrix and external disturbances. Using an adaptive fuzzy approximation method, an uncertain nonlinear model is estimated. Then, by a mixed \( H_2 \) and \( H_\infty \) attitude control design, the effect of external disturbance and fuzzy approximation error on spacecraft attitude can be restrained and the tracking error as well as consumed energy of the controller is minimized. By virtue of the skew symmetric property of the spacecraft system and adequate choice of state variable transformation, this adaptive fuzzy control problem can be reduced to solving two algebraic equations instead of a pair of coupled time-varying differential equations. Finally, experimental simulation results are presented to demonstrate the effectiveness of the proposed design methods.
II. INTRODUCTION

The attitude control of spacecraft has received extensive attention in recent decades, and several methods of spacecraft attitude control have been developed to treat this problem. Based on linearization using coordinate transformation and nonlinear feedback, a controller for attitude control has been derived in [1]. The feasibility of applying the feedback linearization technique to spacecraft attitude control and momentum management problem has also been discussed in [2, 3]. A sliding manifold approach [4, 5] has also been used for spacecraft attitude control where upper bounds of the unknown system parameters are needed. The rule-based fuzzy control technologies have been applied to attitude control of spacecraft systems [6–9]. Neural network technology has also been applied to attitude control of spacecraft systems in [10]. More relevant to this study, optimal control theory has been used for attitude control of spacecraft systems [11–13]. The approach of $H_{\infty}$ control has been applied to the space station attitude and momentum control problem while considering the linearized equations of motion [14]. Recently, Chen, et al. [15] developed a nonlinear $H_{\infty}$ control design to treat the spacecraft attitude tracking control problem under parameter perturbation and external noise. However, only robust performance but the optimization on control energy is considered in the $H_{\infty}$ control scheme. For precise attitude control of the spacecraft, more effort is needed.

Over the past ten years, mixed $H_2/H_{\infty}$ optimal control has been studied for linear systems [16–19]. The main purpose of this type of control is to design an $H_2$ optimal control for the worst-case external disturbance whose effects on system output must be attenuated below a desired value (i.e., to design an $H_2$ optimal control under $H_{\infty}$ disturbance attenuation constraint). This control design has the advantages of both $H_2$ optimal control performance and $H_{\infty}$ robust control performance and is very suitable for optimal tracking control of systems under uncertain external disturbance. Along this line, based on the nonlinear two-player Nash differential game theorem, Wu, et al. [20] also developed a unified design for $H_2$, $H_{\infty}$ and mixed $H_2/H_{\infty}$ attitude control of spacecraft with known parameters.

In practical situations, system parameter variations of the spacecraft such as those generated by payload motion, vehicle docking and the rotation of solar arrays, etc. are inevitable. This makes the dynamics of spacecraft become uncertain or unknown. Since a fuzzy logic system can be tuned to approximate any nonlinear dynamic system, a fuzzy control technique has been used in this study to efficiently eliminate the plant uncertainties, via an adaptive learning method [21–23]. For the attitude control problem, one can develop an adaptive fuzzy scheme that will continuously adjust the controller to compensate unknown dynamics and environmental effects to ensure stable and robust performance.

We design an attitude controller with an adaptive law of fuzzy parameters so that $H_2$ optimal control performance can be achieved under a desired $H_{\infty}$ disturbance attenuation constraint when the inertia matrix of the spacecraft is uncertain (or unknown) and external disturbances exist. The proposed controller is comprised of two terms: one being a self-tuning fuzzy logic system, which is employed to optimally eliminate the uncertain dynamics in order to enhance the tracking robustness, and the other being a mixed $H_2/H_{\infty}$ control algorithm, which is employed to attenuate the worst case effects of both the residue of fuzzy elimination and the exogenous disturbance on the tracking error within a desired level. A pair of coupled time-varying differential equations with an adaptive constraint must first be solved to design the controller. By adequate choice of a state transformation and use of the skew-symmetric property of spacecraft systems, two time-varying coupled differential equations can be transformed into two corresponding algebraic coupled equations and the adaptive constraint holds. Then, through Cholesky factorization [24], these two coupled algebraic equations can be easily solved. As a result, the structure of the controller becomes very simple and the control gains depend on the disturbance attenuation level $\gamma$ which is assigned according to mission requirement.

III. MATHEMATICAL MODEL OF SPACECRAFT SYSTEM

Consider a spacecraft moving in a circular orbit. The coordinate systems used in the attitude control are shown in Fig. 1. The inertial fixed reference coordinates $(e_1, e_2, e_3)$ with its origin at the center of the Earth is used to determine the orbital position of the spacecraft. The orbital coordinate reference $(o_1, o_2, o_3)$ is rotating about the $o_3$ axis with respect to the fixed inertial coordinate system $(e_1, e_2, e_3)$ at the orbital rate $\omega_0$. The axes of this reference frame are chosen such that the roll axis $o_1$ is in the flight direction, the pitch axis $o_2$ is perpendicular to the orbital plane, and the yaw axis $o_3$ points toward the Earth. The last reference system used is the body fixed reference frame $(b_1, b_2, b_3)$. The orientation...
of the spacecraft with respect to the reference frame \((\alpha_1, \alpha_2, \alpha_3)\) is obtained, in this work, by a yaw-pitch-roll \((\theta_3 - \theta_2 - \theta_1)\) sequence of rotations. The origins of both orbit coordinates and body fixed coordinates are at the center of mass of the spacecraft.

The nonlinear equations of motion, in terms of components along the body fixed control axes, are given by the attitude kinematics and the spacecraft dynamics and can be written as follows [25, 26].

**Attitude kinematics:**

\[
\omega = R(\theta) \dot{\theta} - \omega_c(\theta).
\] (1)

**Spacecraft dynamics:**

\[
J \ddot{\omega} = [h \times] \omega + \tau_a + \tau_d
\] (2)

where

\[
R(\theta) = \begin{bmatrix}
1 & 0 & -\sin \theta_2 \\
0 & \cos \theta_1 & \sin \theta_1 \cos \theta_2 \\
0 & -\sin \theta_1 & \cos \theta_1 \cos \theta_2 \\
\end{bmatrix},
\]

\[
\omega_c(\theta) = \omega_0 \begin{bmatrix}
\cos \theta_3 \cos \theta_1 + \sin \theta_2 \sin \theta_1 \\
-\sin \theta_3 \cos \theta_1 + \sin \theta_2 \sin \theta_1 \\
\cos \theta_3 \sin \theta_1 + \sin \theta_2 \sin \theta_1 \\
\end{bmatrix},
\]

\[
J = \begin{bmatrix}
J_{11} & J_{12} & J_{13} \\
J_{21} & J_{22} & J_{23} \\
J_{31} & J_{32} & J_{33} \\
\end{bmatrix},
\]

\[
\tau_g = 3\omega_0^2 [c \times] J c
\]

with \(c = \begin{bmatrix}
\sin \theta_1 \cos \theta_2 \\
\cos \theta_1 \cos \theta_2 \\
\cos \theta_1 \sin \theta_2 \\
\end{bmatrix},
\]

and \(\tau_d = \tau_{aero} + \tau_{solar}\).

**Remark 1** If the control torque is limited by the saturation of actuator, the spacecraft dynamics (2) is of the following form

\[
J \ddot{\omega} = [h \times] \omega + \tau_a + \tau_{a,sat} + \tau_{aero} + \tau_{solar}
\] (3)

where \(\tau_{a,sat} = [\text{sat}(\tau_{a1}) \text{sat}(\tau_{a2}) \text{sat}(\tau_{a3})]^T\) and

\[
\text{Sat}(\tau_{ai}) = \begin{cases}
S_{ai}, & \tau_{ai} \geq S_{ai} \\
\tau_{ai}, & |\tau_{ai}| < S_{ai} , \quad \text{for} \quad i = 1, 2, 3,
\end{cases}
\]

\[-S_{ai}, & \tau_{ai} \leq -S_{ai}.
\]

In order to apply the proposed design method, the dynamic (3) can be put into the form of (2) with \(\tau_d = \tau_{aero} + \tau_{solar} + \tau_{a,sat} - \tau_d\). In this situation, the deviation \(\tau_{a,sat} - \tau_d\) can be considered as a part of the external disturbance.

**Remark 2** This description of (1) is defined for all \((\theta_1, \theta_2, \theta_3)\) except \(\theta_2 = \pm(2n + 1)\pi/2\) for any integer \(n\). The singularity (i.e., the determinant of matrix \(R(\theta)\) becomes zero at \(\theta_2 = \pm(2n + 1)\pi/2\) arises owing to the choice of the set of rotations which define the orientation of the spacecraft relative to the orbital frame. However, when the orientation corresponding to the singularity at \(\theta_2 = \pm\pi/2\) lies in the control region of attitude angles, another set of rotations can be defined to eliminate this singularity [27]. In this paper, we are interested in the trajectories in the bounded region \(\Omega\).

Differentiating (1), gives

\[
\dot{\omega} = R(\theta) \ddot{\theta} + \left( \frac{d}{dt} R(\theta) \right) \dot{\theta} - \left( \frac{d}{dt} \omega_c(\theta) \right).
\] (4)

Substituting \(\omega\) and \(\dot{\omega}\) from (1) and (4) into (2) and premultiplying (4) by the matrix \(R^T(\theta)\), we obtain

\[
M(\theta) \ddot{\theta} + C(\theta, \dot{\theta}) \dot{\theta} + G(\theta, \dot{\theta}) = u + d
\] (5)

where

\[
M(\theta) = R^T(\theta) J R(\theta)
\]

\[
C(\theta, \dot{\theta}) = R^T(\theta) J \left( \frac{d}{dt} R(\theta) \right) - R^T(\theta) [h \times] R(\theta)
\]

\[
G(\theta, \dot{\theta}) = -R^T(\theta) J \left( \frac{d}{dt} \omega_c(\theta) \right) + R^T(\theta) [h \times] \omega_c(\theta)
\]

\[
-3\omega_0^2 R^T(\theta) [c \times] J c
\]

\[
u = R^T(\theta) \tau_g, \quad d = R^T(\theta) \tau_d.
\]

**Assumption.** The inertia matrix \(J\) is an unknown and is symmetric positive definite. Moreover, the external disturbance \(\tau_d\) is bounded but unknown.

Under the above assumption, the spacecraft system (5) has the following properties.

P1. The matrix \(M(\theta)\) is symmetric positive definite.

P2. The matrix \(\frac{1}{2} (d/dt) M(\theta) - C(\theta, \dot{\theta})\) is skew-symmetric [28], that is

\[
x^T \left( \frac{1}{2} \frac{d}{dt} M(\theta) - C(\theta, \dot{\theta}) \right) x = 0 \quad \forall \quad x \in \mathbb{R}^3.
\] (6)
P3. The external disturbance $d$ in the spacecraft system (5) is bounded.

In this paper, the desired attitude reference trajectory is assumed to be available as bounded functions of time in terms of angle vector $\hat{\theta} \in C^2$ (the class of twice continuously differentiable functions), its corresponding velocity vector $\dot{\hat{\theta}}$, and acceleration vector $\ddot{\hat{\theta}}$. Thus, the attitude tracking error is defined as follows:

$$e =: \begin{bmatrix} \hat{\theta} \\ \dot{\hat{\theta}} \end{bmatrix} = \begin{bmatrix} \theta - \theta_r \\ \dot{\theta} - \dot{\theta}_r \end{bmatrix}.$$  (7)

To get the tracking error dynamic equation of the spacecraft system, a filtered link of tracking error $r$ and the state-space transformation matrix $T$ are introduced, respectively, as [29]:

$$r(t) = \lambda_1 \hat{\theta} + \lambda_2 \ddot{\hat{\theta}}$$  (8a)

$$T = \begin{bmatrix} \lambda_1 I_{3x3} & \lambda_2 I_{3x3} \\ 0_{3x3} & I_{3x3} \end{bmatrix}$$  (8b)

for some positive constants $\lambda_1$ and $\lambda_2$ which should be adequately determined later. Then, we have

$$\dot{r} = -M^{-1}(\theta)C(\theta, \dot{\theta})r + \lambda_2 M^{-1}(\theta)[-F(e, t) + (u + d)]$$  (9)

where

$$F(e, t) = M(\theta) \begin{bmatrix} \hat{\theta} \\ \dot{\theta} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + C(\theta, \dot{\theta}) \begin{bmatrix} \hat{\theta} \\ \dot{\theta} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + G(\theta, \dot{\theta}).$$

Thus, by (7)-(9), the tracking error dynamic equation of spacecraft system can be obtained as

$$\dot{e}(t) = T^{-1} \begin{bmatrix} \ddot{r}(t) \\ \dot{r}(t) \end{bmatrix} = A_F(e, \theta)e(t) + B_F(e, \theta)[-F(e, t) + (u + d)]$$  (10)

where

$$A_F(e, \theta) = \begin{bmatrix} 0_{3x3} & I_{3x3} \\ -\lambda_2 M^{-1}(\theta)C(\theta, \dot{\theta}) - \lambda_1 M^{-1}(\theta)C(\theta, \dot{\theta}) - \lambda_1 I_{3x3} \end{bmatrix}$$

$$B_F(e, \theta) = \begin{bmatrix} 0_{3x3} \\ M^{-1}(\theta) \end{bmatrix}.$$  

IV. REVIEW OF FUZZY LOGIC SYSTEM AND PROBLEM FORMULATION

In practical spacecraft systems, since parameter perturbation, payload motion, vehicle docking, the rotation of solar arrays, and external disturbance are inevitable, the nonlinear function $F(e, t)$ in (10) is uncertain and unknown. Several robust design algorithms [30–32] can be employed as robust controllers to override the upper norm bound of $|F(e, t)|$. Due to high nonlinearity and uncertainty of $F(e, t)$, it is however, not easy to estimate the upper norm bound. Furthermore, it is also not easy to suppress the effect of external disturbance $d(t)$ by using, for example, a VSS (variable structure systems) control or a dead-Zone control based on the upper bound of external disturbance to treat this robust control problem. If the upper bound is overestimated, it will lead to a conservative control with more chattering. However, if the upper bound is underestimated, it will lead to instability of the control system. Therefore, these robust control methods may lead to conservative and imprecise results in practical applications.

In this work, the control signal $u$ in (10) is divided into two parts as follows:

$$u = u_e + u_f(e, \Theta).$$  (11)

In this situation, the spacecraft dynamics in (10) can be written as

$$\dot{e} = A_F(e, \theta)e + B_F(e, \theta)u_e + B_F(e, \theta)u_f$$

$$\times [u_f(e, \Theta) - F(e, t)] + B_F(e, \theta)d$$  (12)

where $u_f(e, \Theta)$ is a fuzzy logic system and $\Theta$ is the fuzzy update vector which is discussed in the following paragraph. In (12), the fuzzy logic system $u_f(e, \Theta)$ is tuned to approximate the nonlinear function $F(e, t)$ as closely as possible. Furthermore, the control signal $u_e(t)$ is used to attenuate the effects of residue of $u_f(e, \Theta) - F(e, t)$ as well as the external disturbance $d(t)$ on the attitude state from the mixed $H_2/H_{\infty}$ perspective.

The basic configuration of the fuzzy logic system for spacecraft attitude control is shown in Fig. 2. The fuzzy logic system $u_f(e, \Theta)$ in this work performs a mapping from $U \subset R^8$ to $V \subset R^3$. Let $U = U_1 \times U_2 \times U_3 \times U_4 \times U_5 \times U_6$ where $U_i \subset R$, for $i = 1, 2, \ldots, 6$. The fuzzy rule base consists of a collection of fuzzy If-Then rules as follows:

$$R^{(l)}: If e_1 is F_{1}^{l}, e_2 is F_{2}^{l}, e_3 is F_{3}^{l},$$

$$e_4 is F_{4}^{l}, x_5 is F_{5}^{l} and e_6 is F_{6}^{l}$$

Then $u_f$ is $G^{l}$, for $l = 1, 2, \ldots, M$  (13)

where $e = (e_1, e_2, e_3, e_4, e_5, e_6)^T \in U$ and $u_f \in V$ are the input and output of the fuzzy logic system, respectively. In our design, the fuzzy inference engine performs a mapping from fuzzy sets in $U \subset R^8$ to fuzzy sets in $V \subset R^3$, based upon the fuzzy If-Then rules in the fuzzy rule base and the compositional rule of inference. The fuzzifier maps a crisp point $e \in U$ into a fuzzy set $F_{e}$ in $U$. The defuzzifier maps a fuzzy set $F_{e}$ in $V$ into a crisp point $u_f \in V$. More information can be found in [33].
The fuzzy logic system contained in Fig. 2 comprises a very rich class of static system mapping from $U \subset R^k$ to $V \subset R^3$, since many different choices are available within each block, and in addition, many combinations of these choices can result in useful subclasses of fuzzy logic systems. One subclass of fuzzy logic systems is used here as building blocks of the fuzzy approximation controller $u_f(e, \Theta)$ for adaptive cancellation of nonlinear function $F(e, t)$ and is described by the following important result.

**Lemma 33** If the fuzzy basis functions are defined as

$$
\xi_i(e) = \frac{\prod_{j=1}^6 \mu_{F_j}(e_j)}{\sum_{k=1}^M \prod_{j=1}^6 \mu_{F_j}(e_j)} \quad \text{for} \quad i = 1, 2, 3
$$

and $l = 1, 2, ..., M$

(14)

Then the fuzzy logic systems with center-average defuzzifier, product inference and singleton fuzzifier for spacecraft attitude control systems, are of the following form:

$$
u_f(e, \Theta) = \begin{bmatrix} u_{f1}(e, \Theta) \\ u_{f2}(e, \Theta) \\ u_{f3}(e, \Theta) \end{bmatrix} = \begin{bmatrix} \xi_1^T(e) \Theta_1 \\ \xi_2^T(e) \Theta_2 \\ \xi_3^T(e) \Theta_3 \end{bmatrix} = \Xi(e) \Theta
$$

(15)

where

$$
\begin{align*}
\Theta &= \begin{bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{bmatrix} \\
\Xi(e) &= \begin{bmatrix} \xi_1^T(e) & 0_{1 \times M} & 0_{1 \times M} \\ 0_{1 \times M} & \xi_2^T(e) & 0_{1 \times M} \\ 0_{1 \times M} & 0_{1 \times M} & \xi_3^T(e) \end{bmatrix}
\end{align*}
$$

and $\Theta = [\Theta_1 \quad \cdots \quad \Theta_M]^T$, $\xi_i(e) = [\xi_{i1}(e) \quad \cdots \quad \xi_{iM}(e)]^T$, for $i = 1, 2, 3$ with that $\Theta_1$ is the point at which the given membership function $\mu_{F_1}(\Theta_1)$ achieves its maximum value, and we assume that $\mu_{F_1}(\Theta_1) = 1$.

**Remark 3**

(1) There are two main reasons for using the fuzzy logic system (15) as a basic building block of adaptive fuzzy controller. First, the fuzzy logic system (15) is constructed from the fuzzy IF-THEN rules of (13), using specific fuzzy inference, fuzzification, and defuzzification strategies. Therefore, linguistic information from a human expert can be directly incorporated into the controllers. Second, the fuzzy logic systems of the form (15) have been proven in [33] to be universal approximations, i.e., for any given real continuous function $F(e, t)$ defined on the compact set $U$, there exists a fuzzy logic system of the form (15) such that it can uniformly approximate $F(e, t)$.

(2) In this design, the membership functions $\mu_{F_1}$ are specified according to the designer’s experience or knowledge about the uncertainties of the spacecraft systems and the parameter $\Theta$ is to be tuned according to the attitude tracking error $e(t)$.

However, in practical control design, the number of fuzzy basis functions of fuzzy logic system is chosen as small as possible for the convenience of computation and implementation. In this situation, an adaptive law must be developed to tune the parameter $\Theta$, in order to make a fuzzy logic system $u_f(e, \Theta)$ with adequate dimension to eliminate $F(e, t)$ in the following manner.

Let us define the following optimal update of parameter of elimination [33]

$$
\Theta^* = \arg \min_{\Theta \in \Omega_\Theta} \max_{e \in \Omega_e} \| \Xi(e) \Theta - F(e, t) \|
$$

(16)

where $\| \cdot \|$ denotes the Euclidean norm, i.e., $\| e \| = \sqrt{e^T e}$, $\Omega_\Theta$ and $\Omega_e$ denote the sets of $\Theta$ and $e$ that have suitable bounds, respectively. Obviously, $\Theta^*$ exists and is a constant. Furthermore, in $\Omega_\Theta$ and $\Omega_e$, $F(e, t) = \Xi(e) \Theta^* + e(t)$ with $e(t) \leq \bar{e}$ for some $\bar{e} > 0$. To guarantee that $\Theta$ lies inside the bounded set $\Omega_\Theta$, the projection algorithm is introduced to prevent the divergence of $\Theta$ in the next section.
Then, the dynamic equation in (12) can be rewritten as follows:
\[ \dot{\epsilon} = A_T(e,t)\epsilon + B_T(e,t)u_e + B_T(e,t)\Xi(e)\dot{\Theta}(t) + B_T(e,t)\xi(t)w(t) \]  
(17)
where \( w(t) = d(t) - \epsilon(t) \) denotes the sum of optimal fuzzy approximation error and the external disturbance, and \( \dot{\Theta}(t) \) is defined as
\[ \dot{\Theta}(t) = \Theta(t) - \Theta^* . \]  
(18)

The proposed design procedure is divided into two steps. In the first step, the adaptive fuzzy logic system \( u_T(e,\Theta) \) is tuned by way of \( \Theta(t) \) to optimally approximate and eliminate the uncertain term \( F(e,t) \) of spacecraft systems. However, the sum of approximation error and external disturbance \( w(t) \) cannot be estimated and eliminated. Therefore, in the second step, the control signal \( u_T(e,\Theta) \) is specified such that the worst case effect of the uncertainty \( w(t) \) on the attitude state \( e(t) \) must be attenuated as much as possible and it takes the minimum control energy.

That is, the attitude control problem considered in this paper can be stated as follows.

**Adaptive Fuzzy Mixed \( H_2/H_{\infty} \) Attitude Control Problem**: Consider the nonlinear spacecraft dynamic system (17). Given a desired disturbance attenuation level \( \gamma > 0 \) and weighting matrices \( Q_1, Q_2, W_1, \) and \( W_2 \), the adaptive fuzzy mixed \( H_2/H_{\infty} \) attitude control problem is said to be solved if there exists a control law \( u_T(e,\Theta) \) and an update law of \( \Theta(t) \) such that the following \( H_2 \) (quadratic) optimal tracking performance [16–19]

\[
\min_{u_T,\Theta} \left[ e^T(t)Q_1e(t) + \dot{\Theta}_T(t)S_1\dot{\Theta}_T(t) + \int_0^t \left[ e^T(\tau)Q_2e(\tau) + u^*_T(\tau)W_2u_T(\tau) \right] d\tau \right]
\]  
(19a)
can be achieved for all \( t \in [0,\infty) \) and for some positive definite matrices \( Q_T = Q_T^T > 0 \) and \( S_1 = S_1^T > 0 \) under the following \( H_{\infty} \) disturbance attenuation constraint
\[
e^T(t)Q_1e(t) + \dot{\Theta}_T(t)S_1\dot{\Theta}_T(t) + \int_0^t \left[ e^T(\tau)Q_2e(\tau) + u^*_T(\tau)W_2u_T(\tau) \right] d\tau \leq \epsilon^T(0)Pe(0) + \dot{\Theta}_T(0)S_1\dot{\Theta}_T(0) + \gamma^2 \int_0^t w^T(\tau)w(\tau) d\tau, \quad \forall \ w(t) \in L_2[0, t],
\]  
(19b)
for some positive definite matrices \( Q_T = Q_T^T > 0, S_1 = S_1^T > 0 \) and \( P = P^T > 0 \). That is, not only the \( H_2 \) optimal tracking performance is achieved, but also the effect of uncertainty \( w(t) \) on the attitude state \( e(t) \) and control \( u_T(e,\Theta) \) must be attenuated below a desired level \( \gamma^2 \) from the energy point of view.

**REMARK 4** Let
\[
J_1(u_T, w) = e^T(t)Q_1e(t) + \dot{\Theta}_T(t)S_1\dot{\Theta}_T(t) + \int_0^t \left[ e^T(\tau)Q_2e(\tau) + u^*_T(\tau)W_2u_T(\tau) - \gamma^2 w^T(\tau)w(\tau) \right] d\tau
\]  
(20a)
\[
J_2(u_T, w) = e^T(t)Q_1e(t) + \dot{\Theta}_T(t)S_1\dot{\Theta}_T(t) + \int_0^t \left[ e^T(\tau)Q_2e(\tau) + u^*_T(\tau)W_2u_T(\tau) \right] d\tau.
\]  
(20b)

Then, the mixed \( H_2/H_{\infty} \) attitude control performances (20a) and (20b) are equivalent to finding the control law \( u^*_T(e,t) \), update law of \( \Theta(t) \) and the worst case disturbance \( w^*(e,t) \) such that [34]
\[
J_1(u^*_T(e,t), w^*(e,t)) \geq J_1(u_T(e,t), w), \quad \forall \ w \in L_2[0, t].
\]  
(21a)
\[
J_2(u^*_T(e,t), w^*(e,t)) \leq J_2(u_T(e,t), w'), \quad \forall \ u_T \in L_2[0, t].
\]  
(21b)

This mixed performances with fuzzy parameter adjustment can be viewed as an attitude control design approach that minimizes an \( H_2 \) tracking cost function under the \( H_{\infty} \) disturbance attenuation constraint on the spacecraft attitude controlled system associated with the adaptive fuzzy approximation scheme.

V. ADAPTIVE FUZZY MIXED \( H_2/H_{\infty} \) ATTITUDE CONTROL DESIGN

A. Sufficient Conditions of Mixed \( H_2/H_{\infty} \) Attitude Control Problem

In this section, we present sufficient conditions for the existence of solution of the adaptive fuzzy mixed \( H_2/H_{\infty} \) attitude control problem. For the convenience of design, we take \( W_i = W_2 = W \) and \( S_1 = S_2 = S \) throughout this study. By using the standard technique of completing the squares, we have the following theorem.

**THEOREM** For the spacecraft attitude control system in (5), if the attitude controller \( u(e,t) \) is chosen as
\[
u(e,t) = u^*_T(e,t) + \Xi(e)\Theta(t)
\]  
(22a)
with the mixed \( H_2/H_{\infty} \) attitude control law \( u^*_T(e,t) \) and the fuzzy update law of \( \Theta(t) \) are as:
\[
u^*_T(e,t) = -W^{-1}B_T(e,t)P(e,t)e
\]  
(22b)
\[\dot{\Theta}(e,t) = -S^{-1}\Xi^T(e)B_T(e,t)P(e,t)e \]  
(22c)
where $P_1(e,t)$ and $P_2(e,t)$ are the solutions of the following coupled time-varying differential equations

\[
\dot{P}_1(e,t) + P_1(e,t)A_T(e,t) + A_T(e,t)^T P_1(e,t) + Q_1
- [P_1(e,t)B_T(e,t), P_2(e,t)B_T(e,t)]
\times \begin{bmatrix}
-1 & I_{3x3} & W^{-1} \\
0 & -1 & W^{-1} \\
-1 & -1 & W^{-1}
\end{bmatrix}
\begin{bmatrix}
B_T(e,t)^T P_1(e,t) \\
B_T(e,t)^T P_2(e,t)
\end{bmatrix} = 0
\]

\tag{23a}
\]

\[
\dot{P}_2(e,t) + P_2(e,t)A_T(e,t) + A_T(e,t)^T P_2(e,t) + Q_2
- [P_1(e,t)B_T(e,t), P_2(e,t)B_T(e,t)]
\times \begin{bmatrix}
-1 & I_{3x3} & W^{-1} \\
0 & -1 & W^{-1} \\
-1 & -1 & W^{-1}
\end{bmatrix}
\begin{bmatrix}
B_T(e,t)^T P_1(e,t) \\
B_T(e,t)^T P_2(e,t)
\end{bmatrix} = 0
\]

\tag{23b}
\]

and the adaptive fuzzy constraint

\[
B_T^T(e,t)P_1(e,t) = B_T^T(e,t)P_2(e,t)
\]

\tag{23c}
\]

with $P_1(e,t) = P_1^T(e,t) \geq 0$, $P_2(e,t) = P_2^T(e,t) \geq 0$ and the terminal conditions $P_1(e(t_f), t_f) = Q_{1f}$ and $P_2(e(t_f), t_f) = Q_{2f}$. Then the adaptive fuzzy mixed $H_2/H_{\infty}$ attitude control problem is solved by (22a)–(22c) with the worst case uncertainty

\[
w^*(e,t) = \frac{1}{\gamma^2} B_T^T(e,t)P_1(e,t)e.
\]

\tag{24}
\]

**Proof:** See the Appendix.

**Remark 5** (1) If only the adaptive fuzzy $H_2$ attitude control design is considered, the desired uncertainty attenuation constraint is negligible, i.e., $\gamma \rightarrow \infty$. In this situation, the criterions (21a) and (21b) are combined into the following problem:

\[
\min_{u(e,t)} J(u_{\infty}, d) = \min_{u(e,t)} \left[ e^T(t_f)Q_f e(t_f) + \dot{\Theta}^T(t_f)S\dot{\Theta}(t_f) + \int_0^{t_f} [e^T(t)Q e(t) + u^T(t)W u(t)]dt \right].
\]

\tag{25}
\]

In this situation, the solution of the $H_2$ attitude controller is given as [29]

\[
u(e,t) = \Xi(e(t))\Theta(t) + u^*_2(e,t),
\]

\tag{26a}
\]

where

\[
u^*_2(e,t) = -W^{-1}B_T^T(e,t)P(e(t))e(t)
\]

\tag{26b}
\]

\[
\dot{\Theta}(e,t) = -S^{-1}\Xi^T(e(t))B_T^T(e,t)P(e(t))e(t)
\]

\tag{26c}
\]

and $P(e,t) = P_T(e,t) \geq 0$ is the solution of the following time-varying differential equation

\[
\dot{P}(e(t), t_f) + P(e(t))A_T(e(t), t_f) + A_T(e(t), t_f)^T P(e(t)) + Q
- P(e(t))B_T(e(t))W^{-1}B_T^T(e(t))P(e(t)) = 0
\]

\tag{26d}
\]

\[
P(e(t_f), t_f) = Q_f.
\]

\tag{26e}
\]

The above result is the same as the time-varying differential equation in (23a) or (23b) with $P(e(t)) = P_1(e,t) = P_2(e,t)$, $Q = Q_1 = Q_2$ and $Q_f = Q_{1f} = Q_{2f}$.

(2) If only the adaptive fuzzy $H_{\infty}$ attitude control design is considered, the criterions (21a) and (21b) are combined into the following dynamic game problem [35, 36]

\[
J(u^*_2(e(t), w), w) \leq J(u^*_2(e(t), w^*(e(t))), w) \leq J(u_{\infty}(w), w^*(e(t)))
\]

\tag{27a}
\]

with

\[
J(u_{\infty}, w) = e^T(t_f)Q_f e(t_f) + \dot{\Theta}^T(t_f)S\dot{\Theta}(t_f)
\]

\[+ \int_0^{t_f} [e^T(t)Q e(t) + u^*_2(t)W u(t) - \gamma^2 w^T(t)w(t)]dt.
\]

\tag{27b}
\]

In this situation, the solution of the $H_{\infty}$ attitude controller of the spacecraft system is given as

\[
u(e,t) = \Xi(e(t))\Theta(t) + u^*_2(e(t), t_f)
\]

\tag{28a}
\]

where

\[
u^*_2(e(t), t_f) = -W^{-1}B_T^T(e(t), t_f)P(e(t))e(t)
\]

\tag{28b}
\]

\[
\dot{\Theta}(e,t) = -S^{-1}\Xi^T(e(t))B_T^T(e(t), t_f)P(e(t))e(t)
\]

\tag{28c}
\]

and $P(e,t) = P_T(e,t) \geq 0$ is the solution of the following time-varying differential equation

\[
\dot{P}(e(t), t_f) + P(e(t))A_T(e(t), t_f) + A_T(e(t), t_f)^T P(e(t)) + Q
- P(e(t))B_T(e(t), t_f)W^{-1}B_T^T(e(t), t_f)P(e(t)) = 0
\]

\tag{29a}
\]

\[
P(e(t_f), t_f) = Q_f.
\]

\tag{29b}
\]

This result is the same as the time-varying differential equation in (23a) with $P(e(t)) = P_1(e,t) = P_2(e,t)$, $Q = Q_1$ and $Q_f = Q_{1f}$.

From the above analysis, the main work in the design of adaptive fuzzy mixed $H_2/H_{\infty}$ attitude control of the spacecraft system is reduced to solving the time-varying coupled differential equations in (23a) and (23b) under the adaptive fuzzy constraint (23c).

**B. Solution of Time-Varying Differential Equations**

In general, however, it is difficult to solve $P_1(e,t)$ and $P_2(e,t)$ in the coupled time-varying differential
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\section*{Section 4: Adaptive Fuzzy Constraint Satisfaction}

Equations (23a) and (23b), especially satisfying the adaptive fuzzy constraint in (23c) simultaneously. Fortunately, in the spacecraft system, the differential equations (23a) and (23b) can be further simplified to algebraic matrix equations by adequately selecting the nonlinear function matrices \( P_1(e,t) \) and \( P_2(e,t) \) and by using the skew-symmetric property in (6). Moreover, with the selected matrices \( P_1(e,t) \) and \( P_2(e,t) \), the condition (23c) can be satisfied.

Because the state transformation (8b) has been involved in the process of design, without loss of generality, we suggest the solutions \( P_1(e,t) \) and \( P_2(e,t) \) of the coupled differential equations (23a) and (23b) can be put in more explicit forms as the following [29]

\begin{align}
P_1(e,t) &= T^T BM(e,t)B^T + \begin{bmatrix} K_1 & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix} \quad (30a) \\
\dot{P}_2(e,t) &= T^T BM(e,t)B^T + \begin{bmatrix} K_2 & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix} \quad (30b)
\end{align}

where \( B = [I_{3\times3}, 0_{3\times3}]^T, K_1, \) and \( K_2 \) are some positive definite symmetric constant matrices. In the following paragraphs, it is demonstrated that under some conditions, these suggested matrices \( P_1(e,t) \) and \( P_2(e,t) \) are the solutions of the differential equations (23a) and (23b). Furthermore, the constant matrices \( T, K_1, \) and \( K_2 \) can be solved from a pair of coupled algebraic equations.

Consider the second and third terms on the left-hand side of the differential equations (23a) and (23b). Using the skew-symmetric property in equation (6) and the selected relations in (30a)-(30b), we get

\begin{align}
\dot{P}_1(e,t) + P_1(e,t)A_T(e,t) + A_T^T(e,t)P_1(e,t) &= \begin{bmatrix} 0_{3\times3} & K_1 \\ K_1 & 0_{3\times3} \end{bmatrix} \quad (31a) \\
\dot{P}_2(e,t) + P_2(e,t)A_T(e,t) + A_T^T(e,t)P_2(e,t) &= \begin{bmatrix} 0_{3\times3} & K_2 \\ K_2 & 0_{3\times3} \end{bmatrix} \quad (31b)
\end{align}

It can also be easily checked that

\begin{align}
B_T^T(e,t)P_1(e,t) &= \lambda_2 B^T T \\
B_T^T(e,t)P_2(e,t) &= \lambda_2 B^T T
\end{align}

which satisfy the condition in (23c).

By using the results of (31a)-(31b), the coupled differential equations (23a) and (23b) can be reduced to the following coupled algebraic equations

\begin{align}
\begin{bmatrix} 0_{3\times3} & K_1 \\ K_1 & 0_{3\times3} \end{bmatrix} + Q_1 - \lambda_2^2 T^T B \begin{pmatrix} W^{-1} - \frac{1}{\gamma^2} I_{3\times3} \end{pmatrix} B^T T &= 0 \quad (33a)
\end{align}

Then the mixed \( H_2/H_{\infty} \) control law, fuzzy update law and the worst case uncertainty can be rewritten as

\begin{align}
u(e,t) &= -\lambda_2 W^{-1} (\lambda_1 \dot{\theta} + \lambda_2 \ddot{\theta}) \quad (34a) \\
\dot{\theta}(e,t) &= -\lambda_2 S^{-1} \Xi T(e)(\lambda_1 \dot{\theta} + \lambda_2 \ddot{\theta}) \quad (34b) \\
w^*(e,t) &= \frac{\lambda_2}{\gamma^2} (\lambda_1 \dot{\theta} + \lambda_2 \ddot{\theta}) \quad (34c)
\end{align}

From the above analysis, matrices \( P_1(e,t) \) and \( P_2(e,t) \) in (30a) and (30b) are the solutions of the coupled differential equations (23a), (23b) and the adaptive fuzzy constraint (23c) if matrices \( K_1, K_2, \) and \( T \) satisfy the coupled algebraic equations (33a) and (33b) simultaneously. Furthermore, the positive definite symmetric property of \( K_1 \) and \( K_2 \) must be satisfied. In order to guarantee the solvability, further assumptions and constraints on the weighting matrices \( Q_1, Q_2, \) and \( W \) are required.

For the simplicity of design, let

\begin{align}
W = a^2 I_{3\times3} \quad S = b_{6\times6} \quad \text{and} \quad Q_2 = \alpha Q_1
\end{align}

where \( a > 0, b > 0, \alpha > 0 \) and the positive definite symmetric matrix \( Q_1 \) is selected as

\begin{align}
Q_1 = \begin{bmatrix} q_{11} I_{3\times3} & q_{12} I_{3\times3} \\ q_{12} I_{3\times3} & q_{22} I_{3\times3} \end{bmatrix}
\end{align}

Using the definitions of \( T \) in (11b) and the forms in (35) and (36), the coupled algebraic equations in (33a) and (33b) can be solved by the following equalities

\begin{align}
\lambda_1 &= \frac{\sqrt{\gamma^2 q_{11}}}{\sqrt{q_{22}(\gamma^2 - a^2)}} \quad (37a) \\
\lambda_2 &= \frac{q_{22} \lambda_1}{q_{11}} \quad (37b) \\
\alpha &= \frac{\gamma^2 - 2a^2}{\gamma^2 - a^2} \quad (37c) \\
K_1 &= (q_{11} q_{22} - q_{12}) I_{3\times3} \quad (37d) \\
K_2 &= \alpha K_1 \quad (37e)
\end{align}

with \( 0 < a < \gamma/\sqrt{2} \) and \( q_{12} < q_{11}, q_{22} \).

From the above analysis, the solution to the adaptive mixed \( H_2/H_{\infty} \) attitude control problem is concluded in the following corollary:

\textbf{COROLLARY} (The adaptive fuzzy mixed \( H_2/H_{\infty} \) attitude control). Given a desired disturbance
attenuation level $\gamma > 0$, let the weighting matrix $W$, $S$, $Q_1$, and $Q_2$, be taken as in (35) and (36) with $\alpha$ satisfying the requirement in (37c) and $q_{12} < q_{11}q_{22}$. If the constant $a$ in the weighting matrix $W$ satisfying

$$0 < a < \frac{\gamma}{\sqrt{2}}$$

then the following mixed $H_2/H_\infty$ attitude controller and the worst case uncertainty solve the adaptive fuzzy mixed $H_2/H_\infty$ attitude control problem

$$u(t) = \Xi(e)\Theta(t) + u^*_e(t)$$

$$w(t) = \frac{a}{\gamma \sqrt{\gamma^2 - a^2}}(q_{11}\dot{\theta} + q_{22}\dot{\gamma})$$

where

$$u^*_e(t) = -\frac{\gamma}{a\sqrt{\gamma^2 - a^2}}(q_{11}\dot{\theta} + q_{22}\dot{\gamma})$$

$$\dot{\theta}(t) = -\frac{a\gamma}{b\sqrt{\gamma^2 - a^2}}\Xi^T(e)(q_{11}\dot{\theta} + q_{22}\dot{\gamma})$$

$$\text{where}$$

$$u^*_e(t) = -\frac{\gamma}{a\sqrt{\gamma^2 - a^2}}(q_{11}\dot{\theta} + q_{22}\dot{\gamma})$$

$$\dot{\theta}(t) = -\frac{a\gamma}{b\sqrt{\gamma^2 - a^2}}\Xi^T(e)(q_{11}\dot{\theta} + q_{22}\dot{\gamma})$$

VI. DESIGN ALGORITHM

Based on the above discussion, the proposed spacecraft attitude control via adaptive fuzzy approximation design can be outlined as the following design algorithm.

Step 1 Decide the fuzzy architecture $\Xi(e)$ in (15).

Step 2 Choose a desired level of disturbance attenuation, $\gamma > 0$.

Step 3 Select the weighting matrices, $W = a^2I_{3 \times 3}$ and $S = bI_{6 \times 6}$ such that $0 < a < \gamma / \sqrt{2}$ and $b > 0$,

$$Q_1 = \begin{bmatrix} q_{11}I_{3 \times 3} & q_{12}I_{3 \times 3} \\ q_{12}I_{3 \times 3} & q_{22}I_{3 \times 3} \end{bmatrix}$$

with $q_{12} < q_{11}q_{22}$, and $Q_2 = (\gamma^2 - 2a^2)/(\gamma^2 - a^2)Q_1$.

Step 4 Obtain the corresponding mixed $H_2/H_\infty$ applied torque associated with the fuzzy update law for the spacecraft system in (1) and (2)

$$\tau_a = R^{-T}(\theta) \left[ -\frac{\gamma}{a\sqrt{\gamma^2 - a^2}}(q_{11}\dot{\theta} + q_{22}\dot{\gamma}) + \Xi(e)\Theta \right]$$

$$\dot{\theta}(t) = -\frac{a\gamma}{b\sqrt{\gamma^2 - a^2}}\Xi^T(e)(q_{11}\dot{\theta} + q_{22}\dot{\gamma})$$

Remark 6 Following the Remark 5 and by the same techniques in the above paragraphs, we have the following results.

(1) For the adaptive fuzzy $H_2$ attitude control design case, the applied torque and the fuzzy update law for the spacecraft system in (1) and (2) are

$$\tau_a = R^{-T}(\theta) \left[ -\frac{1}{a}(q_{11}\dot{\theta} + q_{22}\dot{\gamma}) + \Xi(e)\Theta \right]$$

$$\dot{\theta}(t) = -\frac{a}{b}\Xi^T(e)(q_{11}\dot{\theta} + q_{22}\dot{\gamma})$$

with $a > 0$ and $b > 0$.

(2) For the adaptive fuzzy $H_\infty$ attitude control design case, the applied torque and the fuzzy update law for the spacecraft system in (1) and (2) are

$$\tau_a = R^{-T}(\theta) \left[ -\frac{\gamma}{a\sqrt{\gamma^2 - a^2}}(q_{11}\dot{\theta} + q_{22}\dot{\gamma}) + \Xi(e)\Theta \right]$$

$$\dot{\theta}(t) = -\frac{a\gamma}{b\sqrt{\gamma^2 - a^2}}\Xi^T(e)(q_{11}\dot{\theta} + q_{22}\dot{\gamma})$$

with $0 < a < \gamma$ and $a > 0$.

VII. SIMULATION RESULTS

To substantiate the performance of the adaptive fuzzy mixed $H_2/H_\infty$ attitude control design, simulations of the attitude control on the spacecraft system are considered in this section. For the convenience of simulation, we consider the spacecraft at its nominal 600 km circular orbit environment with the orbital rate $\omega_0 = 0.0011$ rad/s. Since the solar arrays are designed to point toward the sun as much as possible, the practical parameter variation of the moments of inertia $I_i$ and the products of inertia $I_{ij}$ ($i \neq j, i, j = 1, 2, 3$) when the spacecraft orbits a cycle are presented in Fig. 3. The external disturbances $\tau_{aero}$ and $\tau_{solar}$ in the body frame are presented in Fig. 4.

These data are generated by a simulator according to certain practical parameter variations and external disturbances of the ROCSAT-1 spacecraft at its nominal circular orbit environment. In this simulation, the inertia matrix $J$ is assumed to be unknown and is represented as

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}$$

Besides, the output torque vector of the reaction wheel is limited in amplitude due to saturation. Thus, we set the saturation values $S_{\omega i} = 0.05N - m, i = 1, 2, 3$ in (3).

Moreover, the spacecraft with initial altitude values at $\theta(0) = (\pi/45, \pi/45, \pi/45)^T$ and $\dot{\theta}(0) = (0, 0, 0)^T$ is required to track a hypothetical desired attitude trajectory of the form $\dot{\theta}_r + K_\theta \dot{\theta}_r + K_p \theta_r = K_u r$; with the initial conditions $\dot{\theta}_r(0) = (\pi/180, \pi/180, \pi/180)^T$ and $\dot{\theta}_r(0) = (0, 0, 0)^T$, as well as the coefficient
and 6 state variables, the following 126 membership functions (i.e., $M = 7$) are selected:

\[
\mu_{F_{ij}^1} = \exp[-(e_j - 3a_i)^2], \quad \mu_{F_{ij}^2} = \exp[-(e_j - 2a_i)^2] \\
\mu_{F_{ij}^3} = \exp[-(e_j - a_i)^2], \quad \mu_{F_{ij}^4} = \exp[-e_j^2] \\
\mu_{F_{ij}^5} = \exp[-(e_j + a_i)^2], \quad \mu_{F_{ij}^6} = \exp[-(e_j + 2a_i)^2] \\
\mu_{F_{ij}^7} = \exp[-(e_j + 3a_i)^2]
\]

for $i = 1, 2, 3,$ and $j = 1, 2, 3, 4, 5, 6$ where $a_1 = \pi/135$, $a_2 = \pi/135$, $a_3 = \pi/135$, $a_4 = \pi/1350$, $a_5 = \pi/1350$ and $a_6 = \pi/1350$. The fuzzy rules in the following form are included in the fuzzy rule bases:

$R^{(1)}$: If $e_1$ is $F_{11}$, $e_2$ is $F_{21}$, $e_3$ is $F_{31}$, $e_4$ is $F_{41}$, $e_5$ is $F_{51}$ and $e_6$ is $F_{61}$, then $u_1$ is $G^{(1)}$, 

$R^{(2)}$: If $e_1$ is $F_{12}$, $e_2$ is $F_{22}$, $e_3$ is $F_{32}$, $e_4$ is $F_{42}$, $e_5$ is $F_{52}$ and $e_6$ is $F_{62}$, then $u_2$ is $G^{(2)}$, 

$R^{(3)}$: If $e_1$ is $F_{13}$, $e_2$ is $F_{23}$, $e_3$ is $F_{33}$, $e_4$ is $F_{43}$, $e_5$ is $F_{53}$ and $e_6$ is $F_{63}$, then $u_3$ is $G^{(3)}$, 

$R^{(4)}$: If $e_1$ is $F_{14}$, $e_2$ is $F_{24}$, $e_3$ is $F_{34}$, $e_4$ is $F_{44}$, $e_5$ is $F_{54}$ and $e_6$ is $F_{64}$, then $u_4$ is $G^{(4)}$, 

$R^{(5)}$: If $e_1$ is $F_{15}$, $e_2$ is $F_{25}$, $e_3$ is $F_{35}$, $e_4$ is $F_{45}$, $e_5$ is $F_{55}$ and $e_6$ is $F_{65}$, then $u_5$ is $G^{(5)}$, 

$R^{(6)}$: If $e_1$ is $F_{16}$, $e_2$ is $F_{26}$, $e_3$ is $F_{36}$, $e_4$ is $F_{46}$, $e_5$ is $F_{56}$ and $e_6$ is $F_{66}$, then $u_6$ is $G^{(6)}$, 

$R^{(7)}$: If $e_1$ is $F_{17}$, $e_2$ is $F_{27}$, $e_3$ is $F_{37}$, $e_4$ is $F_{47}$, $e_5$ is $F_{57}$ and $e_6$ is $F_{67}$, then $u_7$ is $G^{(7)}$, 
for $i = 1, 2, 3$. Denote

\[
D = \sum_{k=1}^{7} \prod_{j=1}^{6} \mu_{F_{ij}^{(k)}}(e_j)
\]
and we have

$$\Xi(e) = \begin{bmatrix} 0_{1 \times 7} & 0_{1 \times 7} & 0_{1 \times 7} \\ 0_{1 \times 7} & \xi_1^T(e) & 0_{1 \times 7} \\ 0_{1 \times 7} & 0_{1 \times 7} & \xi_5^T(e) \end{bmatrix}, \quad \Theta = \begin{bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{bmatrix}$$

where

$$\Theta_i = [\Theta_{i1} \quad \Theta_{i2} \quad \ldots \quad \Theta_{i7}]^T, \quad \xi_i(e) = \left[ \frac{\prod_{j=1}^{6} \mu_{F_{i,j}}}{D} \quad \frac{\prod_{j=1}^{6} \mu_{F_{i,j}}}{D} \quad \ldots \quad \frac{\prod_{j=1}^{6} \mu_{F_{i,j}}}{D} \right]^T$$

for $i = 1, 2, 3$.

**Step 2** The attenuation levels $\gamma$ with respect to three cases considered in this simulation are chosen as follows:

- **Case 1** select $\gamma = 0.6$.
- **Case 2** select $\gamma = 0.4$.
- **Case 3** select $\gamma = 0.2$.

**Step 3** Select the weighting matrices as $W = (\gamma^2/20)I_{3 \times 3}$ (i.e., $a = \gamma/\sqrt{20}$), $S = I_{6 \times 6}$ (i.e., $b = 1$), $Q_1 = I_{6 \times 6}$ and $Q_2 = (\gamma^2 - 2a^2)/(\gamma^2 - a^2)Q_1$ (i.e., $\alpha = (\gamma^2 - 2a^2)/(\gamma^2 - a^2)$).

**Step 4** Obtain the corresponding adaptive fuzzy mixed $H_2/H_{\infty}$ applied torque associated with the fuzzy update law as equations in (40a) and (40b), respectively.

The simulation results are shown in Figs. 5 and 6 for the attitude angles as well as angle rates and the applied torques for above three cases, respectively. It is obvious that a smaller $\gamma$ may yield a better tracking performance in attenuating the effect of external disturbances $d(t)$. The simulation result in Fig. 7 shows the tracking performances $\int_0^t e^T(t)Q_1 e(t) + \alpha_e^2(t)W_d e(t)dt$ of the steady state responses over $t = 6000$ s. It can be found that the effects of external disturbances to the tracking errors are attenuated, even when the external disturbances are at a maximum at $t = 3000$ s.
B. Simulation 2

In order to verify the ability for disturbance attenuation of the proposed method, three control designs are considered: adaptive fuzzy $H_2$ attitude control design, adaptive fuzzy $H_\infty$ attitude control design, and the adaptive fuzzy mixed $H_2/H_\infty$ attitude control design, all with the fuzzy approximation schemes. In each of the three controller designs, we choose the weighting matrix $S$ to be the identity matrix $I_{6\times6}$, i.e., $b = 1$. Other simulation parameters are selected as follows.

Adaptive fuzzy $H_2$ optimal attitude control design: select $W = 0.4I_{3\times3}$ (i.e., $a = 0.2$) and $Q = I_{6\times6}$.

Adaptive fuzzy $H_\infty$ attitude control design: select $\gamma = 0.2$, $W = (\gamma^2/2)I_{3\times3}$ (i.e., $a = 0.2/\sqrt{2}$) and $Q = I_{6\times6}$.

Adaptive fuzzy mixed $H_2/H_\infty$ attitude control design: select $\gamma = 0.2$, $W = (\gamma^2/\sqrt{2})I_{3\times3}$ (i.e., $a = \gamma/\sqrt{20}$), $Q_1 = I_{6\times6}$ and $Q_2 = (\gamma^2 - 2a^2)/(\gamma^2 - a^2)Q_1$ (i.e., $\alpha = (\gamma^2 - 2a^2)/(\gamma^2 - a^2)$).

For the adaptive fuzzy $H_2$ attitude control case, the applied torque and update law in (41a) and (41b) are used. For the adaptive fuzzy $H_\infty$ attitude control case, the applied torque and update law in (42a) and (42b) are used. And for the adaptive fuzzy mixed $H_2/H_\infty$ attitude control case, the applied torque and update law in (40a) and (40b) are used. Then, the simulation results are shown in Figs. 8 to 10. The attitude angles and angle rates are represented in Fig. 8 for the adaptive fuzzy $H_2$, $H_\infty$, and mixed $H_2/H_\infty$ attitude controller design. As the results of these three proposed methods reveal, the adaptive fuzzy mixed $H_2/H_\infty$ attitude control causes quicker decay responses and has superior ability to diminish the effect of external disturbance under unknown parameters. These results can be expected from the fact that the adaptive fuzzy mixed $H_2/H_\infty$ attitude controller is designed to achieve the adaptive fuzzy $H_2$ optimal attitude control under an adaptive fuzzy $H_\infty$ disturbance attenuation constraint. Furthermore, the adaptive fuzzy $H_2$ attitude controller causes the smallest disturbance attenuation abilities (with respect to slower decay responses) among the three methods. This is reasonable since the adaptive fuzzy $H_2$ attitude control is designed without consideration of the external disturbance and therefore does not, in general, guarantee any robust performance in the face of disturbance. The applied torques $\tau_1$, $\tau_2$, and $\tau_3$ are represented in Fig. 9 and the tracking performances $\int_0^t e'(t)Qe(t) + u_2'(t)Wu_2(t)dt$ are plotted in Fig. 10 for the three cases. These simulation results show that the tracking performance of the adaptive fuzzy $H_\infty$ attitude controller is better than the tracking performance of adaptive fuzzy $H_2$ attitude control and
Fig. 8. Attitude angles and angle rates: --- for adaptive fuzzy $H_2$ control design, $a = 0.2$; -- for adaptive fuzzy $H_\infty$ control design, $\gamma = 0.2, a = 0.2/\sqrt{2}$; --- for adaptive fuzzy mixed $H_2/H_\infty$ control design, $\gamma = 0.2, a = 0.2/\sqrt{2}$; --- for desired attitude trajectories.

Fig. 9. Applied torques: --- for adaptive fuzzy $H_2$ control design, $a = 0.2$; -- for adaptive fuzzy $H_\infty$ control design, $\gamma = 0.2, a = 0.2/\sqrt{2}$; --- for adaptive fuzzy mixed $H_2/H_\infty$ control design, $\gamma = 0.2, a = 0.2/\sqrt{2}$.
FIG. 10. Attitude tracking performances: —— for adaptive fuzzy
$H_2$ control design, $a = 0.2$, —— for adaptive fuzzy $H_\infty$ control
design, $\gamma = 0.2$, $a = 0.2/\sqrt{2}$, —— for adaptive fuzzy mixed $H_2/H_\infty$
control design, $\gamma = 0.2$, $a = 0.2/\sqrt{20}$.

worse than the tracking performance of adaptive fuzzy
mixed $H_2/H_\infty$ attitude control.

VIII. CONCLUSION

In this paper, an adaptive fuzzy approximation technique and a mixed $H_2/H_\infty$ attitude control
technique are used for rough tuning and fine tuning,
respectively, to treat the robust tracking control of
nonlinear spacecraft systems with external disturbance
and uncertain inertia matrix. By adaptive fuzzy
approximation method, the uncertain nonlinear
model is estimated. Then, by the mixed $H_2$ and
$H_\infty$ attitude control design, the effect of external
disturbance and fuzzy approximation error to the
spacecraft attitude can be restrained and the tracking
error as well as consumed energy of the controller is
minimized. Unlike the conventional nonlinear mixed
$H_2/H_\infty$ control design which is based on solving two
coupled differential equations, a general solution
can be obtained by the proposed method via
skew symmetric property and state transformation
techniques. The structure of the controller is very
simple and the controller gain depends on the
disturbance attenuation level $\gamma$ which is assigned
globally according to mission requirement. According
to the simulation results, the adaptive fuzzy mixed
$H_2/H_\infty$ attitude controller has more excellent ability
to diminish the effects of external disturbance to get
better tracking performance than the adaptive fuzzy
$H_2$ attitude controller or the adaptive fuzzy $H_\infty$
attitude controller. Furthermore, the adaptive fuzzy mixed
$H_2/H_\infty$ attitude controller achieves robust tracking
performance for the spacecraft attitude control
systems. From the simulation results, the proposed
design algorithm exhibits significant advantages
for the attitude control of the spacecraft under unknown parameters and large external
disturbance.

APPENDIX. PROOF OF THE THEOREM

Let us first consider the cost function $J_2(u_e,w)$ in
(20b). It can be rearranged as

$$
J_2(u_e,w) = e^T(t)Q_2e(t) + \tilde{\Theta}^T(t)S\tilde{\Theta}(t)
+ \int_0^t [e^T(t)Q_2e(t) + u^T(t)Wu(t)]
+ \frac{d}{dt}[e^T(t)P_2(e,t)e(t) + \tilde{\Theta}^T(t)S\tilde{\Theta}(t)]
\right] dt
- e^T(t)P_2(e(t),t)u(t) + e^T(t)P_2(e(t),0)w(t).
$$

By the terminal condition $P_2(e(t),t) = Q_2$, we obtain

$$
J_2(u_e,w) = e^T(0)P_2(e(0),0)e(0) + \tilde{\Theta}^T(0)S\tilde{\Theta}(0)
+ \int_0^t [e^T(t)Q_2e(t) + u^T(t)Wu(t)]
+ \tilde{\Theta}^T(t)P_2(e,t)e(t) + e^T(t)P_2(e,t)e(t)
+ \tilde{\Theta}^T(t)P_2(e,t)e(t) + \tilde{\Theta}^T(t)S\tilde{\Theta}(t)
+ \tilde{\Theta}^T(t)S\tilde{\Theta}(t)] dt.
$$

Substituting the attitude tracking error dynamic
equation (17) into (44) leads to

$$
J_2(u_e,w) = e^T(0)P_2(e(0),0)e(0) + \tilde{\Theta}^T(0)S\tilde{\Theta}(0)
+ \int_0^t [e^T(t)P_2(e,t)e(t) + P_2(e,t)\dot{A}_e(t)]
+ A^T_2(e,t)P_2(e,t) + Q_2)\xi(t)
+ u^T(t)Wu(t) + u^T(t)B_2^T(e,t)P_2(e,t)e(t)
+ \tilde{\Theta}^T(t)P_2(e,t)e(t)
+ w^T(t)B_2^T(e,t)P_2(e,t)e(t)
+ e^T(t)P_2(e,t)e(t)
+ \tilde{\Theta}^T(t)P_2(e,t)e(t)
+ \tilde{\Theta}^T(t)S\tilde{\Theta}(t)
+ \tilde{\Theta}^T(t)S\tilde{\Theta}(t)] dt.
$$

Thus, by the worst case uncertainty $w^*(e,t)$ in (24),
the fuzzy update law $\dot{\tilde{\Theta}}(e,t)$ in (22c) with the fact that
$\tilde{\Theta}(e,t) = \tilde{\Theta}(e,t)$, and the differential equation (23b), we have

$$
J_2(u_e,w^*(e,t)) = e^T(0)P_2(e(0),0)e(0) + \tilde{\Theta}^T(0)S\tilde{\Theta}(0)
+ \int_0^t [(u_e(t) + W^{-1}B_2^T(e,t)P_2(e,t)e(t))^T W
\times (u_e(t) + W^{-1}B_2^T(e,t)P_2(e,t)e(t))] dt.
$$
which, by the control law in (22b), results in
\[
J_2(u^*_e(e(t)), w^*(e(t))) = e^T(0)P_1(e(0), 0)e(0) + \tilde{\Theta}^T(0)S\tilde{\Theta}(0).
\] (47)

Then, we have
\[
J_2(u^*_e(e(t)), w^*(e(t))) \leq J_2(u_e, w^*(e(t))) \quad \forall \; u_e(t) \in L_2[0, t_f].
\] (48)

Similarly, by the terminal condition \( P_1(e(t_f), t_f) = Q_{1f} \),
the cost function \( J_1(u_e, w) \) in (20a) can be rewritten as
\[
J_1(u_e, w) = e^T(0)P_1(e(0), 0)e(0) + \tilde{\Theta}^T(0)S\tilde{\Theta}(0)
\]
\[+ \int_0^{t_f} \left\{ e^T(t)(P_2(e(t)) + P_3(e(t))A_p(e(t)
+ A_1^T(e(t), P_2(e(t)) + Q_1)e(t) + u^*_e(t))^2 + u^*_e(t))^2
+ e^T(t)BR_1(e(t), P_2(e(t)) + Q_1)e(t)
+ e^T(t)(R_1(e(t), P_2(e(t)) + Q_1)e(t)
+ e^T(t)B_2(e(t), P_2(e(t)) + Q_1)e(t)
+ e^T(t)B_3(e(t), P_2(e(t)) + Q_1)e(t) + \tilde{\Theta}^T(t)S\tilde{\Theta}(t)
+ \tilde{\Theta}^T(t)\Xi^T(t)B_2(e(t), P_2(e(t)) + Q_1)e(t)
+ e^T(t)P_4(e(t), B_1(e(t))\Xi(e(t))\tilde{\Theta}(t))dt. \] (49)

It can be deduced from the differential equation (23a),
the optimal control \( u^*_e(e(t) \) in (22b),
the adaptive law \( \tilde{\Theta}(e(t) \) in (22c)
and the adaptive fuzzy constraint in (23c)
that
\[
J_1(u^*_e(e(t), w) = e^T(0)P_1(e(0), 0)e(0) + \tilde{\Theta}^T(0)S\tilde{\Theta}(0)
\]
\[− \int_0^{t_f} \left\{ \left( \gamma w(t) - \frac{1}{\gamma} BR_1(e(t), P_2(e(t)) + Q_1)e(t) \right)^T
\times \left( \gamma w(t) - \frac{1}{\gamma} BR_1(e(t), P_2(e(t)) + Q_1)e(t) \right) \right\} dt. \] (50)

Then, we can conclude that
\[
J_1(u^*_e(e(t), w^*(e(t))) = e^T(0)P_1(e(0), 0)e(0) + \tilde{\Theta}^T(0)S\tilde{\Theta}(0)
\] (51)

and
\[
J_1(u^*_e(e(t), w^*(e(t))) \geq J_1(u^*_e(e(t), w), \quad \forall \; w(t) \in L_2[0, t_f].
\] (52)

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