Mixed $H_2/H_\infty$ Fuzzy Output Feedback Control Design for Nonlinear Dynamic Systems: An LMI Approach

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Abstract—This study introduces a mixed $H_2/H_\infty$ fuzzy output feedback control design method for nonlinear systems with guaranteed control performance. First, the Takagi and Sugeno fuzzy model is employed to approximate a nonlinear system. Next, based on the fuzzy model, a fuzzy observer-based mixed $H_2/H_\infty$ controller is developed to achieve the suboptimal $H_2$ control performance with a desired $H_\infty$ disturbance rejection constraint. A robust stabilization technique is also proposed to override the effect of approximation error in the fuzzy approximation procedure. By the proposed decoupling technique and two-stage procedure, the outcome of the fuzzy observer-based mixed $H_2/H_\infty$ control problem is parameterized in terms of two eigenvalue problems (EVP’s)—one for observer and the other for controller. The EVP’s can be solved very efficiently using the linear matrix inequality (LMI) optimization techniques. A simulation example is given to illustrate the design procedures and performances of the proposed method.

Index Terms—Fuzzy control design, linear matrix inequalities, mixed $H_2/H_\infty$.

I. INTRODUCTION

THE MIXED $H_2/H_\infty$ control design is appealing for control engineers since it combines the merits of both the $H_2$ optimal control and the $H_\infty$ robust control. Mixed $H_2/H_\infty$ control problems for linear systems have been extensively studied by many researchers [20]–[26]. The outcome of their works is parameterized in terms of coupled nonlinear Riccati equations. Although numerical methods to solve the Riccati equations do exist, at this time there are no effective procedures for solving these cross-coupled Riccati equations. Khargonekar et al. give an algorithmic solution to the mixed $H_2/H_\infty$ control problem for linear systems based on a convex optimization approach [25]. More recently, LMI techniques have been employed to deal with the mixed $H_2/H_\infty$ control problem for linear systems [13], [14].

Recently, the mixed $H_2/H_\infty$ control problem for nonlinear systems has been proposed in [27], [28] where the optimal controller is developed to minimize the $H_2$ control performance under a desired $H_\infty$ disturbance rejection constraint. In [27], the mixed $H_2/H_\infty$ control problem for nonlinear systems is characterized in terms of cross-coupled Hamilton–Jacobi–Issacs partial differential equations. Until now, it is still very difficult to solve cross-coupled Hamilton–Jacobi–Issacs partial differential equations either analytically or numerically. Since the optimal solution for the mixed $H_2/H_\infty$ control problem of nonlinear systems is hardly obtained, it turns out to be interesting in seeking the suboptimal solution for the mixed $H_2/H_\infty$ control problem of nonlinear systems. Furthermore, state variables are often unavailable in nonlinear systems. In this situation, mixed $H_2/H_\infty$ output feedback control is more appealing for practical application. In this study, based on a suboptimal approach, a fuzzy observer-based mixed $H_2/H_\infty$ control design is proposed to achieve the suboptimal $H_2$ control performance under a desired $H_\infty$ disturbance rejection constraint.

In the past few years, there has been rapidly growing interest in fuzzy control of nonlinear systems, and there have been many successful applications. In spite of these successes, it has become evident that many basic issues remain to be addressed. The most important issue for fuzzy control systems is how to achieve a systematic design with the guarantee of stability and performance and recently there have been significant research efforts on the issue in fuzzy control systems [2], [9], [10], [17], [19]. In other studies, a nonlinear plant was approximated by a Takagi–Sugeno fuzzy linear model [1], and then a model-based fuzzy control was developed to stabilize the Takagi–Sugeno fuzzy linear model with guaranteed $H_2$ optimal performance [11], [12] or $H_\infty$ performance [4], [6], [7], [12], [18]. For practical control systems, a simple fuzzy control design with guaranteed $H_2$ control performance under a desired $H_\infty$ disturbance rejection constraint is more appealing for nonlinear systems. In this work, the Takagi and Sugeno fuzzy model is used to approximate a nonlinear system. Then, a fuzzy observer-based mixed $H_2/H_\infty$ controller is developed to achieve the suboptimal $H_2$ control performance under the $H_\infty$ robustness constraint. A robust stabilization technique is also introduced to eliminate the effect of approximation error due to fuzzy approximation. The proposed method attempts to combine the fuzzy observer technique with a mixed $H_2/H_\infty$ fuzzy control scheme to obtain a simple and practical output feedback algorithm to achieve both $H_\infty$ robustness performance and $H_2$ control performance for nonlinear systems.

The outcome of the fuzzy observer-based mixed $H_2/H_\infty$ control problem is parameterized in terms of two eigenvalue problems (EVP’s). The eigenvalue problem (EVP) is to minimize the maximum eigenvalue of a matrix that depends affinely on a variable, subject to some linear matrix inequality (LMI)
constraints [32]. Since solving LMI is a convex optimization problem, such formulations offer a numerically tractable means of attacking problems that lack an analytical solution [13]. Consequently, the control problem is preferred to be formulated as an LMI problem. In this study, a decoupling technique is developed so that the mixed $H_2/H_\infty$ control problem and mixed $H_2/H_\infty$ observer problem can be treated by a two-stage design procedure, i.e., solving the observer parameters at first and then solving the control parameters. Therefore, the nonlinear mixed $H_2/H_\infty$ output-feedback design problem was also formulated as two EVP’s (LMI problems), one for observer and the other for controller. As a benefit of LMI formulation, the EVP’s can be solved very efficiently by convex optimization techniques to complete the mixed $H_2/H_\infty$ fuzzy control design. However, the LMI approach can be fairly conservative in some cases i.e., a common Lyapunov function for different objectives, where suboptimal solution is obtained instead of optimal one [13].

The primary contribution of this paper is that the mixed $H_2/H_\infty$ output feedback control design is extended from linear systems toward nonlinear systems with the guaranteed performances using fuzzy observer-based control scheme. A robust stabilization technique is also proposed to override the effect of the approximation error due to fuzzy approximation. Furthermore, a two-stage algorithm based on decoupling and LMI optimization techniques is developed to solve the mixed $H_2/H_\infty$ fuzzy output feedback control problem.

A simulation example is provided to illustrate both the design procedures and the performances of the proposed method. The simulation results show that the desired performances can be achieved by the proposed method.

The paper is organized as follows. The problem formulation is presented in Section II. In Section III, a fuzzy observer-based mixed $H_2/H_\infty$ control is introduced. In Section IV, a simulation example is provided to demonstrate the design effectiveness and to confirm the desired performance. Finally, concluding remarks are made in Section V.

II. PROBLEM FORMULATION

Consider the following nonlinear system

\[ \dot{x}(t) = f(x(t)) + g(x(t))u(t) + w(t) \]
\[ y(t) = h(x(t)) + v(t) \]

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^{n \times 1}$ denotes the state vector, $u(t) = [u_1(t), u_2(t), \ldots, u_m(t)]^T \in \mathbb{R}^{m \times 1}$ denotes the control input, $u(t)$ denotes unknown but bounded disturbance, $v(t)$ denotes bounded measurement noise, the vector fields $f(x(t))$ and $g(x(t))$ and $h(x(t))$ are smooth functions with $f(0) = 0$, and $y(t) \in \mathbb{R}^{p \times 1}$ is output of the system.

A fuzzy linear dynamic model has been proposed by Takagi and Sugeno [1] to represent local linear input/output relations of nonlinear systems. This fuzzy linear model is described by fuzzy If-Then rules and will be employed here to deal with the control design problem for the nonlinear system in (1). The $i$th rule of the fuzzy linear model for the nonlinear system (1) is of the following form [1], [4]–[7], [10]–[12]

**Plant Rule:**

If $z_1(t)$ is $F_{i1}$ and \( \ldots \) and $z_g(t)$ is $F_{ig}$

Then $\dot{x}(t) = A_ix(t)B_iu(t) + u(t)$

\[ y(t) = C_ix + v(t) \text{ for } i = 1, 2, \ldots, L \] (2)

where $F_{ij}$ is the fuzzy set, $A_i \in \mathbb{R}^{m \times n}$, $B_i \in \mathbb{R}^{m \times n}$, $C_i \in \mathbb{R}^{p \times n}$, $L$ is the number of If-Then rules; and $z_1(t)$, $z_2(t)$, \( \ldots \), $z_g(t)$ are the premise variables.

The final output of the fuzzy system is inferred as follows [1], [4], [10]:

\[ \dot{x}(t) = \sum_{i=1}^{L} \mu_i(z(t))(A_ix(t) + B_iu(t)) + u(t) \]

\[ y(t) = \sum_{i=1}^{L} \mu_i(z(t))C_ix(t) + v(t) \]

\[ = \sum_{i=1}^{L} h_i(z(t))C_ix(t) + v(t) \] (3)

where

\[ \mu_i(z(t)) = \prod_{j=1}^{g} F_{ij}(z_j(t)) \]

\[ h_i(z(t)) = \frac{\mu_i(z(t))}{\sum_{i=1}^{L} \mu_i(z(t))} \]

\[ z(t) = [z_1(t), z_2(t), \ldots, z_g(t)] \] (4)

and $F_{ij}(z_j(t))$ is the grade of membership of $z_j(t)$ in $F_{ij}$.

In this paper, we assume

\[ \mu_i(z(t)) \geq 0, \text{ for } i = 1, 2, \ldots, L \]

and

\[ \sum_{i=1}^{L} \mu_i(z(t)) > 0 \]

for all $t$. Therefore, we get

\[ h_i(z(t)) \geq 0 \text{ for } i = 1, 2, \ldots, L \] (5)

and

\[ \sum_{i=1}^{L} h_i(z(t)) = 1. \] (6)
System (1) can be rearranged as the following equivalent system:

\[ \dot{x}(t) = f(x(t)) + g(x(t))u(t) + w(t) \]

\[ = \sum_{i=1}^{L} h_i(z(t))[A_i x(t) + B_i u(t)] \]

\[ + \left\{ f(x(t)) - \sum_{i=1}^{L} h_i(z(t))A_i x(t) \right\} \]

\[ + \left\{ g(x(t)) - \sum_{i=1}^{L} h_i(z(t))B_i u(t) \right\} + w(t) \]

\[ = \sum_{i=1}^{L} h_i(z(t))[A_i x(t) + B_i u(t)] \]

\[ + \{ \Delta f + \Delta g \} + w(t) \]

\[ y(t) = h(x(t)) = \sum_{i=1}^{L} h_i(z(t))C_i x(t) \]

\[ + \left( h(x(t)) - \sum_{i=1}^{L} h_i(z(t))C_i x(t) \right) + v(t) \]

\[ = \sum_{i=1}^{L} h_i(z(t))C_i x(t) + \Delta h + v(t) \quad (7) \]

where

\[ \Delta f = f(x(t)) - \sum_{i=1}^{L} h_i(z(t))A_i x(t) \quad (8) \]

\[ \Delta g = (g(x(t)) - \sum_{i=1}^{L} h_i(z(t))B_i u(t) \quad (9) \]

and

\[ \Delta h = h(x(t)) - \sum_{i=1}^{L} h_i(z(t))C_i x(t) \quad (10) \]

denote the approximation error between the nonlinear system (1) and the fuzzy model (3).

Remark 1: If we assume \( f = n \) and \( z_1(t) = x_1(t), z_2(t) = x_2(t), \ldots, z_n(t) = x_n(t) \), then the plant rule can be represented as

**Plant Rule i:**

If \( x_1(t) = F_{i1} \) and \( \ldots \) and \( x_n(t) = F_{in} \).

Then

\[ \dot{x}(t) = A_i x(t) + B_i u(t) + w(t) \]

\[ y(t) = C_i x + v(t), \quad \text{for } i = 1, 2, \ldots, L. \quad (11) \]

Suppose the following fuzzy linear observer is proposed to deal with the state estimation of nonlinear system (7):

**Observer Rule i:**

If \( z_1(t) = F_{i1} \) and \( \ldots \) and \( z_n(t) = F_{in} \).

Then

\[ \dot{x}(t) = A_i \dot{x}(t) + B_i \dot{u}(t) + L_i(y(t) - \dot{y}(t)) \quad (12) \]

where \( L_i \) is the observer gain for the \( i \)th observer rule and \( \dot{y}(t) = \sum_{i=1}^{L} h_i(z(t))C_i \dot{x}(t) \).

The overall fuzzy observer is represented as follows:

\[ \dot{x}(t) = \sum_{i=1}^{L} h_i(z(t))[A_i \dot{x}(t) + B_i u(t) \]

\[ + L_i(y(t) - \dot{y}(t))] \]

\[ = \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t))h_j(z(t))[A_i \dot{x}(t) + B_i u(t) \]

\[ + L_i C_j(x(t) - \dot{x}(t)) + L_i \Delta h + L_i \dot{v}(t)]. \quad (13) \]

Let us denote the estimation error as

\[ e(t) = x(t) - \dot{x}(t). \quad (14) \]

By differentiating (14), we get

\[ \dot{e}(t) = \dot{x}(t) - \dot{x}(t) \]

\[ = f(x(t)) + g(x(t))u(t) \]

\[ - \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t))h_j(z(t))[A_i \dot{x}(t) + B_i u(t) \]

\[ + L_i C_j(x(t) + L_i \Delta h + L_i \dot{v}(t) + u(t)] \]

\[ = \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t))h_j(z(t))[A_i \dot{x}(t) + B_i u(t) \]

\[ + L_i C_j(x(t) + L_i \Delta h + L_i \dot{v}(t) + u(t)] \]

\[ = \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t))h_j(z(t))[A_i - L_i C_j]e(t) \]

\[ \cdot - L_i \Delta h - L_i \dot{v}(t) + \Delta f + \Delta g + u(t). \quad (15) \]

Then, the augmented system can be written as the following form:

\[ \begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t))h_j(z(t))[A_i \dot{x}(t) + B_i u(t) + L_i C_j x(t) + L_i \Delta h + L_i \dot{v}(t)] \\ -L_i \Delta h - L_i \dot{v}(t) + \Delta f + \Delta g + u(t) \end{bmatrix} \quad (16) \]

Suppose the following fuzzy controller is employed to deal with the above control system design

**Control Rule j:**

If \( z_1(t) = F_{j1} \) and \( \ldots \) and \( z_n(t) = F_{jn} \).

Then

\[ u(t) = K_j \dot{x}(t) \quad \text{for } j = 1, 2, \ldots, L. \quad (17) \]
Hence, the fuzzy controller is given by
\[
    u(t) = \frac{\sum_{j=1}^{L} \mu_j(z(t))(K_j \dot{x}(t))}{\sum_{j=1}^{L} \mu_j(z(t))} = \sum_{j=1}^{L} h_j(z(t))(K_j \dot{x}(t))
\] (18)

where \( K_j \) (for \( j = 1, 2, \ldots, L \)) are the control parameters.

After manipulation, (16) can be expressed as the following form:
\[
\begin{bmatrix}
    \dot{x}(t) \\
    \dot{e}(t)
\end{bmatrix}
= \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t))
\times \begin{bmatrix}
    (A_i + B_i K_j) & L_i C_j \\
    0 & (A_i - L_i C_j)
\end{bmatrix}
\times \begin{bmatrix}
    \dot{x}(t) \\
    \dot{e}(t)
\end{bmatrix}
+ \begin{bmatrix}
    \sum_{i=1}^{L} h_i(z(t)) L_i \Delta h \\
    -\sum_{i=1}^{L} h_i(z(t)) L_i \Delta h
\end{bmatrix}
+ \begin{bmatrix}
    0 \\
    \Delta f
\end{bmatrix}
+ \begin{bmatrix}
    0 \\
    \Delta g
\end{bmatrix}.
\] (19)

Let us denote
\[
    \dot{x}(t) = \begin{bmatrix}
        \dot{x}(t) \\
        \dot{e}(t)
    \end{bmatrix},
    \Delta f = \begin{bmatrix}
        0 \\
        \Delta f
    \end{bmatrix},
    \Delta g = \begin{bmatrix}
        0 \\
        \Delta g
    \end{bmatrix}
\]

Therefore, the augmented system defined in (19) can be expressed as the following form:
\[
\dot{\tilde{x}}(t) = \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t)) h_j(z(t)) \begin{bmatrix}
    A_{ij} \dot{x}(t) + \tilde{E}_i \tilde{u}(t)
\end{bmatrix}
+ \Delta h + \Delta \tilde{f} + \Delta \tilde{g}.
\] (21)

**Assumption:** There exist bounding matrices \( \Delta A, \Delta B \) and \( \Delta C \) such that
\[
    ||\Delta f|| \leq ||\Delta A x(t)||
\] (22)
where $\Omega_j = [\Delta BK_j, 0]$ for $j = 1, 2, \cdots, L$ and
\[
(\Delta \hat{h})^T(\Delta \hat{h}) \leq 2 \left( \sum_{i=1}^{L} h_i(z(t)) L_i \Delta \hat{h} \right)^T \times \left( \sum_{i=1}^{L} h_i(z(t)) L_i \Delta \hat{h} \right)
\]
In this study, we assume that $\hat{u}(t)$ is uncertain but bounded. However, the effect of $\hat{u}(t)$ will deteriorate the control performance of the fuzzy control system and even lead to instability of the nonlinear control system. Therefore, how to eliminate the effect of $\hat{u}(t)$ to guarantee control performance is also an important design purpose of fuzzy control systems. Since $H_\infty$ control is the most important control design to efficiently eliminate the effect of $\hat{u}(t)$ on the control system, it will be employed to deal with the robust performance control in (1). Let us consider the following $H_\infty$ control performance [29], [30]
\[
\int_0^{t_f} \hat{x}(t)^T \hat{Q}_2 \hat{x}(t) \, dt \leq \hat{x}(0)^T P_1 \hat{x}(0)
\]
where $t_f$ is terminal time of control, $\rho$ is a prescribed attenuation level, the weighting matrix $\hat{Q}_2 = \hat{Q}_2^T > 0$ and $\hat{R}_2 = \hat{R}_2^T > 0$ are specified beforehand according to the design purpose.

Based on the fuzzy model (7), the design purpose of this paper is to seek a suboptimal solution for the following mixed $H_2/H_\infty$ fuzzy control problem:

Find a fuzzy controller for the augmented nonlinear system (21) such that the suboptimal quadratic control performance (29) is achieved with a desired $H_\infty$ disturbance rejection constraint in (28). Furthermore, the closed-loop system
\[
\hat{x}(t) = \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t)) h_j(z(t)) A_{ij} \hat{x}(t) + \Delta \hat{h}
+ \Delta \hat{f} + \Delta \hat{y}
\]
is locally quadratically stable at the equilibrium $\hat{x}(t) = 0$ in the absence of external disturbance $\hat{u}(t)$.

III. FUZZY OBSERVER-BASED MIXED $H_2/H_\infty$ CONTROL DESIGN

The design purpose in this study is to specify the fuzzy control in (18) to achieve the mixed $H_2/H_\infty$ control performance in (29) and (28) simultaneously. In general, the conventional separation principle in linear system theory cannot be applied directly to design the nonlinear mixed $H_2/H_\infty$ controller and nonlinear mixed $H_2/H_\infty$ observer separately. In this situation, more efforts have to be made to deal with the fuzzy observer-based nonlinear mixed $H_2/H_\infty$ control design. The design procedure is discussed step by step as the following. First, the $H_\infty$ robust fuzzy control design and the $H_2$ fuzzy control design are discussed. Then, the problem of fuzzy observer-based mixed $H_2/H_\infty$ control design is parameterized in terms of two EVP’s.

From the analysis above, the first step in the design process of the fuzzy control system is to specify a fuzzy observer-based controller such that the system is robustly stabilized and the effect of external disturbance $\hat{u}(t)$ is efficiently attenuated, thus achieving $H_\infty$ control performance in (28) with a prescribed attenuation level $\rho$.

Let us choose a Lyapunov function for the system of (21) as
\[
V(\hat{x}(t)) = \hat{x}(t)^T P_1 \hat{x}(t)
\]
where the weighting matrix $P_1$ is the same as that in (28).

By differentiating (31), we obtain
\[
\dot{V}(\hat{x}(t)) = \hat{x}(t)^T P_1 \dot{\hat{x}}(t) + \hat{x}(t)^T P_1 \ddot{\hat{x}}(t)
\]
where $\hat{x}(t)$ is terminal time of control, $\rho$ is a prescribed attenuation level, the weighting matrix $\hat{Q}_2 = \hat{Q}_2^T > 0$ and $\hat{R}_2 = \hat{R}_2^T > 0$ are specified beforehand according to the design purpose.

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\hat{x}(t) = \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t)) h_j(z(t)) A_{ij} \hat{x}(t) + \Delta \hat{h}
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\[
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\]
where $\hat{x}(t)$ is terminal time of control, $\rho$ is a prescribed attenuation level, the weighting matrix $\hat{Q}_2 = \hat{Q}_2^T > 0$ and $\hat{R}_2 = \hat{R}_2^T > 0$ are specified beforehand according to the design purpose.
Then, we obtain the following result.

**Theorem 1:** In the nonlinear augmented system (21), if \( z(t) \) is the common solution of the following matrix inequalities

\[
\begin{align*}
\sum_{i=1}^{L} h_i(z(t)) &\in \mathbb{R}^n \\
\sum_{j=1}^{L} h_j(z(t)) &\in \mathbb{R}^n \\
\end{align*}
\]

for \( t \). Then the \( H_{\infty} \) control performance of (28) is guaranteed for a prescribed \( \rho^2 \).

**Proof:** By (33), we obtain

\[
\begin{align*}
\sum_{i=1}^{L} h_i(z(t)) &\in \mathbb{R}^n \\
\sum_{j=1}^{L} h_j(z(t)) &\in \mathbb{R}^n \\
\end{align*}
\]

Integrating (36) from \( t = 0 \) to \( t = t_f \) yields

\[
\int_0^{t_f} \dot{V}(t) dt \leq \dot{V}(0) - \frac{1}{\rho^2} \int_0^{t_f} \dot{V}(t) dt + \rho^2 \int_0^{t_f} \dot{x}(t) dt.
\]

Therefore, the \( H_{\infty} \) control performance is achieved with a prescribed \( \rho^2 \).

After the \( H_{\infty} \) controller is specified to efficiently eliminate the effect of external disturbance \( r(t) \), in the second step of our design, we consider the \( H_2 \) performance index in (29). Since there exist approximation errors \( \Delta \hat{f}, \Delta \hat{g} \) and \( \Delta \hat{h} \), it is very difficult to directly solve the optimal \( H_2 \) control problem. In this

\[
\begin{align*}
\sum_{i=1}^{L} h_i(z(t)) &\in \mathbb{R}^n \\
\sum_{j=1}^{L} h_j(z(t)) &\in \mathbb{R}^n \\
\end{align*}
\]
situation, a suboptimal $H_2$ control design is proposed to deal with this problem by minimizing the upper bound of the $H_2$ performance index in (29). From (29), we obtain

$$J_2(u) = \int_0^{t_f} \left\{ \dot{x}(t)Q_2\dot{x}(t) + u^T(t)R_2u(t) \right\} dt$$

$$
= \dot{x}^T(0)P_2\dot{x}(0) - \dot{x}^T(t_f)P_2\dot{x}(t_f)
+ \int_0^{t_f} \left\{ \dot{x}^T(t)Q_2\dot{x}(t) + u^T(t)R_2u(t)
+ \frac{d}{dt}(\dot{x}^T(t)P_2\dot{x}(t)) \right\} dt
\leq \dot{x}^T(0)P_2\dot{x}(0)
+ \int_0^{t_f} \left\{ \dot{x}^T(t)Q_2\dot{x}(t) + u^T(t)R_2u(t)
+ \frac{d}{dt}(\dot{x}^T(t)P_2\dot{x}(t)) \right\} dt
\leq \dot{x}^T(0)P_2\dot{x}(0) + \int_0^{t_f} \dot{x}^T(t)Q_2\dot{x}(t)
+ \left( \sum_{j=1}^{L} h_j(z(t))[K_j,0]z(t) \right)^T \tilde{K}_2
\times \left( \sum_{j=1}^{L} h_j(z(t))[K_j,0]z(t) \right)
+ \sum_{i=1}^{L} h_i(z(t))h_j(z(t))\bar{A}_{ij}\tilde{z}(t) + \Delta \hat{h}
+ \Delta \hat{f} + \Delta \hat{g} \right) P_2\dot{x}(t)
+ \dot{x}^T(t)P_2 \left( \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t))h_j(z(t))\bar{A}_{ij}\tilde{z}(t) \right)
+ \Delta \hat{h} + \Delta \hat{f} + \Delta \hat{g} \right) dt
$$

$$
\leq \dot{x}^T(0)P_2\dot{x}(0) + \int_0^{t_f} \dot{x}^T(t)Q_2\dot{x}(t)
+ \left( \sum_{j=1}^{L} h_j(z(t))\tilde{K}_j\tilde{z}(t) \right)^T \tilde{K}_2
\times \left( \sum_{j=1}^{L} h_j(z(t))\tilde{K}_j\tilde{z}(t) \right)
+ \sum_{i=1}^{L} h_i(z(t))h_j(z(t))\bar{A}_{ij}\tilde{z}(t)
+ \Delta \hat{h} + \Delta \hat{f} + \Delta \hat{g} \right) P_2\dot{x}(t)
+ \dot{x}^T(t)P_2 \left( \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t))h_j(z(t))\bar{A}_{ij}\tilde{z}(t) \right)
+ \Delta \hat{h} + \Delta \hat{f} + \Delta \hat{g} \right) dt
$$

where $\tilde{K}_j = [K_j,0]$.

If

$$
\left( \bar{A}_{ij}^T P + P \bar{A}_{ij} + \tilde{K}_j^T \tilde{K}_j + 2\Xi_\ell \Xi_i + \Phi^T \Phi + \Omega_j^T \Omega_j + 3P \tilde{P}_2 + \tilde{Q}_2 \right) < 0
$$

we obtain the upper bound of the $H_2$ performance as follows:

$$
J_2(u) \leq \dot{x}^T(0)P_2\dot{x}(0).
$$

Therefore, we can minimize the upper bound of $J_2(u)$ to obtain a suboptimal $H_2$ control design based on the following minimization problem:

$$
\min_{P_2} \dot{x}^T(0)P_2\dot{x}(0)
\text{subject to } P_2 > 0 \text{ and (40).}
$$

From the above results, $H_2$ and $H_{\infty}$ fuzzy controllers are developed separately. In the following, we focus on the mixed $H_2/H_{\infty}$ fuzzy control design problem. A suboptimal fuzzy observer-based mixed $H_2/H_{\infty}$ controller is developed to satisfy both the suboptimal $H_2$ performance in (29) and the $H_{\infty}$ robust performance constraint in (28). From the analysis above, the mixed $H_2/H_{\infty}$ fuzzy control problem can be formulated as the following optimization problem:

$$
\min_{P} \dot{x}^T(0)P\dot{x}(0)
\text{subject to } P = P^T = P_1 = P_2 > 0
$$

$$
\left( \bar{A}_{ij}^T P + P \bar{A}_{ij} + 2\Xi_\ell \Xi_i + \Phi^T \Phi + \Omega_j^T \Omega_j + 3P \tilde{P}_2 + \tilde{Q}_2 \right) < 0
$$

and

$$
\left( \bar{A}_{ij}^T P + P \bar{A}_{ij} + \tilde{K}_j^T \tilde{K}_j + 2\Xi_\ell \Xi_i + \Phi^T \Phi + \Omega_j^T \Omega_j + 3PP + \tilde{Q}_2 \right) < 0.
$$
Furthermore, the attenuation level $r^2$ can be minimized so that the performance degradation due to $\hat{w}(t)$ is minimized, i.e.,

$$\min_{\hat{P}} r^2$$

subject to (42)–(45).

Before solving the mixed $H_2/H_\infty$ suboptimal control problem in (42)–(45), the stability of the closed-loop system in (30) must be guaranteed at the equilibrium $\hat{x}(t) = 0$. Let us define a Lyapunov function for the system of (30) as

$$V(\hat{x}(t)) = \hat{x}^T(t)P\hat{x}(t)$$

(47)

where the weighting matrix $P$ is the same as that in (47).

By differentiating (47), we obtain

$$\dot{V}(\hat{x}(t)) = \hat{x}^T(t)P\dot{\hat{x}}(t) + \dot{\hat{x}}^T(t)P\hat{x}(t)$$

$$= \left( \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t))h_j(z(t))\overline{A}_{ij}\hat{x}(t) + \Delta\hat{h} + \Delta\hat{f} \right)^T P\hat{x}(t)$$

$$+ \hat{x}^T(t)P \left( \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t))h_j(z(t))\overline{A}_{ij}\hat{x}(t) + \Delta\hat{h} + \Delta\hat{f} + \Delta\hat{g} \right)$$

$$= \left( \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t))h_j(z(t))\overline{A}_{ij}\hat{x}(t) \right)^T P\hat{x}(t)$$

$$+ (\Delta\hat{h} + \Delta\hat{f} + \Delta\hat{g})^T P\hat{x}(t)$$

$$+ \hat{x}^T(t)P \left( \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t))h_j(z(t))\overline{A}_{ij}\hat{x}(t) \right)$$

$$+ \hat{x}^T(t)P (\Delta\hat{h} + \Delta\hat{f} + \Delta\hat{g})$$

$$\leq \left( \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t))h_j(z(t))\overline{A}_{ij}\hat{x}(t) \right)^T P\hat{x}(t)$$

$$+ \hat{x}^T(t)P \left( \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t))h_j(z(t))\overline{A}_{ij}\hat{x}(t) \right)$$

$$+ 2 \left( \sum_{i=1}^{L} h_i(z(t))\overline{\varepsilon}_i \hat{x}(t) \right) \left( \sum_{i=1}^{L} h_i(z(t))\overline{\varepsilon}_i \hat{x}(t) \right)$$

$$+ (\Phi\hat{x}(t))^T (\Phi\hat{x}(t)) + 3\hat{x}^T(t)P\hat{x}(t)$$

$$+ \left( \sum_{i=1}^{L} h_i(z(t))\Omega_i \hat{x}(t) \right) \left( \sum_{i=1}^{L} h_i(z(t))\Omega_i \hat{x}(t) \right)$$

$$\leq \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t))h_j(z(t))\overline{A}_{ij}\hat{x}(t) \hat{x}(t) + \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t))h_j(z(t))\overline{A}_{ij}\hat{x}(t) \hat{x}(t)$$

Note that in the above derivation, the inequalities in (25)–(27) have been used.

Then, we obtain the following result:

**Theorem 2:** In the nonlinear closed-loop system (30), if there exists a common solution $P = P^T > 0$ for the mixed $H_2/H_\infty$ suboptimal control problem in (42)–(45). Then the closed-loop system in (30) is locally quadratically stable at the equilibrium $\hat{x} = 0$.

**Proof:** From (48), we obtain

$$\dot{V}(\hat{x}) \leq \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t))h_j(z(t))\overline{A}_{ij}\hat{x}(t) P + P\overline{A}_{ij}$$

$$+ 2\varepsilon_i^T \varepsilon_i + \Phi^T \Phi + \Omega_j^T \Omega_j + 3PP\hat{x}(t)$$

(49)

By (44), we get

$$\dot{V}(\hat{x}) \leq \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t))h_j(z(t))\overline{A}_{ij}\hat{x}(t) P + P\overline{A}_{ij}$$

$$\times \left[ -\hat{Q}_1 - \frac{1}{r^2} P\hat{E}_i \hat{E}_i^T P \right] \hat{x}(t) < 0.$$ 

(50)

This completes the proof.

From the analysis above, the most important work of the fuzzy observer-based mixed $H_2/H_\infty$ control problem is how to solve the common solution $P = P^T > 0$ from the minimization problem (42)–(45). Since (44) and (45) are not convex in general, it is not easy to analytically determine common solution $P = P^T > 0$ for (42)–(45). Fortunately, (42)–(45) can be transferred into a minimization problem subject to some linear matrix inequalities (LMI's) called eigenvalue problem (EVP) [32] by the following procedures. The EVP can be solved in a computationally efficient manner using a convex optimization technique such as the interior point method.

The following decoupling technique is employed to simplify the design problem. For the convenience of design, we assume

$$P = \begin{bmatrix} \hat{P}_{11} & 0 \\ 0 & \hat{P}_{22} \end{bmatrix}$$

(51)

and

$$\hat{Q}_1 = \begin{bmatrix} \hat{Q}_{11} & 0 \\ 0 & \hat{Q}_{22} \end{bmatrix}, \quad \hat{Q}_2 = \begin{bmatrix} \hat{Q}_{21} & 0 \\ 0 & \hat{Q}_{22} \end{bmatrix}.$$ 

(52)

These choices are suitable for the separate design of fuzzy controller and fuzzy observer. By substituting (51) and (52) into (44) and (45), we obtain

$$\begin{bmatrix} A^T P_{11} + P_{11} A + (B_k K_j)^T P_{11} + P_{11} B_k K_j \\
+ 2(L_i \Delta C)^T (L_i \Delta C) + \Delta A^T \Delta A + \hat{Q}_{11} \\
+ K_j^T \Delta B^T \Delta B K_j + P_{11}(31 + r^2 L_i L_i^T) + P_{11} \{ (L_i C_j)^T P_{11} + 2(L_i \Delta C)^T (L_i \Delta C) + \Delta A^T \Delta A \\
- \rho^2 P_{22} L_i L_i^T P_{22} \} + 2(L_i \Delta C)^T (L_i \Delta C) + (3 + r^2) P_{22} \hat{Q}_{22} + \Delta A^T \Delta A + \rho^2 P_{22} L_i L_i^T P_{22} \} \end{bmatrix} < 0$$ 

(53)
and

\[
\begin{bmatrix}
A_T^TP_{11} + P_{11}A_i + (B_iK_j^T)^TP_{11} + P_{11}B_iK_j \\
+2(L_i\Delta C)^T(L_i\Delta C) + \Delta A_T^T\Delta A \\
+K_j^T(\tilde{R}_2 + \Delta B^T\Delta B)K_j + 3P_{11} + \tilde{Q}_{211}
\end{bmatrix}
\]

\[
(L_iC_j)^TP_{11} + 2(L_i\Delta C)^T(L_i\Delta C) + \Delta A_T^T\Delta A \\
P_{11}L_iC_j + 2(L_i\Delta C)^T(L_i\Delta C) + \Delta A_T^T\Delta A
\]

\[
\{A_i - L_iC_j\}^TP_{22} + P_{22}(A_i - L_iC_j) + \tilde{Q}_{222} \\
+2(L_i\Delta C)^T(L_i\Delta C) + \Delta A_T^T\Delta A + 3P_{22}P_{22}
\]

By introducing a new matrix

\[
W = \begin{bmatrix}
W_{11} & 0 \\
0 & I
\end{bmatrix} = \begin{bmatrix}
P_{11}^{-1} & 0 \\
0 & I
\end{bmatrix}
\]

(55)

where \(W_{11} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \) and multiplying it into (53) and (54), we get (56), shown at the bottom of the next page, and

\[
W = \begin{bmatrix}
\{A^T_{11}P_{11} + P_{11}A_i + (B_iK_j^T)^TP_{11} + P_{11}B_iK_j \\
+2(L_i\Delta C)^T(L_i\Delta C) + \Delta A_T^T\Delta A \\
+K_j^T(\tilde{R}_2 + \Delta B^T\Delta B)K_j + 3P_{11} + \tilde{Q}_{211}
\end{bmatrix}
\]

\[
\{A_i - L_iC_j\}^TP_{22} + P_{22}(A_i - L_iC_j) + \tilde{Q}_{222} \\
+2(L_i\Delta C)^T(L_i\Delta C) + \Delta A_T^T\Delta A + 3P_{22}P_{22} \}
\]

\[
W_{11} \begin{bmatrix} A_{11} & A_i & B_{11} \\ A_i^T & A_i & B_{11}^T \\ B_{11} & B_{11} & D_{11} \end{bmatrix} W_{11} + \begin{bmatrix} W_{11} & 0 & 0 \\ 0 & W_{11} & 0 \\ 0 & 0 & W_{11} \end{bmatrix} \begin{bmatrix} L_jC_j \end{bmatrix} = \begin{bmatrix} \tilde{R}_2 + \Delta B^T\Delta B \end{bmatrix} \begin{bmatrix} L_iC_j \end{bmatrix} + \begin{bmatrix} \tilde{Q}_{211} \end{bmatrix} \]

(57)

With \(Y_j = K_jW_{11} \) and \(Z_i = P_{22}L_i \) and by the Schur complements [32], the matrix inequalities (56) and (57) can be rearranged as the following forms:

\[
\begin{bmatrix}
G_{11} & W_{11} & Y_j^T & G_{11}^T \\
W_{11} & G_{22} & 0 & 0 \\
Y_j & 0 & G_{33} & 0 \\
G_{44} & 0 & 0 & G_{44}
\end{bmatrix} < 0
\]

(58)

and

\[
\begin{bmatrix}
H_{11} & W_{11} & Y_j^T & H_{11}^T \\
W_{11} & H_{22} & 0 & 0 \\
Y_j & 0 & H_{33} & 0 \\
H_{44} & 0 & 0 & H_{44}
\end{bmatrix} < 0
\]

(59)

where

\[
G_{11} = W_{11}A_i^T + A_iW_{11} + (B_iY_j^T) + B_iY_j \\
+ (3I + \rho^{-2}L_iL_i^T) \\
G_{22} = -(L_i\Delta C)^T(L_i\Delta C) + \Delta A_T^T\Delta A + \tilde{Q}_{211} \}

G_{44} = A_i^T P_{22} + P_{22}A_i - (Z_iC_j)^T - Z_iC_j \\
+ \begin{bmatrix} \tilde{R}_2 + \Delta B^T\Delta B \end{bmatrix} \begin{bmatrix} L_iC_j \end{bmatrix}
\]

(56)
Therefore, the mixed $H_2/H_\infty$ suboptimal fuzzy control problem can be reformulated as the following optimization problem:

$$\min_P \mathbb{E}^T(0)P\mathbb{E}(0)$$
subject to $P = P^T > 0$, (58) and (59),

$$\tag{60}$$

Since four parameters $P_{11}, P_{22}, K_i, L_i$ should be determined from (58) and (59) which are not convex, there are no effective algorithms for solving them simultaneously until now. By the choice of (51) and (52), the control problem and observer problem are decoupled and can be solved separately by the following two-stage procedures which solve the observer parameters first and then solve the control parameters. These will be discussed in detail in the following.

Note that (58) and (59) implies that

$$A_k^T P_{22} + P_{22} A_k - (Z_i C_j)^T - Z_i C_j + \Delta A^T \Delta A + (3 + \rho^2) P_{22} P_{22} + \rho^2 Z_i Z_i^T + \hat{Q}_{122} < 0 \tag{61}$$

and

$$A_k^T P_{22} + P_{22} A_k - (Z_i C_j)^T - Z_i C_j + \Delta A^T \Delta A + 3 P_{22} P_{22} + \hat{O}_{222} < 0 \tag{62}$$

respectively, which are still not convex related to solving the mixed $H_2/H_\infty$ suboptimal fuzzy observer design. By the Schur complements, (61) and (62) can be transformed into the following linear matrix inequalities (LMI’s) if $\rho^2$ is given in advance

$$\mathbf{\begin{bmatrix} A_k^T P_{22} + P_{22} A_k - (Z_i C_j)^T - Z_i C_j + \Delta A^T \Delta A + (3 + \rho^2) P_{22} P_{22} + \rho^2 Z_i Z_i^T + \hat{Q}_{122} & P_{22} & Z_i \\ P_{22} & -\{(3 + \rho^2) I\}^{-1} & 0 \\ Z_i & 0 & -\rho^2 I \end{bmatrix}} < 0$$

and

$$\mathbf{\begin{bmatrix} A_k^T P_{22} + P_{22} A_k - (Z_i C_j)^T - Z_i C_j + \Delta A^T \Delta A + 3 P_{22} P_{22} + \hat{O}_{222} & P_{22} \\ P_{22} & -\{(3 + \rho^2) I\}^{-1} & 0 \end{bmatrix}} < 0$$

$$\tag{63}$$

$$\tag{64}$$

for $i, j = 1, 2, \ldots, L$.

Solving LMI’s is convex optimization problem which can be easily tractable numerically. The parameters $P_{22}$ and $Z_i$ (thus observer parameter $L_i = P_{22}^{-1} Z_i$) can be obtained by solving the following eigenvalue problem (EVP)

$$\min_{P_{22}} \mathbb{E}^T(0)P_{22}\mathbb{E}(0)$$
subject to $P_{22} = P_{22}^T > 0$, (63) and (64).

After solving the above mixed $H_2/H_\infty$ fuzzy observer problem, substituting $P_{22}$ and $L_i$ into (58) and (59), (58) and (59) become convex linear matrix inequalities (LMI’s). Similarly, we can easily solve $W_{11}$ and $Y_j$ (thus control parameter $K_j = Y_j W_{11}^{-1}$) from the following EVP:

$$\min_{W_{11}} \mathbb{E}^T(0)W_{11}^{-1}\mathbb{E}(0)$$
subject to $W_{11} = W_{11}^T > 0$, (58) and (59).

Therefore, the mixed $H_2/H_\infty$ suboptimal fuzzy output feedback control problem in (46) is equivalent to solving

$$\min_{P_i} \rho^2$$
subject to (65)–(66),

$$\tag{67}$$

This problem can be solved as follows. First, we solve the mixed $H_2/H_\infty$ fuzzy observer problem in (65) and the mixed $H_2/H_\infty$ fuzzy control problem in (66) with initial attenuation level $\rho^2$ (for example, $\rho^2 = 1$). Then, by decreasing the attenuation level $\rho^2$ and solving the EVP in (65) and (66) repeatedly, the procedures are terminated until $P_{22} > 0$ and $W_{11} > 0$ cannot be found.

**Design Procedures:** The fuzzy observer-based suboptimal mixed $H_2/H_\infty$ control is summarized as follows.

**Step 1)** Select membership functions and fuzzy plant rules in (2).

**Step 2)** Given an initial attenuation level $\rho^2$, select weighting matrices $\hat{Q}_i$, $\hat{Q}_2$, and $\hat{Q}_3$ according to the design purpose and specify perturbation matrices $\Delta A$, $\Delta B$, and $\Delta C$.

**Step 3)** Solve the EVP for observer in (65) to obtain $P_{22}$ and $Z_i$ (thus $L_i = P_{22}^{-1} Z_i$ can also be obtained).

**Step 4)** Substitute $P_{22}$, $Z_i$, and $L_i$ into (58) and (59) and then solve the EVP for controller in (66) to obtain $W_{11}$ and $Y_j$ (thus $P_{11} = W_{11}^{-1}$, $K_j = Y_j W_{11}^{-1}$ can also be obtained).

**Step 5)** Decrease $\rho^2$ and repeat Steps 3–5 until $W_{11}$ and $P_{22}$ can not be found.

**Step 6)** Check the assumptions of $h_j(z(t))$ and $h_i(z(t))$.

If they are not satisfied, adjust (expand) the bounds for all elements in $\Delta A$, $\Delta B$, and $\Delta C$ and then repeat Steps 3–6).

**Step 7)** To confirm the mixed $H_2/H_\infty$ performance and stability of the closed-loop system, substitute $P =
diag \((P_{11}, P_{22})\) and \(\rho^2\) into (44), (45) and verify these inequalities.

Step 8) Construct the fuzzy observer (13).

Step 9) Construct the fuzzy controller (18).

Remark 2:

1) Less conservative results can be obtained for the mixed \(H_2/H_\infty\) fuzzy control problem in (42)–(45) by utilizing the \(S\)-procedure [32]. Since approximation error between original nonlinear system and fuzzy model is considered, only weak stability (i.e., the stability in the sense of Lyapunov) can be guaranteed via the \(S\)-procedure (see details in Appendix). For stabilization problem, the stability in the sense of Lyapunov is not strong enough. The quadratical stability is obtained in this study.

2) In the case that \(y(t) = Cx(t)\), i.e., output depends on the state linearly and measurement noises are ignored, the mixed \(H_2/H_\infty\) fuzzy control problem in (42)–(45) can be made less conservative as solving the following optimization problem:

\[
\min_P \quad \tilde{x}^T(0)P\tilde{z}(0)
\]

subject to

\[
P = P^T = P_1 = P_2 > 0
\]

\[
\overline{A}_{ij}^T P + P\overline{A}_{ij} + \Phi^T \Phi + \Omega_j^T \Omega_j
\]

\[
+ \left(2 + \frac{1}{\rho^2}\right) P P + \tilde{Q}_1 < 0
\]

and

\[
\overline{A}_{ij}^T P + P\overline{A}_{ij} + \tilde{K}_j^T \tilde{R}_2 \tilde{K}_j + \Phi^T \Phi
\]

\[
+ \Omega_j^T \Omega_j + 2PP + \tilde{Q}_2 < 0.
\]

3) By the choice of decoupling \(P = \text{diag} \,(P_{11}, P_{22})\), the nonconvex problem in (42)–(45) can be transformed into two convex LMI problems (65) and (66) which can be solved separately (based on two-stage procedure). The matrix inequalities (58) and (59) are not convex. However, they become convex if \(P_{22}\) and \(L_i\) are given in advance. We can solve \(P_{22}\) and \(L_i\) from the mixed fuzzy observer problem in (65) first and then substitute \(P_{22}\) and \(L_i\) into (58) and (59). Therefore, (58) and (59) become convex linear matrix inequalities (LMI's). Then, we can easily solve \(W_{11}\) and \(K_j\) from (58) and (59).

4) Since an upper bound for the approximation error is used, the simplification of common \(P = \text{diag} \,(P_{11}, P_{22})\) is considered and the two-stage procedure of LMI approach is employed in the design procedures, the attenuation level \(\rho^2\) obtained from (67) is only an upper bound (suboptimal attenuation level) on the \(H_\infty\) performance. Actually, the true optimal attenuation level \(\rho^2\) on the \(H_\infty\) performance is difficult to evaluate. This attenuation level \(\rho^2\) can be made smaller if the more tight bounds on fuzzy approximation errors in (22)–(24) can be obtained.

5) Based on the design procedures, the solution \(P = \text{diag} \,(P_{11}, P_{22})\) solved according to (67) until it satisfies (44), (45), the mixed \(H_2/H_\infty\) performance and stability are therefore guaranteed.

6) EVP can be solved very efficiently by the convex optimization technique such as interior point algorithm [32]. Software packages such as LMI optimization
toolbox in Matlab [34] have been developed for this purpose and can be employed to easily solve the EVP.

IV. SIMULATION EXAMPLE

To illustrate the proposed mixed $H_2/H_{\infty}$ fuzzy control approach, a control problem of balancing an inverted pendulum on a cart is considered. For this example, the state equations of the inverted pendulum are given by

$\dot{x}_1 = x_2$

$\dot{x}_2 = \frac{1}{(M+m)(J+m^2) - (ml \cos x_1)^2} \times [-f_1(M+m)x_2 - (ml^2) \sin x_1 \cos x_1 + f_0 n_1 \cos x_1 + (M+m)mgl \sin x_1 - ml \cos x_1 u + w_1$

$\dot{x}_3 = x_4$

$\dot{x}_4 = \frac{1}{(M+m)(J+m^2) - (ml \cos x_1)^2} \times [-f_1 m x_2 \cos x_1 + (J + ml^2)ml^2 \sin x_1 + f_0 (J + ml^2) x_4 - m^2 gl^2 \sin x_1 \cos x_1 + (J + ml^2) n_1] + w_2$

$y_1 = x_1$

$y_2 = x_3$

(68)

where $x_1$ denotes the angle (rad) of the pendulum from the vertical, $x_2$ is the angular velocity (rad/s), $x_3$ is the displacement (m) of the cart, and $x_4$ is the velocity (m/s) of the cart. In addition, $g = 9.8 m/s^2$ is the gravity constant, $m$ is the mass (kg) of the pendulum, $M$ is the mass (kg) of the cart, $f_0$ is the friction factor (N/m/s) of the cart, $f_1$ is the friction factor (N/rad/s) of the pendulum, $l$ is the length (m) from the center of mass of the pendulum to the shaft axis, $J$ is the moment of inertia (kgm$^2$) of the pendulum, $w_1$ and $w_2$ are external disturbances and $u$ is the force (N) applied to the cart. The design parameters are $m = 0.3$ (kg), $M = 15$ (kg), $l = 0.3$ (m), $J = 0.005$ (kgm$^2$), $f_0 = 10$ (N/m/s), and $f_1 = 0.007$ (N/rad/s).

Now, following the design procedures in the above section, the fuzzy observer-based suboptimal mixed $H_2/H_{\infty}$ control is determined by the following steps.

Step 1) To use the fuzzy control approach, we must have a fuzzy model that represents the dynamics of the nonlinear plant. Therefore, we first represent the system (68) by a Takagi-Sugeno fuzzy model. To minimize the design effort and complexity, we try to use as few fuzzy rules as possible. Hence, we approximate the system by the following four-rule fuzzy model:

Rule 1: IF $x_1$ is about 0

THEN $\dot{x} = A_1 x + B_1 u + w$

$y = C_1 x$

Rule 2: IF $x_1$ is about $\pm \pi/9$

THEN $\dot{x} = A_2 x + B_2 u + w$

$y = C_2 x$
Fig. 4. The plots of $\| f(x(t)) - \sum_{i=1}^{4} h_i (x_1(t)); A_i x(t) \|$ (dashed line) and $\| \Delta A x(t) \|$ (solid line).

Rule 3: IF $x_1$ is about $\pm \frac{2\pi}{9}$
THEN $\dot{x} = A_3 x + B_3 u + w$
$y = C_3 x$

Rule 4: IF $x_1$ is about $\pm \frac{\pi}{3}$
THEN $\dot{x} = A_4 x + B_4 u + w$
$y = C_4 x$

where

$$w = [0, w_1, 0, w_2]^T$$

$$A_1 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & -0.2224 & 0 & 1.8692 \\
0 & 0.0013 & 0 & -0.6646 \\
-0.1649 & 0 & 0 & 0
\end{bmatrix}$$

$$B_1 = \begin{bmatrix}
0 \\
-0.1809 \\
0 \\
0.0665
\end{bmatrix}$$

$$B_2 = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}$$

$$B_3 = \begin{bmatrix}
0 \\
-0.1422 \\
0 \\
0.0600
\end{bmatrix}$$

$$B_4 = \begin{bmatrix}
0 \\
0 \\
0 \\
0.0656
\end{bmatrix}$$

$$C_i = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}$$ (for $i = 1, 2, 3, 4$).

Step 2) Select

$$\Delta A = \begin{bmatrix}
0 & 0.0140 & 0 & 0 \\
0.3204 & 0.0031 & 0 & 0.0129 \\
0 & 0 & 0 & 0.0140 \\
0.0009 & 0 & 0 & 0.0002
\end{bmatrix}$$

$$\Delta B = \begin{bmatrix}
0 \\
0.0015 \\
0 \\
0.0011
\end{bmatrix}.$$ 

Since $\Delta \theta = 0$ in this example, $\Delta C = 0$ is chosen.

Steps 3–5) The suboptimal $\rho^2 = 0.9$ is found using the LMI optimization toolbox in Matlab. In this case, we
obtain the common solution for $P_{22}$ and $W_{11}$ as
follows:

$$P_{22} = \begin{bmatrix}
10.4800 & -0.2034 & 1.9309 & -0.0179 \\
-0.2034 & 0.2634 & 0.0250 & -0.1200 \\
1.9309 & 0.0250 & 5.6042 & -0.3559 \\
-0.0179 & -0.1200 & -0.3559 & 0.5535
\end{bmatrix}$$

and

$$W_{11} = \begin{bmatrix}
4.2119 & -20.729 & 19.147 & -27.584 \\
-20.729 & 104.10 & -28.731 & 11.200 \\
19.147 & -28.731 & 589.75 & -365.73 \\
-27.584 & 11.200 & -365.73 & 405.94
\end{bmatrix}$$

Step 6) The assumptions of

$$\left\| \frac{df(x)}{dt} \right\|_{\infty} \leq H_{2} ||x(t)||$$

and

$$\left\| g(x) - \sum_{j=1}^{4} \sum_{i=1}^{4} h_{i}(z(t)) h_{j}(z(t)) B_{j} K_{j} \hat{\theta}(t) \right\| \leq \sum_{j=1}^{4} h_{j}(z(t)) \Delta BK_{j} \hat{\theta}(t)$$

are satisfied (refer to Figs. 4 and 5).

Step 7) The observer parameters are found to be

$$L_{1} = \begin{bmatrix}
64.2833 & -13.8878 \\
109.4052 & 6.5878 \\
-14.0718 & 100.3782 \\
7.2555 & 64.3720
\end{bmatrix}$$

$$L_{2} = \begin{bmatrix}
64.0383 & -14.0171 \\
109.3448 & 6.8346 \\
-14.1824 & 100.3219 \\
7.7578 & 64.5457
\end{bmatrix}$$

$$L_{3} = \begin{bmatrix}
63.6645 & -14.2031 \\
108.5267 & 7.2790 \\
-14.3895 & 100.2351 \\
8.7431 & 64.7800
\end{bmatrix}$$

$$L_{4} = \begin{bmatrix}
63.4491 & -14.3335 \\
106.5370 & 7.6004 \\
-14.5146 & 100.2069 \\
9.7377 & 64.8450
\end{bmatrix}$$

Then, we construct the mixed $H_{2}/H_{\infty}$ fuzzy observer as

$$\dot{\hat{x}}(t) = \sum_{i=1}^{4} h_{i}(x(t))[A_{i}\hat{\theta}(t) + B_{i}u(t)] + L_{4}(y(t) - \hat{y}(t)).$$

Step 8) The control parameters are found to be

$$K_{1} = [113 \quad 22.5 \quad 4.08 \quad 10.7 \times 10^{3}],$$

$$K_{2} = [95.4 \quad 1.90 \quad 3.44 \quad 9.05 \times 10^{3}].$$
Therefore, we obtain the mixed $H_2/H_{\infty}$ fuzzy control law

$$u(t) = \sum_{i=1}^{4} h_i(x_i(t))K_i\hat{x}(t).$$

Figs. 2–5 present the simulation results for the fuzzy observer-based mixed $H_2/H_{\infty}$ control. The initial condition is assumed to be $(x_1(0), x_2(0), x_3(0), x_4(0), \hat{x}_1(0), \hat{x}_2(0), \hat{x}_3(0), \hat{x}_4(0))^T = (\pi/4, 0, 0, 0, 0, 0, 0, 0)^T$, and the external disturbance $u_{d1}$ and $u_{d2}$ are assumed to be a periodically square wave with amplitude $\pm 0.5$ and period $0.5$ s in the simulations. Fig. 2 shows the trajectories of the states $x_1$ and $x_2$ (including the estimated states $\hat{x}_1$, $\hat{x}_2$ and external disturbance $u_{d1}$). Fig. 3 shows the trajectories of the states $x_3$ and $x_4$ (including the estimated states $\hat{x}_3$, $\hat{x}_4$ and external disturbance $u_{d2}$). The simulation results show that the suboptimal fuzzy observer-based mixed $H_2/H_{\infty}$ controller can balance the inverted pendulum with large external disturbances and that the desired performance can be achieved.

V. CONCLUSION

In this paper, fuzzy observer technique and mixed $H_2/H_{\infty}$ control scheme are combined to solve the mixed $H_2/H_{\infty}$ output feedback control problem of nonlinear systems. A fuzzy observer-based mixed $H_2/H_{\infty}$ nonlinear control scheme has been proposed for the first time from output feedback perspective. With the proposed fuzzy control method, the mixed $H_2/H_{\infty}$ output feedback control design by LMI approach can be extended from linear systems toward nonlinear systems. Furthermore, robust stabilization is guaranteed to override the effect of fuzzy approximation error.

By the proposed decoupling technique and two-stage procedure, the mixed $H_2/H_{\infty}$ fuzzy control problem is parameterized in terms of two EVP’s, one for observer and the other for controller, respectively. The corresponding EVP’s can be solved very efficiently by convex optimization technique with the aid of the LMI optimization toolbox in Matlab. By employing this mixed $H_2/H_{\infty}$ fuzzy output feedback control scheme, the performance of fuzzy control design for nonlinear output feedback control systems can be significantly improved. Therefore, the proposed design algorithm is appropriate for practical design of nonlinear system controls with external disturbances.

The proposed design method is simple and the number of membership functions for the proposed control law can be extremely small. However, because of the efficient elimination of the external disturbance by the $H_{\infty}$ robust control and the relatively good performance of the $H_2$ optimal control, the proposed method is more practical than other control methods. Based on the convex optimization techniques, an iterative design procedure is also proposed to achieve suboptimal mixed $H_2/H_{\infty}$ output feedback control design for the nonlinear systems. Simulation results indicate that the desired mixed $H_2/H_{\infty}$ performance for nonlinear systems can be achieved using the proposed method.

APPENDIX

Here, we show that the result is less conservative by utilizing the $S$-procedure [32], [33]. From (28), we obtain

$$\int_0^{t_f} \bar{x}^T(t)\bar{Q}_i\bar{x}(t) \, dt \leq \bar{x}(0)^T P \bar{x}(0) + \int_0^{t_f} \left[ \bar{x}^T(t)\bar{Q}_i\bar{x}(t) + \frac{d}{dt}(\bar{x}^T(t)P\bar{x}(t)) \right] \, dt.$$

Then we get

$$\int_0^{t_f} \bar{x}^T(t)\bar{Q}_i\bar{x}(t) \, dt \leq \bar{x}(0)^T P \bar{x}(0) + \rho^2 \int_0^{t_f} \bar{w}^T(t)\bar{w}(t) \, dt.$$

Therefore, the $H_{\infty}$ control performance is achieved with a prescribed $\rho^2$.

To ensure the constraint on the variable $P$ satisfying (69) and assumptions in (25)–(27),
Applying the SS-procedure, (69) and (25)–(27) are equivalent to the existence of $\tau \geq 0$ such that

$$
\begin{bmatrix}
A_{ij}^T P + P A_{ij} + Q_i \\
\tau (2\Xi_i \Xi_i + \Phi_i^T \Phi_i + \Omega_i^T \Omega_i)
\end{bmatrix} P \\
-\tau I \\
0
\end{bmatrix} \begin{bmatrix}
P \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
0 \\
-P \Phi_i \\
P
\end{bmatrix} \leq 0.
$$

(70)

Therefore, if there exist $\tau \geq 0$ and $P = P^T > 0$ is the common solution of the matrix inequalities in (70) for $i, j = 1, 2, \ldots, L$, then the $H_\infty$ control performance of (28) is guaranteed for a prescribed $\rho^2$.

We omit the result of $H_2$ performance since the procedure is similar to that of $H_\infty$ performance. Next, we discuss the stability of the closed-loop system (30) at the equilibrium $\tilde{x}(t) = 0$. Let us define a Lyapunov function as (47), then we obtain

$$
\dot{V}(\tilde{x}) = \tilde{x}^T(t) P \dot{\tilde{x}}(t) + \dot{\tilde{x}}^T(t) P \tilde{x}(t)
$$

$$
= \sum_{j=1}^L h_j(z(t)) \sum_{j=1}^L h_j(z(t))
$$

$$
\begin{bmatrix}
\tilde{x}^T(t) \\
\Delta \tilde{h}_i \\
\Delta \tilde{j}_j
\end{bmatrix}
$$

$$
A_{ij}^T P + P A_{ij} P P P
$$

$$
\times
$$

$$
\begin{bmatrix}
\dot{\tilde{x}}^T(t) \\
\Delta \tilde{h}_i \\
\Delta \tilde{j}_j
\end{bmatrix}
$$

If there exists a common solution $P = P^T > 0$ for the following matrix inequalities:

$$
\begin{bmatrix}
A_{ij}^T P + P A_{ij} P P P \\
P \\
P
\end{bmatrix} \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} \leq 0
$$

(71)

then we get $\dot{V}(t) \leq 0$.

Therefore, the closed-loop system is locally stable in the sense of Lyapunov at the equilibrium $\tilde{x} = 0$. Note that (70) and (71) only hold for nonstrict inequalities.

REFERENCES


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