Robust discrete-time adaptive control subject to a control input amplitude constraint

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Subject to a control input amplitude constraint, an adaptive algorithm is proposed to control discrete-time plants with unmodelled dynamics. By introducing an additional feedback signal, it is shown that the uniform boundedness of all signals in the adaptive loop can be guaranteed. The nominal plant is assumed to be stable but unnecessarily minimum phase. Various properties of this adaptive algorithm are analysed. It is shown that the performance which can be achieved with no control input amplitude constraint in the non-adaptive case (i.e. the case when the true nominal plant is known a priori) is asymptotically well maintained in the adaptive system. The analysis is supported by computer simulation results.

1. Introduction

In a control system, the dynamic range of practical actuators is usually limited; for example, a valve saturates when it is fully open or closed, the throttle position can only go so far, and the control surfaces in an aircraft or missile can be deflected only to a certain angle from their nominal positions. Saturation usually occurs when the reference signal is abnormally large and/or changes abruptly. Since the saturation would result in wind-up and instability problems, the rational solution is to develop an algorithm that takes this into account. This can be done using optimal control theory. However, such a design method is quite complicated, and the corresponding control law is also complex. Therefore, it is practical to use a simple heuristic method.

In PID control, the common philosophy to overcome saturation is to ‘turn off’ the integral action as soon as the actuator saturates. Two anti-windup schemes using this philosophy were given by Franklin et al. (1986). Other anti-windup schemes for PID controllers have been given by Krikels (1980), Krikels and Barkas (1984), Glattfelder and Schaufelberger (1983). However, this philosophy is applicable only to PID controllers. Åström and Wittenmark (1984) proposed an observer-based controller which can be allowed to be unstable. This controller can assure that all signals in the closed loop system are uniformly bounded if the plant output is uniformly bounded. Chen and Wang (1988), by taking the saturated system as one which consists of a linear time-invariant nominal part and a nonlinear time-varying perturbation part and using the Bellman–Gronwall lemma to analyse the system in the time domain, determined the range which is allowed for the input of the actuator (i.e. the control input), but it is not practical to know where the control input goes in advance.

In the adaptive system, saturation always occurs during the starting period.
since the identified model has not been aligned with the true plant. In many papers, e.g. Abramovitch and Franklin (1987), Miller and Davison (1987), Payne (1986), various techniques have been proposed to solve this problem, but the results of these papers are not satisfactory for practical use, as the system response may be sluggish, so only BIBO stability is achieved, or some severe constraints are put on the plant. Abramovitch and Franklin (1987), with a control input amplitude constraint, employed a self-tuning algorithm to ensure that the tracking error converges asymptotically to zero without identifying the plant, and assumed the linear part to be stable and that the response may be sluggish. In Miller and Davison (1987), the stability of an adaptive pole-placement controller with a saturating actuator is guaranteed under an assumption which is not easily judged. In Payne (1986), the constrained optimization problem is defined and solved. BIBO stability is achieved under the condition that the plant is stable. But, if the tracking property is to be achieved, the assumption of the plant being minimum phase is also required.

In this paper, our philosophy to solve the problem of the control input amplitude being constrained is to let the control input follow the original trajectory as if the saturation never occurred. Based on this philosophy, an additional feedback signal is introduced. It is shown that all signals in the adaptive loop are uniformly bounded. The true nominal plant is assumed to be stable but unnecessarily minimum phase. Various properties of this adaptive algorithm are analysed. It is also shown that as long as the sizes of unmodelled dynamics and disturbance are small, the performance that can be achieved when there is no control input amplitude constraint in the non-adaptive case, is asymptotically well maintained in the adaptive system if the original trajectory of the control input is eventually within the constrained amplitude. The analysis is supported by computer simulation results.

The contents of this paper are organized as follows. In § 2, a description of the plant is presented. In § 3, the proposed control algorithm is introduced and its function in the non-adaptive system is revealed. In § 4, the proposed adaptive algorithm is established and various properties of this algorithm are proved. Some computer simulation results and a conclusion are given in § 5 and 6 respectively.

2. The plant model

Let $u(t)$ and $y(t)$ be the plant input and output respectively. Also let $G_{\alpha}(z)$ be the nominal transfer function, which is the best estimate, in some sense, of the true plant behaviour, and let $G(z)$ denote the true transfer function of the plant. $G(z)$ is assumed to be strictly proper. The most commonly used methods to represent the plant with unmodelled dynamics are as follows

$$G(z) = G_{\alpha}(z) + \mu_{\alpha} \Delta_{\alpha}(z) \quad (2.1a)$$

$$G(z) = G_{\alpha}(z)(1 + \mu_{m} \Delta_{m}(z)) \quad (2.1b)$$

and

$$G(z) = G_{\alpha}(z)(1 + \mu_{m} \Delta_{m}(z)) + \mu_{\alpha} \Delta_{\alpha}(z) \quad (2.1c)$$

where $\mu_{\alpha}$ and $\mu_{m}$ are small and constant, and $\Delta_{\alpha}(z)$ and $\Delta_{m}(z)$ are stable transfer functions. Moreover, $\Delta_{\alpha}(z)$ is strictly proper, but $\Delta_{m}(z)$ is not necessar-
ily proper, \( \mu_a \Delta_a \) and \( \mu_m \Delta_m \) are called the additive perturbation and multiplicative perturbation, respectively. In general, these modelling uncertaintys are caused by neglecting some fast dynamics such as actuators and sensors, or the parasitic dynamics of the plant itself. Thus, \( \Delta_a(z) \) and \( \Delta_m(z) \) contain fast stable modes of the plant. We consider the case that \( G_o(z) \) is a rational transfer function and

\[
G_o(z) = \frac{B(\theta, z^{-1})}{A(\theta, z^{-1})}
\]

where

\[
\theta = [a_1, \ldots, a_n, b_1, \ldots, b_m]^T
\]

\[
A(\theta, z^{-1}) = 1 + a_1 z^{-1} + \cdots + a_n z^{-n} = A(z^{-1})
\]

\[
B(\theta, z^{-1}) = b_1 z^{-1} + \cdots + b_n z^{-n} = B(z^{-1})
\]

For brevity, let \( d \) denote the unit delay operator and then the true plant can be represented by the following input/output relation

\[
A(d)y(t) = B(d)u(t) + \mu H(d)u(t) + A(d)w(t)
\]

where \( w(t) \) is the uniformly bounded disturbance, \( \mu = \max(\mu_a, \mu_m) \), and \( \mu H(d) \) is the modelling uncertainty which can be assumed to be BIBO stable from (2.1) and the assumptions on \( \Delta_a(z) \) and \( \Delta_m(z) \).

Let \( u_c(t) \) be the control input. Since we are going to deal with the problem of the adaptive control with constrained control input amplitude, the relation between \( u_c \) and \( u \) is given as follows

\[
u = \begin{cases}
  u_c, & \text{if } -\bar{u} \leq u_c \leq \bar{u} \\
  \bar{u}, & \text{if } u_c > \bar{u}, \quad 0 < u, \quad \bar{u} < \infty \\
  -\bar{u}, & \text{if } u_c < -\bar{u}
\end{cases}
\]

Before going to the next section, we make some necessary assumptions about the plant as follows.

**Assumption A:**

(i) \( A(d) \) and \( B(d) \) are coprime.

(ii) \( B(1) \neq 0 \).

(iii) \( A(d) \) is a stable polynomial (i.e. \( A(d) \neq 0 \) for \( |d| \leq 1 \)).

**Remark 1.1:**

(i) Since the pole-placement technique is used in this paper, Assumption A (i) is standard.

(ii) It is expected that the constant reference signal can be tracked, so \( (d - 1) \) will be a factor of the denominator of the controller by the internal model principle and this is the reason for Assumption A (ii) being made, since the internal stability must be satisfied for the general control system. Certainly, other types of reference signals can be tracked, but Assumption A (ii) should be modified accordingly by the internal model principle. \[\square\]
3. The proposed control algorithm with known nominal plant

Since we will adopt the equivalent certainty principle (Goodwin and Sin 1984) to establish the adaptive control algorithm, for ease of studying the adaptive system, it is advantageous to investigate the situation in which the nominal plant is known a priori. For clarity, first we will analyse the function of the proposed control algorithm in the ideal case (i.e. when neglecting the modelling uncertainties and the disturbance), and then go to the non-ideal case.

Ideal case: Let $A^*(d)$ be a prespecified $(2n + 1)$th order Hurwitz polynomial in $d$. The roots of $A^*(d)$ are the desired closed loop poles. Also, let $C(d) = S(d)/R(d)$ be the controller, where $R(d) = 1 + r_1 d + \cdots + r_{n+1} d^{n+1}$ and $S(d) = s_0 d + s_2 d^2 + \cdots + s_{n+1} d^{n+1}$. The relation between $A(d)$, $B(d)$, $R(d)$, $S(d)$ and $A^*(d)$ can be represented by the following diophantine equation

$$A(d)R(d) + B(d)S(d) = A^*(d) R(d) = R'(d)(d - 1) \tag{3.1}$$

Note that $R(d)$ and $S(d)$ are uniquely determined since $A(d)(d - 1)$ and $B(d)$ are coprime by Assumption A.

Our philosophy for solving the problem of the control input amplitude being constrained is to let $u_c(t)$ track the original trajectory as if the saturation never occurred. This goal can be achieved by an additional feedback signal $y_0(t)$ which is obtained by passing $(u_c - u)(t)$ through a ‘fictitious plant’. This situation is shown graphically in Fig. 1. In Fig. 1, $r(t)$ is the uniformly bounded reference input. After observing Fig. 1, we have the following equations in the ideal case:

$$A(d)y_a(t) = B(d)(u_c - u)(t) \tag{3.2a}$$

$$R(d)u_c(t) = S(d)(r_c - y_a - y)(t) \tag{3.2b}$$

$$A(d)y(t) = B(d)u(t) \tag{3.2c}$$

By (3.1), it is easily verified that

$$u_c(t) = \frac{A(d)S(d)}{A^*(d)} r(t) \triangleq u_c^*(t) \tag{3.3a}$$

$$y_a(t) = \frac{B(d)S(d)}{A^*(d)} r(t) \triangleq y^*(t) \tag{3.3b}$$

where $u_c^*(t)$ and $y^*(t)$ are the nominal control input and nominal output respectively. It is clear that $y_a(t)$ is used to compensate the ‘loss’ of $u_c^*(t)$ such that $u_c(t)$ can follow $u_c^*(t)$ even if saturation occurs. After a short period, say

![Figure 1. The compensation of retaining $u_c^*$.](image-url)
$T_c$, $u_c^*(t)$ should lie in $[-u, \bar{u}]$ as is usually the case. Thus, $(u_c - u)(t) = 0$ after $T_c$ and then as $t \to \infty$, $y_e(t) \to 0$ and $(y - y^*)(t) \to 0$ exponentially. \hfill \Box

**Non-ideal case:** At this point, some notations are introduced to let our work proceed conveniently and transparently: $R^+$ denotes the set of non-negative real numbers and $Z^+$ denotes the set of non-negative integers. Let $f(t)$ be a time signal. If $f(t)$ is uniformly bounded, $\|f\|_\infty \leq \sup_{t \leq 0} |f(t)|$. The truncated signal $f_T(t)$ is defined as follows

$$f_T(t) \doteq \begin{cases} f(t), & \text{if } t \leq T \\ 0, & \text{if } t > T \end{cases}$$

i.e. $f_T(t)$ is obtained by truncating $f(t)$ at $T$. We also define

$$f^*_T(t) \doteq \begin{cases} f(t), & \text{if } t > T \\ 0, & \text{if } t \leq T \end{cases}$$

Note that if $f(t)$ is a uniformly bounded signal, then $\|f\|_\infty \leq \|f_T\|_\infty$ and $\|f\|_\infty \geq \|f^*_T\|_\infty$. For brevity, let $\varepsilon(t, t_0, \gamma)$ denote any exponentially decaying signal and also satisfy the condition that there exists a constant $c_1 \in R^+$ such that $|\varepsilon(t, t_0, \gamma)| \leq c_1 \gamma^{-t_0}, \forall t \geq t_0, 0 \leq \gamma < 1$.

From (2.3), (3.2 a), and (3.2 b) and by noting that (3.2 a) and (3.2 b) are also available for the non-ideal case, it is obtained that

$$A^*(d)u_c(t) = A(d)S(d)r(t) - \mu S(d)H(d)u(t) - S(d)A(d)w(t) \quad (3.4)$$

We introduce a proposition here. \hfill \Box

**Proposition 3.1:** If $\limsup_{t \to \infty} \|w_i^e\|_\infty < \varepsilon_w$, where $\varepsilon_w \in R^+$, then there exist constants $c_2, c_3 \in R^+$ such that

$$\limsup_{t \to \infty} \|(u_c - u_c^*)\|_\infty < c_2 \mu \cdot \max (\bar{u}, u) + c_3 \varepsilon_w$$

**Proof:** It is noted via (2.3) and (3.4) that any signal is uniformly bounded in the closed loop system. From (3.4)

$$u_c(t) - u_c^*(t) = -\mu \frac{S(d)}{A^*(d)} H(d)u(t) - \frac{S(d)A(d)}{A^*(d)} w(t)$$

Since $A^*(d)$ is a Hurwitz polynomial and $H(d)$ is an exponentially stable operator, we have that there exist constants $k_1, k_2 \in R^+$, and $\gamma_1 \in (0, 1)$ such that

$$|u_c(t) - u_c^*(t)| \leq |\varepsilon(t, 0, \gamma_1)| + k_1 \mu \cdot \max (u, \bar{u}) + \sum_{i=0}^{t} k_2 \gamma_1^{t-i} |w(i)|$$

By letting $T_1 < T$, it follows that

$$|(u_c - u_c^*)(T)| \leq |\varepsilon(T, 0, \gamma_1)| + k_1 \mu \cdot \max (u, \bar{u}) + k_2 \|w_{T_1}\|_\infty \cdot \gamma_1^{T-T_1} \cdot (1 - \gamma_1) + k_2/(1 - \gamma_1) \cdot \sup_{T_1 < t < T} w(t)$$

Also since $\limsup_{t \to \infty} \|w_i^e\|_\infty < \varepsilon_w$, there exists $0 < T_2 < \infty$ such that $\|w_{T_2}\|_\infty < \varepsilon_w$, $\forall T_3 \geq T_2$. By letting $T_1 = T_2$, it is implied that
\[ \| (u_c - u_a^s) \|_\infty < k_1 \mu \max (u, a) + k_2 / (1 - \gamma_1) \cdot \varepsilon_w, \quad \text{as } T \to \infty \]

Consequently, this proposition can be established by letting \( k_1 = c_2, k_2 / (1 - \gamma_1) = c_3 \).

\[ \square \]

Proposition 3.1 tells us that \( u_c(t) \) is near to \( u_a^s(t) \) eventually if \( u \) is small and \( w(t) \) is small eventually. By using Proposition 3.1, we have the main result for the non-adaptive case as follows.

**Theorem 3.1:** If \( \limsup_{T \to \infty} \| (u_a^s)_{T}^* \|_\infty + c_2 \mu \max (a, u) + c_3 \varepsilon_w \leq \min (a, u) \), where \( c_2, c_3, \) and \( \varepsilon_w \) are defined in Proposition 3.1, then there exist constants \( c_4, c_5 \in R^+ \) such that \( \limsup_{T \to \infty} \| (y - y^*)_{T}^* \|_\infty < c_4 \mu \max (a, u) + c_5 \varepsilon_w \).

**Proof:** It is noted that \( \| (u_c - u_a^s)_{T}^* \|_\infty \geq \| u_c(t) - u_a^s(t) \|_\infty \), \( T \leq t < \infty \). Thus, \( u_c(t) \leq \| u_c - u_a^s \|_\infty + \| u_a^s(t) \|_\infty \), \( T \leq t < \infty \) and then \( \| u_c^s_{T}^* \|_\infty \leq \| (u_c - u_a^s_{T}^* \|_\infty + \| u_a^s_{T}^* \|_\infty \). By the result of Proposition 3.1, \( u_c \) goes to lie in \([-a, a]\) eventually. Consequently, combining (2.3) and (3.2) and noting that \( u = u_c \) eventually, we have that there exists \( 0 < T_a < \infty \) such that the operation of the closed loop system after \( T_a \) can be described by the following equations

\[ R(d)u(t) = S(d)(r - y)(t) \tag{3.5.a} \]
\[ A(d)y(t) = B(d)u(t) + \mu H(d)u(t) + A(d)w(t) \tag{3.5.b} \]

Meanwhile, from (3.5), it is obtained that

\[ A^*(d)y(t) = B(d)S(d)r(t) + \mu R(d)H(d)u(t) + A(d)R(d)w(t) \tag{3.6} \]

The remaining proof is established by following similar lines as in the proof of Proposition 3.1.

\[ \square \]

Theorem 3.1 tells us that if the nominal input eventually lies in \([-u, u]\), the size of modelling uncertainty is small, \( w \) is eventually small, and \( y \) also keeps near to the nominal output. Moreover, assume further that the reference input is constant, then \( y(t) \) eventually tracks \( r(t) \) well under similar conditions.

**Corollary 3.1:** Suppose that \( r(t) = \tilde{r}, \tilde{r} \) is a constant. If \( A(1) / B(1) \tilde{r} + c_2 \mu \max (a, u) + c_3 \varepsilon_w \leq \min (a, u) \), where \( c_2, c_3, \) and \( \varepsilon_w \) are defined in Proposition 3.1, then \( \limsup_{T \to \infty} \| (y - \tilde{r})_{T}^* \|_\infty < c_4 \mu \max (a, u) + c_5 \varepsilon_w \), where \( c_4 \) and \( c_5 \) are defined in Proposition 3.1.

**Proof:** This result is established immediately from Theorem 3.1 by the facts

\[ \lim_{t \to \infty} u_a^s(t) = \frac{A(1)S(1)}{A^*(1)} \tilde{r} = \frac{A(1)}{B(1)} \tilde{r} \]
\[ \lim_{t \to \infty} y_a^s(t) = \frac{B(1)S(1)}{A^*(1)} \tilde{r} = \frac{B(1)S(1)}{B(1)S(1)} \tilde{r} \]

\[ \square \]

**Remark 3.1:** In fact, in the ideal case, if \( (u_c - u_a^s)(t) = 0 \) after \( T_a \), then \( y_a(t) = \epsilon(t, T_a, \gamma_2) \) and \( y(t) = y_a^s(t) - y_a(t) \). Since \( \gamma_2 \) depends on the roots of \( A(d) \), the plant output response may be sluggish. In spite of this disadvantage, it is interesting that the performance of the system is asymptotically well maintained in the adaptive case. This fact will be shown in § 4.

\[ \square \]
4. Analysis of the adaptive system

In this section, a simple commonly-used projection estimation algorithm with a fixed dead zone is adopted to estimate coefficients of $A(d)$ and $B(d)$. We will demonstrate that this estimation algorithm can give a parameter estimate with standard estimation properties. Moreover, various properties of the proposed control algorithm in the adaptive case will be investigated. It will be shown that the performance would be well maintained as desired in the non-adaptive case if $r(t)$ is nearly a constant or $A^*(d)$ is properly chosen.

The plant (2.3) can be written in a regression form as

$$y(t) = \phi^T(t)\theta + \eta(t) \tag{4.1 a}$$

where

$$\phi(t) = [-y(t - 1), \ldots, -y(t - n), u(t - 1), \ldots, u(t - n)]^T \tag{4.1 b}$$

$$\eta(t) = \mu H(d)u(i) + A(d)w(t) \tag{4.1 c}$$

Since $H(d)$ is a BIBO stable operator and $u(t)$ and $w(t)$ are uniformly bounded, a strictly eventual upper bound on $\eta$ can be obtained as

$$\bar{\eta} = c_6 \mu \cdot \max (\bar{u}, \bar{u}) + c_7 \varepsilon_w, \quad c_6, c_7 \in \mathbb{R}^+$$

where $\varepsilon_w$ is defined in Proposition 3.1 and

$$c_7 = 1 + \sum_{i=1}^{n} |a_i|$$

In other words, there exists $0 < T_S < \infty$ such that

$$\eta(t) < \bar{\eta}, \quad \forall t \geq T_S$$

Let $\hat{\theta}(t) := [\hat{a}_1(t), \ldots, \hat{a}_n(t), \hat{b}_1(t), \ldots, \hat{b}_p(t)]^T$ denote the estimate of the true parameter vector (i.e. $\theta$), where $\hat{a}_i$ and $\hat{b}_j$ are the estimates of $a_i$ and $b_j$ respectively. Let $[\cdot]_i$ denote the $i$th element of a vector and make the following assumption.

**Assumption B1:** There exist two known constant vectors $\theta_{\min}$ and $\theta_{\max}$ such that $[\theta_{\min}]_i \leq [\theta]_i \leq [\theta_{\max}]_i$, $1 \leq i \leq 2n$.

The projection estimation algorithm with a fixed dead zone can be described as follows.

Define the estimation error as

$$e(t) = y(t) - \hat{y}(t), \quad \hat{y}(t) = \phi^T(t)\hat{\theta}(t) \tag{4.2 a}$$

and the dead zone function as

$$D(e, \bar{\eta}) = \begin{cases} 0, & \text{if } |e| \leq \bar{\eta} \\ e - \bar{\eta}, & \text{if } e > \bar{\eta} \\ e + \bar{\eta}, & \text{if } e < -\bar{\eta} \end{cases} \tag{4.2 b}$$

$\hat{\theta}(t)$ is generated by the following equations:

$$\hat{\theta}(t + 1) = \hat{\theta}(t) + \frac{\phi(t)D(e(t), \bar{\eta})}{\bar{\gamma} + \phi^T(t)\phi(t)}, \quad \bar{\gamma} > 0 \tag{4.2 c}$$
\[ [\hat{\theta}(t)]_i = \begin{cases} 
[\hat{\theta}(t)]_i, & \text{if } [\theta_{\text{min}}]_i \leq [\hat{\theta}(t)]_i \leq [\theta_{\text{max}}]_i \\
[\theta_{\text{min}}]_i, & \text{if } [\hat{\theta}(t)]_i \leq [\theta_{\text{min}}]_i \\
[\theta_{\text{max}}]_i, & \text{if } [\hat{\theta}(t)]_i > [\theta_{\text{max}}]_i
\end{cases} \quad (4.2\ d) \]

(4.2) possesses the following basic estimation properties as desired.

**Lemma 4.1:**

- \( P_1: \hat{\theta}(t) \) is uniformly bounded.
- \( P_2: \sum_{t=1}^{\infty} D^2(e(t), \tilde{\eta}) < \infty. \)
- \( P_3: \sum_{t=1}^{\infty} \| \hat{\theta}(t) - \hat{\theta}(t-1) \|^2 < \infty. \)

Moreover, \( P_2 \) implies

\[ \lim_{t \to \infty} D(e(t), \tilde{\eta}) = 0 \]

and \( P_3 \) implies

\[ \lim_{t \to \infty} \| \hat{\theta}(t+k) - \hat{\theta}(t) \|^2 = 0 \text{ for any finite } k. \]

**Proof. (Outline):** Define \( \bar{\theta}(t) = \hat{\theta}(t) - \theta \) and \( \bar{\theta}(t) = \hat{\theta} - \theta. \) Since \( \eta(t) < \tilde{\eta} \) after \( T_5, \forall t \geq T_5, \) from (4.1) and (4.2)

\[ D(e(t), \tilde{\eta}) = -\alpha(t)\phi^T(t)\bar{\theta}(t), \ 0 \leq \alpha(t) \leq 1 \]

Letting \( V(t) = \bar{\theta}^T(t)\bar{\theta}(t), \) we have

\[ V(t+1) - V(t) \leq \bar{\theta}^T(t+1)\bar{\theta}(t+1) - V(t) \leq -\frac{D^2(e(t), \tilde{\eta})}{\tilde{\eta} + \phi^T(t)\phi(t)} \]

Hence all results of this lemma are implied. \( \square \)

Since indirect adaptive control is adopted, an additional assumption should be made as follows.

**Assumption B2:** Let \( \hat{A}(t, d) \triangleq A(\hat{\theta}(t), d) = 1 + \hat{a}_1(t)d + \hat{a}_2(t)d^2 + \cdots + \hat{a}_n(t)d^n, \hat{B}(t, d) \triangleq B(\hat{\theta}(t), d) = \hat{b}_1(t)d + \hat{b}_2(t)d^2 + \cdots + \hat{b}_n(t)d^n, \hat{R}(t, d) \triangleq 1 + \hat{r}_1(t)d + \cdots + \hat{r}_n(t)d^n \) and \( \hat{S}(t, d) \triangleq \hat{s}_1(t)d + \cdots + \hat{s}_n(t)d^n. \)

For any \( t \geq 0, \) \( \hat{A}(t, d) \) and \( \hat{B}(t, d) \) are strictly coprime and \( |\hat{A}(t, 1)|, |\hat{B}(t, 1)| \geq \varepsilon > 0 \) such that the solution \((\hat{R}(t, d), \hat{S}(t, d))\) to the following diophantine equation

\[ \hat{A}(t, d)\hat{R}(t, d) + \hat{B}(t, d)\hat{S}(t, d) = A^*(d), \quad \hat{R}(t, d) = \hat{R}'(t, d)(d - 1) \]

is uniformly bounded and the mapping from \((\hat{A}, \hat{B})\) to \((\hat{R}, \hat{S})\) is continuous.

**Remark 4.1:**

(i) Assumption B2 is consistent with that required in standard indirect adaptive control. In fact, some papers, e.g. Kreisselmeier and Anderson (1986), Kreisselmeier (1986), Larminat (1984), Middleton and Goodwin (1988), have established a few correction procedures for the conventional estimation algorithms such that Assumption B (ii) is satisfied without affecting the basic estimation properties.
(ii) Any estimation algorithm which possesses properties $\mathcal{P}_1 \sim \mathcal{P}_3$ can be adopted in our proposed control algorithm without affecting the following results below.

For brevity and convenience, let $\hat{A} = \hat{A}(t, d)$, $\hat{B} = \hat{B}(t, d)$, $\hat{R} = \hat{R}(t, d)$ and $\hat{S} = \hat{S}(t, d)$, $\mathcal{P}(t, d)$ denotes a set of time-varying operators $P(t, d) = \sum_i P_i(t)(d_i)$ and $Q(t, d) = \sum_i Q_i(t)(d_i)$, define

$$(PQ)(t, d) = \sum_i \sum_j P_i(t)Q_j(t)d_i^d^j = (QP)(t, d)$$

$$(P \cdot Q)(t, d) = \sum_i \sum_j P_i(t)Q_j(t-i)d_i^d^j \neq (Q \cdot P)(t, d)$$

Note that if $P(t, d)$ and $Q(t, d)$ are time-invariant, then $(P \cdot Q)(t, d) = (Q \cdot P)(t, d) = (PQ)(t, d)$. By the equivalent certainty principle and imitating the control algorithm proposed in the non-adaptive case, control input $u_c(t)$ of the adaptive system is generated by the following equation

$$\hat{A}(t, d)y_a(t) = \hat{B}(t, d)(u_c - u)(t) \quad (4.3a)$$
$$\hat{R}(t, d)u_c(t) = \hat{S}(t, d)(r - y_a - y)(t) \quad (4.3b)$$

Thus, the adaptive system is defined by (2.3), (4.2), and (4.3). In the following, it is implicitly assumed that Assumption A, Assumption B1 and Assumption B2 are satisfied. At first, we have the following result.

**Lemma 4.2:** All signals in the adaptive system are uniformly bounded in spite of whether, or whenever, the saturation occurs.

**Proof:** From (2.3), $y(t)$ is uniformly bounded since $A(d)$ is a Hurwitz polynomial by assumption. From (4.3), we have the following equations to describe the relation between $u_c$, $u$, $y$ and $y_a$

$$A^*u_c(t) = [(\hat{A} \hat{R} - \hat{A} \hat{R}) + (\hat{B} \hat{S} - \hat{S} \hat{B})]u_c(t) - (\hat{A} \hat{S} - \hat{S} \hat{A})y_a(t)
+ (\hat{A} \hat{S})r(t) + (\hat{S} \hat{A} - \hat{A} \hat{S})y(t) - \hat{S}e(t) \quad (4.4a)$$

$$A^*y_a(t) = (\hat{R} \hat{B} - \hat{B} \hat{R})u_c(t) + [(\hat{A} \hat{R} - \hat{R} \hat{A}) + (\hat{B} \hat{S} - \hat{S} \hat{B})]y_a(t)
- (\hat{R} \hat{B})u(t) + (\hat{B} \hat{S})(r - y)(t) \quad (4.4b)$$

Putting (4.4a) and (4.4b) together, we have

$$\begin{bmatrix}
A^* & 0 \\
0 & A^*
\end{bmatrix}
\begin{bmatrix}
u_c(t) \\
y_a(t)
\end{bmatrix}
= 
\begin{bmatrix}
(\hat{A} \hat{R} - \hat{A} \hat{R}) + (\hat{B} \hat{S} - \hat{S} \hat{B}) & -\hat{A} \hat{S} + \hat{S} \hat{A} \\
\hat{R} \hat{B} - \hat{B} \hat{R} & (\hat{A} \hat{R} - \hat{R} \hat{A}) + (\hat{B} \hat{S} - \hat{S} \hat{B})
\end{bmatrix}
\begin{bmatrix}
u_c(t) \\
y_a(t)
\end{bmatrix}
+ 
\begin{bmatrix}
-\hat{S}e(t) + (\hat{S} \hat{A} - \hat{A} \hat{S})y(t) + (\hat{A} \hat{S})r(t) \\
-(\hat{R} \hat{B})u(t) + (\hat{B} \hat{S})(r - y)(t)
\end{bmatrix} \quad (4.5)$$

Since $y(t)$, $r(t)$ and $u(t)$ are uniformly bounded, by Assumption B and the estimation properties, (4.5) converges to the following system as $t \to \infty$
\[
\begin{bmatrix}
A^* & 0 \\
0 & A^*
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
u_c(t) \\
y_n(t)
\end{bmatrix}
\end{bmatrix} = \begin{bmatrix}
-\hat{S}e(t) + \hat{A}\hat{S}r(t) \\
-(\hat{R}\hat{B})u(t) + (\hat{B}\hat{S})(r - y)(t)
\end{bmatrix}
\]  
(4.6)

Also note that \(\lim_{t \to \infty} |e(t)| \leq \bar{\eta}\) and all signals are finite during any finite interval from the fact that the coefficients of \(\hat{R}\) and \(\hat{S}\) are uniformly bounded. Thus, it is easily deduced from (4.5) and (4.6) that all signals in the adaptive system are uniformly bounded.

For brevity, let \(\lambda(t)\) denote any asymptotically decaying signal. More precisely, \(\lambda(t) \to 0\) as \(t \to \infty\). Since all signals are uniformly bounded in the adaptive loop, by Lemma 4.1, it is concluded via (4.5) that

\[
A^*(d)u_c(t) = -\hat{S}(t, d)e(t) + (\hat{A}\hat{S})(t, d)r(t) + \lambda(t)
\]  
(4.7)

Whether the saturation occurs depends on the magnitude of \(u_c(t)\) and hence (4.7) is important for our deduction in the following.

**Lemma 4.3:** Suppose \(r(t) = \bar{r}\) and \(A(1)/B(1) \bar{r} \in (-\bar{\mu}, \bar{\mu})\). Then there exist \(\varepsilon_{w1}, \mu_1 > 0\) such that \(\forall \varepsilon_{w1}, \mu \leq \mu_1\)

\[
\limsup_{t \to \infty} u_c(t) \leq \bar{\mu}, \liminf_{t \to \infty} u_c(t) \geq -\bar{\mu}
\]

**Proof:** Since \(r(t)\) is a constant time-function and \(A^{*-1}(d)\) is exponentially stable, (4.7) can be rewritten as

\[
u_c(t) = \frac{\hat{A}(t, 1)}{\hat{B}(t, 1)} \bar{r} + e(t, 0, \gamma'_1) + \lambda(t) + e_1(t)
\]

(4.8)

where \(|e_1(t)| \leq k_3 \bar{\eta}, 0 < k_3 < \infty,\) and \(\gamma'_1\) depends on the roots of \(A^*(d)\). The remaining proof is proceeded by contradiction. Thus, we have two cases.

Case 1. Whatever \(\varepsilon_{w}\) and \(\mu\) are, \(\limsup_{t \to \infty} u_c(t) > \bar{\mu}\).

Case 2. Whatever \(\varepsilon_{w}\) and \(\mu\) are, \(\liminf_{t \to \infty} u_c(t) < -\bar{\mu}\).

If the result of this lemma is not true, either Case 1 or Case 2 occurs. We will prove that Case 1 or Case 2 leads to a contradiction. Since this two cases are similar, only Case 1 is considered.

Case 1 implies that

\[
\limsup_{t \to \infty} \frac{\hat{A}(t, 1)}{\hat{B}(t, 1)} \bar{r} > \bar{\mu}
\]

Thus, there exists an infinite time sequence \(\bar{T} = \{t_i, i \in \mathbb{Z}^+: t_{i+1} > t_i, \forall i\}\) such that \(\forall t_i \in \bar{T}\)

\[
\frac{\hat{A}(t_i, 1)}{\hat{B}(t_i, 1)} \bar{r} > \bar{\mu}
\]

We rewrite (4.8) as

\[
u_c(t) = \frac{\hat{A}(t_i, 1)}{\hat{B}(t_i, 1)} \bar{r} + \left(\frac{\hat{A}(t, 1)}{\hat{B}(t, 1)} - \frac{\hat{A}(t_i, 1)}{\hat{B}(t_i, 1)}\right) \bar{r} + \lambda(t) + e_1(t) + e(t, 0, \gamma'_1)
\]

By Property 3 of Lemma 4.1, given any integer \(k \geq n\)

\[
u_c(t) = \frac{\hat{A}(t_i, 1)}{\hat{B}(t_i, 1)} \bar{r} + \lambda(t) + e_1(t) + \lambda(t) + e(t, 0, \gamma'_1), \forall t \in [t_i, t_i + k]
\]
So, there exist $\varepsilon_w, \mu > 0, t^* < \infty$ such that

$$u = \bar{u}, \forall t \in [t_i, t_i + k], \forall t_i \geq t^*$$

At the same time, $\forall t \in [t_i + n, t_i + k], \forall t_i \geq t^*$, $y(t)$ can be described as

$$y(t) = \frac{B(1)}{A(1)} \bar{u} + \frac{1}{A(d)} (\mu H(d)\bar{u}) + w(t) + \varepsilon(t, t_i, \gamma_2)$$

where $\gamma_2$ depends on the roots of $A(d)$. Thus, $\forall t \in [t_i + n, t_i + k], \forall t_i \geq t^*$

$$\delta(t) = y(i) - \bar{y}(i) = y(t) - [(1 - \tilde{A}(t, d))y(t) + \tilde{B}(t, d)u(t)]$$

$$= -\tilde{B}(t_i, 1)\bar{u} + \tilde{A}(t_i, 1) \frac{B(1)}{A(1)} \bar{u} + \tilde{A}(t_i, d) \left[ \frac{\mu}{A(d)} (H(d)\bar{u}) + w(t) \right]$$

$$+ \varepsilon(t, t_i, \gamma_2) + \lambda(t_i), \forall t \in [t_i + n, t_i + k], \forall t_i \geq t^*$$ (4.9)

By simple manipulations, it is obtained via (4.9) that

$$\frac{\tilde{A}(t_i, 1)}{\tilde{B}(t_i, 1)} = \frac{A(1)}{B(1)} + \frac{1}{\tilde{B}(t_i, 1)} \frac{A(1)}{B(1)} \frac{A(d)}{B(d)} \left[ \varepsilon(t) - \tilde{A}(t_i, d) \left[ \frac{\mu}{A(d)} (H(d)\bar{u}) + w(t) \right] \right]$$

$$+ \varepsilon(t, t_i, \gamma_2) + \lambda(t_i), \forall t \in [t_i + n, t_i + k], \forall t_i \geq t^*$$

Since $k$ can be arbitrarily large, there exist $\varepsilon_w > 0$ and $\mu > 0$ such that

$$\frac{\tilde{A}(t_i, 1)}{\tilde{B}(t_i, 1)} \bar{r} \in [-\mu, \bar{u}]$$

if $t_i$ are sufficiently large. It follows that the infinite time sequence $\bar{T}$ does not exist. This is a contradiction and the proof is completed.

**Remark 4.2:** From (4.9), it is noted that if the DC gain of $G_0(d)$, i.e., $B(1)/A(1)$, is large, then the adaptive system can eventually avoid saturation in the presence of large disturbance and unmodelled dynamics.

Now, we start to investigate the behaviour of the plant output $y(t)$ in the adaptive system.

**Theorem 4.1:** Suppose $r(t) = \bar{r}$ and $A(1)/B(1) \bar{r} \in (-\mu, \bar{u})$. Then there exist constants $c_R, c_\varphi \in \mathbb{R}^+$ such that $\forall \mu \leq \mu_1, \forall \varepsilon_w \leq \varepsilon_{w1}$

$$\lim_{t \to \infty} \sup |y(t) - \bar{r}| < c_R \varepsilon_w + c_\varphi \mu$$

where $\mu_1$ and $\varepsilon_{w1}$ are defined in Lemma 4.3.

**Proof:** By lemma 4.3, it is obtained that $\forall \mu \leq \mu_1, \forall \varepsilon_w \leq \varepsilon_{w1}, u(t)$ can be written as

$$u(t) = u_c(t) + \lambda(t)$$ (4.10)

Thus, (2.3) can be rewritten as

$$A(d)y(t) = B(d)u_c(t) + \mu H(d)u_c(t) + A(d)w(t) + \lambda(t)$$ (4.11)

Given $0 < \tilde{k}, \tilde{t} < \infty$, we can rewrite (4.8) as

$$u_c(t) = \frac{\tilde{A}(\tilde{t}, 1)}{\tilde{B}(\tilde{t}, 1)} \bar{r} + \tilde{\lambda}(\tilde{t}) + \lambda(t) + \varepsilon(t, 0, \gamma_1), \forall t \in [\tilde{t}, \tilde{t} + \tilde{k}]$$ (4.12)
By (4.10) and (4.11), we have
\[
y(t) = \frac{B(1)}{A(1)} \frac{\hat{A}(\hat{i}, 1)}{\hat{B}(\hat{i}, 1)} \hat{r} + \frac{1}{A(d)} \left( \mu H(d) u_c(t) \right) + w(t) + e'_i(t) + \lambda(\hat{r})
+ \lambda(t) + e(t, \hat{i}, \gamma_2) + e(t, 0, \gamma_1'), \forall t \in [\hat{i}, \hat{i} + \tilde{k}]
\] (4.13)
where \( |e'_i(t)| \leq k_3 \tilde{\eta}, 0 < k_3 < \infty \), and
\[
e(t) = -\hat{A}(\hat{i}, 1) \hat{r} + \hat{A}(\hat{i}, 1) \frac{B(1)}{A(1)} \frac{\hat{A}(\hat{i}, 1)}{\hat{B}(\hat{i}, 1)} \hat{r} - \hat{B}(\hat{i}, d)e_1(t)
+ \hat{A}(\hat{i}, d) \left[ \frac{1}{A(d)} \left( \mu H(d) u_c(t) \right) + e'_i(t) + w(t) \right] + e(t, \hat{i}, \gamma_2)
+ e(t, 0, \gamma_1') + \lambda(\hat{r}) + \lambda(t), \forall t \in [\hat{i}, \hat{i} + \tilde{k}]
\] (4.14)
We have following two cases for \( \hat{r} \).

Case i: \( \hat{r} \neq 0 \). It is deduced from (4.14) that there exist constants \( k_4, k_5 \in R^+ \) such that
\[
\left| \frac{B(1)}{A(1)} \frac{\hat{A}(\hat{i}, 1)}{\hat{B}(\hat{i}, 1)} - 1 \right| < k_4 \varepsilon_w + k_5 \mu
\]
if \( \tilde{k} \) and \( \hat{i} \) are sufficiently large. Since \( \hat{r} \) and \( \tilde{k} \) are given arbitrarily, it is concluded via (4.13) that there exists constants \( k_6, k_7 \in R^+ \) such that
\[
\limsup_{t \to \infty} y(t) - \hat{r} \leq k_6 \varepsilon_w + k_7 \mu
\]

Case ii: \( \hat{r} = 0 \). Directly from (4.13), we also have that there exist constants \( k_8, k_9 \in R^+ \) such that
\[
\limsup_{t \to \infty} |y(t)| \leq k_8 \varepsilon_w + k_9 \mu
\]

At this point, the proof of this theorem is completed by taking \( c_8 = \max(k_6, k_8) \) and \( c_9 = \max(k_7, k_9) \). \( \square \)

Let \( \Delta r(t) = r(t) - r(t - 1) \). In general, \( r(t) \) is generated by sampling a continuous-time reference signal. Thus, if the continuous-time reference signal is continuous, then
\[
|r(t) - r(t - 1)| \leq l \cdot \Delta T
\]
where \( 0 \leq l < \infty \) represents the largest rate of variation of the continuous-time reference signal, and \( \Delta T \) is the sampling period. Small \( l \) or \( \Delta T \) results in \( \Delta r(t) \) being small. By Lemma 4.3 and Theorem 4.1, we have the following corollary.

**Corollary 4.1:** Suppose \( \limsup_{t \to \infty} \Delta r(t) < \varepsilon_r \), \( \liminf_{t \to \infty} A(1)/B(1) r(t) > -u \), \( \limsup_{t \to \infty} A(1)/B(1) r(t) < \bar{u} \). Then there exist \( \varepsilon_{u_2}, \mu_2, \varepsilon_{r_1} > 0 \) such that
\[
\forall \mu \leq \mu_2, \forall \varepsilon_w \leq \varepsilon_{w_2}, \forall \varepsilon_r \leq \varepsilon_{r_1},
\limsup_{t \to \infty} |y(t) - r(t)| \leq c_{10} \varepsilon_w + c_{11} \mu + c_{12} \varepsilon_r
\]
where \( c_{10}, c_{11}, c_{12} \in R^+ \) are three constants.
Proof: Given $0 < \tilde{t}, \tilde{k} < \infty$, $u_c(t)$ has another form as
\begin{equation}
u_c(t) = \frac{\hat{A}(\tilde{t}, 1)}{\hat{B}(\tilde{t}, 1)} r(\tilde{t}) + \varepsilon(t, \tilde{t}, \gamma_1') + \lambda(t) + \lambda(\tilde{t}) + e_1(t) + e_2(t), \forall t \in [\tilde{t}, \tilde{t} + \tilde{k}] \tag{4.15}\end{equation}
where $|e_2(\tilde{t})| \leq k_{10} \bar{\varepsilon}_r$, $0 < k_{10} < \infty$. It is claimed that there exist $\varepsilon_w$, $\mu_2$, $\varepsilon_{r1} > 0$ such that $\forall \mu \leq \mu_2$, $\forall \varepsilon_w \leq \varepsilon_w$, $\forall \varepsilon_r \leq \varepsilon_{r1}$
\[
\limsup_{t \to \infty} u_c(t) \leq \bar{u}, \liminf_{t \to \infty} u_c(t) \geq -\mu
\]
Since $\tilde{t}$ is arbitrary and $\tilde{k}$ can be made large such that $\varepsilon(t, \tilde{t}, \gamma_1')$ is small as long as $\varepsilon_r$ is sufficiently small, this claim can be proved by following similar lines as in the proof of Lemma 4.3. Thus, (4.10) is satisfied. Since (4.15) is similar to (4.12), following the similar lines as in the proof of Theorem 4.1, we have that there exist three constants $k_{11}$, $k_{12}$, $k_{13} \in \mathbb{R}^+$ such that
\begin{equation}\|y(t) - r(\tilde{t})\| < k_{11} \varepsilon_w + k_{12} \mu + k_{13} \bar{\varepsilon}_r, \forall t \in [\tilde{t}, \tilde{t} + \tilde{k}] \tag{4.16}\end{equation}
if $\tilde{t}$ and $\tilde{k}$ are sufficiently large. By letting $t = \tilde{t}$, $c_{10} = k_{11}$, $c_{11} = k_{12}$ and $c_{12} = k_{13}$ in (4.16), the result of this corollary is established since $\tilde{t}$ is arbitrary.

Corollary 4.1 tells us that if $r(t)$ is nearly constant, it can be tracked well. But if $\tilde{t}$ is not small, i.e. the continuous-time reference signal varies fast, and $\Delta T$ cannot be made arbitrarily small due to hardware limitations, the result of Corollary 4.1 could not satisfy the requirement. In the non-adaptive case, the roots of $A^*(d)$ are closely related to the system response. Let the roots of $A^*(d)$ be $\{\sigma_1, \ldots, \sigma_{2n+1}\}$ and
\[
\sigma = \max \{1/|\sigma_i|, i = 1, 2, \ldots, 2n + 1\}\]
It is well known that the smaller $\sigma$, the faster the system response. So, $r(t)$ can be tracked well by decreasing $\sigma$. We will establish a similar result in the adaptive system below. Before going to this result, two remarks are given.

(i) Since $A(d)$ is known to be a Hurwitz polynomial, we can constrain $\hat{\theta}(t)$ to a convex region in which $\theta$ lies and $\hat{A}(t, d) \neq 0$ for $|d| \leq 1 + \delta$, $\delta > 0$. Thus, by the basic properties of parameter estimation and the result in Fuchs (1980), $\hat{A}(t, d)^{-1}$ is an exponentially stable operator since $\|\hat{\theta}(t)\|$ is uniformly bounded, $\|\hat{\theta}(t+1) - \hat{\theta}(t)\| \to 0$ as $t \to \infty$, and $\hat{A}(t, d)$ is a strict Hurwitz polynomial for all $t$.

(ii) In the discrete-time system, $|\Delta r(t)|$ can be made small by properly decreasing $\Delta T$ under hardware limitations. It implies that if $k^*$ is a small positive integer, then $|r(t + k^*) - r(t)|$ can be made small. We have the following result.

Theorem 4.2: Given any fixed integer $k^* \geq 2n + 1$. Suppose that $\hat{A}(t, d)$ is a strict Hurwitz polynomial for all $t$, $\liminf_{t \to \infty} \hat{A}(t, 1)/\hat{B}(t, 1) r(t) > -\mu$, $\limsup_{t \to \infty} \hat{A}(t, 1)/\hat{B}(t, 1) r(t) < \bar{u}$ and $k^* \varepsilon_r \leq \varepsilon'$, where $\varepsilon'$ is defined in Corollary 4.1. Then there exist $\varepsilon_{w3}$, $\mu_3$, $\varepsilon_{r1}$, $\sigma_1 > 0$ such that $\forall \mu \leq \mu_3$, $\forall \varepsilon_w \leq \varepsilon_{w3}$, $\forall \varepsilon_r \leq \varepsilon_{r1}$, $\forall \sigma \leq \sigma_1$.
\[ \limsup_{t \to \infty} |y(t) - r(t)| < c_{13} \varepsilon_w + c_{14} \mu + c_{15} \varepsilon_r + c_{16} \sigma \]

where \(c_{13}, c_{14}, c_{15}, c_{16} \in \mathbb{R}^+\) are four constants.

**Proof:** Given any \(0 < \tilde{t} < \infty\), it is obtained via (4.7) that
\[
u_c(t) = (1 - A^*(d))u_c(t) - \hat{S}(t, d)e(t) + (\hat{A} \hat{S})(\tilde{t}, 1)r(\tilde{t}) + (\hat{A} \hat{S})(t, d)(r(t) - r(\tilde{t})) + \lambda(t) + \lambda(\tilde{t}), \forall t \in [\tilde{t}, \tilde{t} + k]\]

Note that \((1 - A^*(d)) \to 0\) as \(\sigma \to 0\), \([(\hat{A} \hat{S})(t, d)(r(t) - r(\tilde{t}))] \leq k_{14} \varepsilon_r^2\), \(0 < k_{14} < \infty\) and \(\hat{A}(t, 1) \hat{S}(t, 1) = \hat{A}(t, 1)/\hat{B}(t, 1) \to \hat{A}(1, 1) \hat{S}(1, 1)\) as \(\sigma \to 0\). Thus, there exist \(\mu_3, \varepsilon_w, \varepsilon_r, \sigma_1 > 0\) such that \(\forall \mu \leq \mu_3, \forall \varepsilon_w \leq \varepsilon_w, \forall \varepsilon_r \leq \varepsilon_r, \forall \sigma \leq \sigma_1\)

\[ \limsup_{t \to \infty} \nu_c(t) \leq \tilde{u}, \liminf_{t \to \infty} \nu_c(t) \geq -\tilde{u} \]

and (4.10) is satisfied. From (4.3 a), (2.3) and the definition of \(e(t)\), it is obtained that
\[ \hat{A}(t, d)(y + y_a)(t) = e(t) + \hat{B}(t, d)\nu_c(t) \quad (4.17) \]

From (4.3 b) and (4.17), we have
\[ A^*(d)(y + y_a)(t) = \hat{R}(t, d)e(t) + (\hat{B} \hat{S})(t, d)r(t) + \lambda(t) \]

and \((y + y_a)(t)\) can be written as
\[
(y + y_a)(t) = (1 - A^*(d))(y + y_a)(t) + \hat{R}(t, d)e(t) + (\hat{B} \hat{S})(\tilde{t}, 1)r(\tilde{t}) + (\hat{B} \hat{S})(t, d)(r(t) - r(\tilde{t})) + \lambda(t) + \lambda(\tilde{t}), \forall t \in [\tilde{t}, \tilde{t} + k]\]

(4.18)

Since \(\hat{A}(t, d)^{-1}\) is an exponentially stable operator (see the previous results prior to this theorem) and \(u_c(t) = u(t) + \lambda(t), y_a(t) = \lambda(t)\). Also, \((\hat{B} \hat{S})(\tilde{t}, 1) = A^*(1)\). Thus, it is concluded via (4.18) that there exist four constants \(k_{14}, k_{15}, k_{16}, k_{17} \in \mathbb{R}^+\) such that
\[ |y(t) - r(\tilde{t})| < k_{14} \varepsilon_w + k_{15} \mu + k_{16} \varepsilon_r + k_{17} \sigma, \forall t \in [\tilde{t}, \tilde{t} + k]\]

(4.19)

if \(\tilde{t}\) sufficiently large. Since \(\tilde{t}\) is arbitrary, the result of this theorem is established by taking \(\tilde{t} = \tilde{t}, c_{13} = k_{14}, c_{14} = k_{15}, c_{15} = k_{16}, c_{16} = k_{17}\) in (4.19).

**Remark 4.3:** Although the conditions in Theorem 4.2 depend on \(\tilde{B}(t)\), they provide a proper way to choose \(r(t)\) if \(r(t)\) can be designed as one wishes.

5. Computer simulation

Consider the following plant
\[ A(d) = 1 + 1.7d + 0.72d^2 \]
\[ B(d) = 1.5d + 2.67d^2 \]
\[ H(d) = \frac{1.1d}{1 + 0.1d} \]
\[ \mu = 0.01 \]
\[ w(t) = 0.01 \sin(0.1t) \]
Design parameters in (4.2) are chosen as follows:

\[
\bar{\eta} = 0.03; \quad \bar{\gamma} = 1; \quad \theta_{\text{min}} = [1.2, 0.5, 0.5, -3.0]^T
\]

\[
\theta_{\text{max}} = [2.5, 1.0, 1.8, -2.0]^T; \quad \hat{\theta}(0) = [1.2, 0.5, 0.5, -3.0]^T
\]

Other parameters are described in the following

\[
A^*(d) = (1 + 0.1d)^2(1 - 0.1d)^3; \quad \bar{u} = 3.5; \quad u = 3.5
\]

For transparency, two cases for \(r(t)\) are simulated. Case (i): \(r(t) = -1, t \in [1, 50]; r(t) = 1.5, t \in [51, 100]; r(t) = 1, t \in [101, 200]\). Case (ii): \(r(t) = -1, t \in [1, 40] \cup [161, 200]; r(t) = 1, t \in [41, 80] \cup [121, 200]\). It is noted that in Case (i) the system is overloaded during \([51, 100]\) since \(|A(1)/B(1)| = 3.42/1.17 \approx 2.92\).

In Case (i), three subcases are simulated. Case (i1) (non-adaptive and without saturation—i.e. no control input amplitude constraint): its simulation results are shown in Fig. 2(a). Case (i2) (non-adaptive and with saturation): its simulation results are shown in Fig. 2(b). Case (i3) (adaptive and with saturation): its simulation results are shown in Fig. 2(c) and Fig. 2(d).

In Case (ii), two subcases are simulated. Case (ii1) (non-adaptive and without saturation): its simulation results are shown in Fig. 3(a). Case (ii2) (adaptive and with saturation): its simulation results are shown in Fig. 3(b) and Fig. 3(c).

By observing these simulation results, our philosophy is verified, i.e. \(u_0(t)\) maintains its original trajectory (under no control input amplitude constraint) approximately, but approximately and asymptotically in the adaptive case. Thus,

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**Figure 2.** (a) Response of case (i1); (b) response of case (i2); (c) parameter estimate; (d) response of case (i3).
if the system is not overloaded eventually, then $y(t)$ also eventually maintains its original trajectory well.

6. Conclusions

In this paper, according to our proposed philosophy (i.e. to let the control input follow the original trajectory as if the saturation never occurred), a simple additional feedback signal is introduced. In the non-adaptive system, as long as the size of unmodelled dynamics and the magnitude of the disturbance are sufficiently small, the performance which can be achieved under no control input amplitude constraint is also exponentially maintained if the original trajectory of the control input eventually goes within the constrained region. These results are shown in §3. In the adaptive system, similar results are also obtained, but the performance is maintained asymptotically. The detailed analysis highlights the function of the proposed additional feedback signal. Computer simulation results given in §5 also support this function.

REFERENCES


