Model reference deadbeat control

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The performance of deadbeat control systems is addressed. Analysis of conventional deadbeat control (CDC) reveals that, in addition to deadbeat tracking performance, other system properties, such as robustness and noise immunity, may need improvement. A model reference deadbeat control (MRDC) is proposed to remedy such deficiencies of CDC systems. Several examples are given to show that the MRDC scheme is better than the CDC scheme in attaining good transient response, noise immunity and robustness if a suitable reference model is selected.

1. Introduction

The deadbeat control problem for a sampled-data system is an important topic in digital control. This problem was initially given in state-space formulation. The state deadbeat control (Kalman 1960) as well as the output deadbeat control (Kučera 1972) theories have made great advances within the last three decades. A historical review of such deadbeat control theory was presented by O'Reilly (1981). On the other hand, based on the polynomial equation approach in the frequency domain, Kučera (1980) presented a simpler and more efficient method to determine a deadbeat (tracking) controller. Moreover, the class of deadbeat controllers was characterized by using a set of diophantine equations in Kučera and Šebek (1984). Chen et al. (1984) solved the deadbeat controller based on the YJB parametrization of all stabilizing controllers (Youla et al. 1976). By using such a YJB parametrization method, it was further established by Zhao and Kimura (1986) that not only the tracking error can be driven to zero within finite steps but also the robustness can be improved by reducing the deadbeat tracking performance. However, later Zhao and Kimura (1988) showed that both optimal robustness and minimal deadbeat tracking performance can be simultaneously attained by using a two-degrees-of-freedom (TDF) scheme for SISO systems. It was further shown by Zhao and Kimura (1989) that the same result still holds for MIMO systems. But the deadbeat controller in their work was not yet compactly characterized, and the properties of system response were not fully analysed.

In general, (conventional) deadbeat control systems tend to be wide bandwidth and thus the system response may not be desirable for tolerating high-frequency unstructured uncertainty and may not effectively filter out the noise component in plant output due to measurement noise. Since plant uncertainty and measurement

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noise, etc., are inevitable in practice, we must shape the system frequency response well to meet the various design specifications. An easy way to design a controller meeting specifications is to choose a suitable reference model and then force the closed-loop system to behave like the reference model. The objective of this paper is to present the model reference deadbeat control (MRDC) approach for which the tracking error between the plant output and the output of a desired reference model driven by the original command signal will vanish within finite steps. We also characterize the class of such model reference deadbeat controllers and investigate the related system performance for different kinds of plants under the MRDC scheme.

This paper is organized as following. The model reference deadbeat control problem is formulated and some preliminaries are given in §2. The solvability of the MRDC problem is discussed and the characterization of model reference deadbeat controllers based on the YJB parametrization is presented in §3. In particular, the conventional deadbeat controller is proved to be a special case under the MRDC scheme. In §4, the deficiencies of system performance under the conventional deadbeat control (CDC) scheme and the relation between the system performance and the reference model under the MRDC scheme will be investigated. The advantages of the MRDC system over CDC and the effect of the reference model upon the system performance are illustrated by several examples in §5. Conclusions are given in §6.

2. Problem statement and preliminaries

Notation and definitions

Let \( \lambda \) denote the unit delay operator \((z^{-1})\). In this paper, all rational functions and polynomials are expressed with indeterminate \( \lambda \), which is to be omitted for notational simplicity. A polynomial is called stable if all its zeros are outside the unit circle of the \( \lambda \)-plane and a rational function is stable if its denominator is a stable polynomial. Let \( L \) be a polynomial and \( R \) be a rational function. A rational function \( R \) is causal if \( R(0) < \infty \). The degree of polynomial \( L \) is denoted as \( \hat{c} L \). The specific polynomials \( L_\cdot \) and \( L_+ \) for \( L \) are defined as \( \hat{c} L = \lambda^d L_\cdot + L_+ \) where \( L_\cdot (0) = 1 \), \( L_\cdot (0) \neq 0 \) and \( L_- \) contains all the zeros of \( L \) which are inside or on the unit circle on the \( \lambda \)-plane. We write \( L | R \) if \( L \) divides the numerator of \( R \).

Let \( r \) be any discrete-time function and its \( Z \)-transform is denoted as \( R(\lambda) \). Let \( D \) be a polynomial. We say \( r \) contains mode \( D \) if polynomial \( D \) divides the denominator of \( R \), and any zero of \( D \) is called a mode of \( r \). Similarly, we say a rational function \( R \) contains mode \( D \) if \( D \) divides its denominator.

2.1. Problem statement

The model reference control structure under a one-degree-of-freedom (ODF) scheme is shown in Fig. 1. The plant \( P \) to be discussed is a single-input-single-output linear time-invariant discrete time system governed by a DARMA model

\[
A y_t = \lambda^d B u_t, \tag{1}
\]

where \( d \) is a positive integer, \( A \) and \( B \) are coprime polynomials with \( A(0) = 1 \) and \( B(0) \neq 0 \). The command signal \( r \), is generated by

\[
D r_t = N \delta_t, \tag{2}
\]
where $\delta_t$ is the unit impulse function, $N$ and $D$ are coprime polynomials. Without loss of generality, we assume that all the zeros of $D$ are on the unit circle. The reference signal is obtained by passing the command signal $r_t$ through a stable and causal reference model $M = F/G$. Some essential assumptions about the reference model are made in the following.

(M1) $F$ and $G$ are coprime, $G$ is stable,
(M2) $F(1)/G(1) = 1$,
(M3) $F$ and $D$ are coprime.

The first assumption (M1) is standard. Assumption (M2) is to make the reference model with unit DC gain. Assumption (M3) is to ensure that the reference model $M$ will not cancel any mode of $r_t$. Let $T$ denote the transfer function from $r_t$ to $y_t$, i.e. $T = PC(1 + PC)^{-1}$. The tracking error $e_t$ is defined as

$$e_t = \frac{F}{G} r_t - Tr_t,$$  \hspace{1cm} (3)

and let $E(\lambda)$ be the $\mathcal{Z}$-transform of $e_t$.

The model reference deadbeat control (MRDC) problem is defined as to find a controller $C$ which stabilizes $P$ and $e_t = 0, \forall t \geq t_0$ for some finite $t_0$, irrespective of the polynomial $N$ in the command signal model (2). The conventional deadbeat control (CDC) problem is defined as above except that $F = G = 1$ is set in (3). A controller which solves the MRDC (CDC) problem is called a model reference (conventional) deadbeat controller. Once the tracking error vanishes within finite steps, $E$ will be a polynomial of finite degree. In this case, $E$ is called the error polynomial. A deadbeat controller is called minimal if the degree of the corresponding error polynomial $E$ is the minimal among the class of deadbeat controllers.

2.2. Preliminaries

The sensitivity and complementary sensitivity functions are defined as

$$S = (1 + PC)^{-1} \hspace{1cm} (4)$$

$$T = PC(1 + PC)^{-1} = 1 - S \hspace{1cm} (5)$$

The following lemma concerns the basic interpolation conditions related to a stabilizing controller.

Figure 1. Model reference deadbeat control scheme.
Lemma 1
Consider the control structure in Fig. 1. The controller $C$ stabilizes $P$ if and only if the following conditions hold

(a) $S$ (and thus $T$) is causal and stable

(b) $A_{-} | S$

(c) $\lambda^{d} B | T$

Proof
Denote the matrix of rational functions $H$ as

$$H = \begin{pmatrix}
(1 + PC)^{-1} & -P(1 + PC)^{-1} \\
C(1 + PC)^{-1} & (1 + PC)^{-1}
\end{pmatrix} = \begin{pmatrix}
S & -PS \\
T/P & S
\end{pmatrix}$$

(9)

By definition in Vidyasagar (1985), the controller $C$ stabilizes $P$ if and only if $C$ is causal and all the entries in $H$ are stable.

(i) We now prove the 'only if' part. Since both $P$ and $C$ are causal, it follows that $S$ and all the entries in $H$ are causal by using (9) and (4). Since $C$ stabilizes $P$, we have $S$, $PS$, $T/P$ are stable. $T$ is then stable following (5). Furthermore, the fact that $PS$ and $T/P$ are stable implies $A_{-} | S$ and $B_{-} | T$, respectively. It remains to show $\lambda^{d} | T$. Now note that $T/P = CS$. Since $C$ and $S$ are causal, so is $T/P$. But $T/P$ is causal implies that $\lambda^{d} | T$.

(ii) Now we prove the 'if' part. Suppose $S$ and $T$ satisfy (6) and the interpolation conditions (7)–(8). From (9), it is easy to see that all the entries in $H$ are causal and stable. To complete the proof it remains to show $C$ is causal. Since $T$ is causal and $\lambda^{d} | T$, it follows $\lambda$ cannot divide the numerator of $S$ by using $S = I - T$. Also, $T$ can be represented by

$$T = PCS = CS\lambda^{d}B/A$$

The above equation and the facts that $\lambda^{d} | T$ and $\lambda$ cannot divide the numerator of $S$ imply that $C$ must be causal.

Under the assumption that $A$ and $B$ are coprime, any controller $C$ which stabilizes plant $P$ (Vidyasagar 1985) is given by

$$C = \frac{X + AV}{Y - \lambda^{d}BV}, \quad Y - \lambda^{d}BV \neq 0$$

(10)

where $V$ is a stable and causal rational functions, $X$ and $Y$ are polynomials solved from the following diophantine equation

$$AY + \lambda^{d}BX = U$$

(11)

with $U$ being any stable polynomial.

3. The model reference deadbeat control
In this section, the structures of both model reference and conventional deadbeat controllers will be clearly characterized. Also, a deadbeat controller which possesses a robust tracking capability will be derived. Note that a different choice
of $U$ in (11) will result in a different class of deadbeat controllers. A reasonable choice of $U$ is $U = G$ since under this situation the poles of the reference model will coincide with the poles of the closed-loop system. In the following, we present the class of deadbeat controllers corresponding to $U = G$. Before deriving the model reference deadbeat controllers we first present the necessary and sufficient conditions about the solvability of the MRDC problem.

Let $D_0$ be the g.c.d. of $D$ and $A_-$ with $D_0(0) = 1$ and denote $\bar{F}$ as $\bar{F} = F - G$.

Lemma 2

The MRDC problem is solvable if and only if

(i) $D$ and $B_-$ are coprime

(ii) $D_0 | \bar{F}$.

Proof

In this proof, only the 'only if' part is provided. The 'if' part will be presented later in the proof of Theorem 1 by directly constructing the deadbeat controllers. The tracking error $e_r$ in (3) can be expressed as

$$e_r = \frac{\bar{F}}{G} r_t + S r_t$$  (12)

Suppose $C$ is a deadbeat controller which solves the MRDC problem. Then, following Lemma 1, there exist some stable rational functions $S'$ and $T'$ so that $T = \lambda^d B_- T'$ and $S = A_- S'$. Equation (12) is then rewritten as

$$e_r = \frac{\bar{F}}{G} r_t - \lambda^d T' B_- r_t$$  (13)

$$= \frac{\bar{F}}{G} r_t + S' A_- r_t$$  (14)

Note, the fact that the MRDC problem is solvable implies $e_r = 0$ as $t \to \infty$. Now, by contradiction, suppose $D$ and $B_-$ are not coprime, the signal $T' B_- r_t$ cannot contain all the modes of $r_t$. On the other hand, that $F$ and $D$ are coprime implies the signal $(F/G)r_t$ will keep all the modes of $r_t$. Recall also that all the zeros of $D$ are on the unit circle. Then, following from (13), we conclude $e_r \neq 0$ as $t \to \infty$ which leads to a contradiction. Thus, it is necessary that $D$ and $B_-$ are coprime to solve the MRDC problem.

On the other hand, suppose that $D_0$ does not divide $\bar{F}$, then, from (14), it is easy to see that $e_r \neq 0$ as $t \to \infty$ by using similar reasoning as in the previous paragraph. So it is also necessary that $D_0 | \bar{F}$ for the MRDC problem to be solvable.

Theorem 1

Assume (i) $D$ and $B_-$ are coprime (ii) $D_0 | \bar{F}$. The MRDC problem is solvable and the deadbeat controller $C$ (corresponding to $U = G$) can be represented as

$$C = \frac{X + VA}{Y - \lambda^d VB}, \quad V = \frac{Q}{B_+ A_-}$$  (15)
where polynomial pairs \((X, Y)\) and \((E_1, Q)\) are arbitrary solutions of the following two diophantine equations.

\[
AY + \lambda^dBX = G \tag{16}
\]
\[
GDE_1 + \lambda^dB \ A_\_Q = (F - AY) \tag{17}
\]

The error polynomial \(E\) is

\[
E = NE_1 \tag{18}
\]

\[\square\]

Proof

By using (10) and setting \(U = G\) in (11), the tracking error \(e_i\) in (3) can be rewritten as \(e_i = NE_1\delta_i\) with

\[
E_1 = \frac{\dot{F} + A(Y - \lambda^dBV)}{GD} \tag{19}
\]

Starting from the above equation, we deduce some causal \(C\) so that \(E_1\) is actually a polynomial. Now (19) can be rewritten as

\[
V = \frac{(\dot{F} + AY) - GDE_1}{(A, B_\_)(\lambda^dA_\_B_\_)}
\]

The above equation indicates that the MRDC problem is solvable if there exist polynomials \(Q\) and \(E_1\) such that

\[
Q = \frac{(\dot{F} + AY) - GDE_1}{\lambda^dA_\_B_\_} \tag{20}
\]

Then \(V\) is stable, causal and

\[
V = \frac{Q}{A_\_B_\_} \tag{21}
\]

From Vidyasagar (1985), it is well known that \(V\) is stable and causal, which implies that \(C\) stabilizes \(P\). Equation (20) can be rewritten as diophantine equation (17). From the assumptions that \((D, B_\_)\) is a coprime pair and \(D_0\|F\_\_\_\_\) it is easy to see that any common factor of \(GD\) and \(\lambda^dA_\_B_\_\) is also a factor of \(\dot{F} + AY\). Thus, the diophantine equation (17) has at least one solution and the MRDC problem is solvable. Moreover, any model reference deadbeat controller is characterized by (15)–(17).

\[\square\]

From (18), the degree of the error polynomial \(E\) can be minimized by choosing \(E_1\) from (17) with the degree constraint \(\delta E_1 < \delta B_\_ + \delta A_\_ + d\). Thus, any minimal deadbeat controller is constructed via (15)–(17) except now (17) is imposed by an additional constraint \(\delta E_1 < \delta B_\_ + \delta A_\_ + d\).

In the following, the class of all conventional deadbeat controllers is presented. It is considered as a special case under the MRDC scheme by setting \(F = G = 1\). This result can be regarded as the generalization of that in Zhao and Kimura (1986).
Corollary 1

Assume \( D \) and \( B \) are coprime. Then any conventional deadbeat controller is represented by (15) where polynomial pairs \((X, Y)\) and \((E_1, Q)\) are solutions of the following two diophantine equations,

\[
AY + \lambda^d BX = 1 \tag{22}
\]

\[
DE_1 + \lambda^d B_+ A_- Q = AY \tag{23}
\]

The error polynomial \( E \) is represented by (18).

Recall that a controller is said to possess a robust tracking property with respect to command signal \( r \), if \( r \) can be asymptotically tracked when the plant is perturbed (within the range that closed-loop stability still remains). It is well known from linear control theory that, for a control system of the ODF scheme, the plant output \( y \), tracks \( r \) if and only if loop gain \( PC \) contains mode \( D \). Now, considering the robust tracking property of the model reference deadbeat control and extending the frequency domain interpolation technique in Wang and Chen (1986), the model reference deadbeat controller containing mode \( D \) will be derived.

Theorem 2

Assume (i) \( D \) and \( B_- \) are coprime (ii) \( D \mid F \). Let \( \hat{F} = F/D \). Then any model reference deadbeat controller \( C \) which contains mode \( D \) is represented as

\[
C = \frac{X + AY}{DY - \lambda^d B} \quad V = \frac{DQ}{A_- B_+} \tag{24}
\]

where polynomial pairs \((X, Y)\) and \((E_1, Q)\) are solutions of the following two diophantine equations

\[
\lambda^d BX + ADY = G \tag{25}
\]

\[
GE_1 + \lambda^d A_- B_+ Q = AY + \hat{F} \tag{26}
\]

The error polynomial \( E \) is \( E = E_1 N \).

Proof

It is easy to see that \( C \) stabilizes \( P \) and \( C \) contains mode \( D \), imply

\[
DA_- \mid S \tag{27}
\]

Since the closed-loop characteristic polynomial is chosen as \( G \), the sensitivity function \( S \) can be represented as

\[
S = \frac{DA_- S'}{G} \tag{28}
\]

for some polynomial \( S' \). By substituting (28) into (12), we have

\[
e_i = \frac{\hat{F} + DA_- S'}{GD} \quad N\delta,
\]

To require \( e_i \) to vanish within finite steps, there must exist polynomial \( E_1 \) such that

\[
GDE_1 - DA_- S' = \hat{F}, \quad \text{or} \quad GE_1 - A_- S' = \hat{F} \tag{29}
\]
and the error polynomial $E$ becomes $E = E_1 N$. But to maintain closed-loop stability, the interpolation condition (8) imposes the other constraint on $S'$. To satisfy condition (8), there must exist a polynomial $T'$ such that

$$\lambda^d B_\perp T' + DA\perp S' = G$$  \hspace{1cm} (30)$$

and the complementary sensitivity function $T'$ is

$$T = \frac{\lambda^d B_\perp \; T'}{G}$$  \hspace{1cm} (31)$$

Thus, any model reference deadbeat controller which contains mode $D$ is characterized by (29) and (30). Note that these two diophantine equations are coupled. To show that the set of equations (29) and (30) are equivalent to (25) and (26), suppose the triple $(E_1, S', T')$ is a solution of (29) and (30). Then, $S'$ and $T'$ can be represented as, for certain polynomial $Q$

$$T' = B_\perp X + DA_\perp Q$$  \hspace{1cm} (32)$$

$$S' = A_\perp Y - \lambda^d B_\perp Q$$  \hspace{1cm} (33)$$

where $(X, Y)$ is a solution of (25). And, by substituting (33) into (29), it is seen that polynomials $E_1$ and $Q$ also satisfy (26). Thus, any solution $(E_1, S', T')$ of (29) and (30) correspond to a solution of (25) and (26).

In addition, any solution of (25) and (26) can be used to construct a solution of (29) and (30) via (32) and (33). Hence, the pair of equations (29) and (30) are equivalent to (25) and (26). Any model reference deadbeat controller which contains mode $D$ is then characterized by (25) and (26). As $(B_\perp, AD)$ and $(G, \lambda^d A_\perp B_\perp)$ are both coprime pairs, (25) and (26) are solvable and thus there at least exists one model reference deadbeat controller which contains mode $D$. Furthermore, such a controller can be computed, by using (28) and (31)–(33), as

$$C = \frac{T}{SP} = \frac{X + VA}{DY - V\lambda^d B}$$

where $V = DQ\perp A_\perp B_\perp$.

\hfill $\Box$

**Corollary 2**

Under the same assumptions in Theorem 2 and supposing the command signal is a step function, i.e., $D = 1 - \lambda$, then the assumption $D \mid \tilde{F}$ is redundant. \hfill $\Box$

\textbf{Proof}

Recall that we have assumed $G(1) = F(1)$ to ensure unit DC gain for the reference model. This assumption implies $(1 - \lambda) \mid F - G$ and thus the assumption $D \mid \tilde{F}$ is automatically satisfied. \hfill $\Box$

4. The system performance

In this section, we focus on discussing the performance of deadbeat control systems. The performance will be evaluated according to properties of the sensitivity and complementary sensitivity functions. Let $S_{MRDC}$ and $T_{MRDC}$ denote the sensitivity and complementary sensitivity functions for the MRDC system,
respectively. From (3) and the fact \( E = E_1N \), we have
\[
S_{\text{MRDC}} = DE_1 - F/G, \quad T_{\text{MRDC}} = F/G - DE_1.
\]
(34)
Then the sensitivity and complementary sensitivity functions for the CDC system, denoted as \( S_{\text{CDC}} \) and \( T_{\text{CDC}} \), respectively, are
\[
S_{\text{CDC}} = DE_1, \quad T_{\text{CDC}} = 1 - DE_1
\]
(35)

4.1. Performance of CDC systems

The sensitivity function of the CDC system is a polynomial \( S_{\text{CDC}} = DE_1 \) (\( T_{\text{CDC}} \) is also a polynomial). When we regard \( S_{\text{CDC}} \) as a function of \( z(z = z^{-1}) \), all the poles of \( S \) are located at \( z = 0 \). On the other hand, to attain deadbeat tracking performance in minimal time, the loop bandwidth of the CDC system becomes wide. Under these situations, we may encounter the following general drawbacks.

1. It may happen that the frequency response of \( T_{\text{CDC}} \) remains a high gain beyond the desired working frequency range. In this case, the response due to the measurement noise in plant output \( y \) may not be smoothly filtered. Therefore, the robustness with respect to the multiplicative unstructured uncertainty (in the high frequency range in most cases) will be poor.

2. Now consider the effect of unstable zeros upon the transient response subject to the step command. It is known from Middleton and Goodwin (1990) that the loop bandwidth must be wide enough to keep the overshoot within an acceptable range for an unstable plant. But when plant has unstable zeros which are close to the unit circle, the undershoot will become worse if the loop bandwidth is not kept within a reasonably narrow range. Thus the transient response will be bad if the CDC scheme is applied to a non-minimum-phase plant with unstable zeros close to the unit circle.

The drawbacks described above mainly arise from the facts that the closed-loop poles are all located at the origin of the \( z \)-plane and the rise-time is too short. To remedy the deficiencies we have to place the closed-loop poles properly and shape the system frequency response well. A simple and effective method is to use the model reference approach for improving system performances.

4.2. Performance of MRDC systems

By comparing the sensitivity function for a conventional deadbeat controller in (35) with (34), we can see that both polynomials \( F \) and \( G \) explicitly appear in (34) but there is no explicit free parameter in (35). In (34), we may use parameters \( F \) and \( G \) to adjust the closed-loop system response. By properly shaping \( T_{\text{MRDC}} \) (or \( S_{\text{MRDC}} \)), MRDC can meet several design specifications, such as loop bandwidth, measurement noise immunity, disturbance rejection and robustness issues, with a suitable compromise (Safonov et al. 1981).

Specifically, if the plant is minimum-phase, unstable and subject to a step command, then the reference model of a wide bandwidth is preferable since the loop bandwidth must be wide enough to keep the overshoot within an acceptable range (Middleton and Goodwin 1990). As an extreme case, the CDC scheme can have better transient response (small overshoot) since it tends to have wider bandwidth. Furthermore, recall that the transient response for a CDC scheme is
characterized by \( e_i = S_{\text{CDC}} r_i \) and \( S_{\text{CDC}} \) is a polynomial. Roughly speaking, a smaller overshoot implies smaller coefficients in the polynomial \( S_{\text{CDC}} \). Thus, the values of the \( H_2 \) norm of \( S \) and \( T \) will be decreased and the noise immunity of the CDC scheme will be better than the MRDC scheme. Although the CDC scheme tends to have better transient response and noise immunity for unstable, minimum-phase plants, it may still result in poor robustness. In this case a trade-off between transient response (and noise immunity) and robustness should be decided according to practical situations.

On the other hand, if the plant is stable but has unstable zeros which are close to the unit circle, it is known from Middleton and Goodwin (1990) that the loop bandwidth must be made narrow enough in order to keep undershoot within a satisfactory range. We may improve the undershoot by choosing an adequately narrow bandwidth for the reference model \( M \). Note that the transient response depends on the polynomial \( E_1 \) in (18). Then, better undershoot implies smaller coefficients of \( E_1 \) in view of (34), and the deviation of \( T_{\text{MRDC}} \) from the desired reference model \( F/G \) will be reduced. Thus, by using a low-pass reference model, the MRDC scheme can give both better transient response (undershoot) and satisfactory noise immunity. Moreover, since the reference model is chosen as a low-pass rational function in this case, the robustness with respect to high frequency uncertainty is also enhanced.

Note that the choice of \( F \) and \( G \) is not a simple job. First, polynomial \( E_1 \) in (34) depends on \( F \) and \( G \). Secondly, the design specifications cannot be expressed in simple and analytical form. In addition, some design specifications, such as robustness and tracking precision, cannot be optimized simultaneously and a trade-off has to be done between them. In general, we need to use an optimization programming method or graphic method for adjusting \( F \) and \( G \) via trial and error procedure.

Note that the degree of the error polynomial in (18) for a minimal model reference deadbeat controller is not affected by the choices of \( F \) and \( G \). Hence, we may simultaneously obtain good system performance and keep the minimal deadbeat tracking capability. In Zhao and Kimura (1986), the system performance is improved by sacrificing the minimal-time deadbeat tracking capability.

5. Simulations

In this section, several examples are given to illustrate the properties of the MRDC systems. Example 1 shows that the MRDC scheme can attain better transient response, noise immunity and robustness than the CDC scheme. Example 2 and Example 3 are used to illustrate the effect of the reference model upon system performance for different kinds of plants.

**Example 1**

The plant to be considered is non-minimum phase with

\[
P = \frac{\lambda(1-5\lambda)}{(1-0.2\lambda)(1-0.5\lambda)}
\]

The command signal is the unit step function, i.e. \( N = 1 \) and \( D = 1 - \dot{z} \). The reference model \( M \) is a low-pass stable transfer function with \( F = 0.2 \) and \( G = 1 - 0.8\dot{z} \). Then the minimal conventional deadbeat controller, denoted as
$C_{\text{CDC}}$, can be computed according to Corollary 1 as

$$C_{\text{CDC}} = \frac{-0.25 + 0.175\lambda - 0.025\lambda^2}{1 + 0.25\lambda - 1.25\lambda^2}$$

and the minimal model reference deadbeat controller, denoted by $C_{\text{MRDC}}$, can be calculated by using Theorem 1 as

$$C_{\text{MRDC}} = \frac{-0.0976 + 0.116\lambda - 0.0431\lambda^2 + 0.0048\lambda^3}{1 - 0.7024\lambda - 0.5357\lambda^2 + 0.2381\lambda^3}$$

(i) Comparison of time responses

Assume that the measured plant output is contaminated by a white gaussian measurement noise with zero mean and variance 0.01. The command signal $r$, and plant output signals corresponding to the MRDC and CDC schemes are plotted in Fig. 2 where $r$, is the solid line, and the outputs $y$, with respect to the MRDC and CDC schemes are in dashed and dotted lines, respectively. Figure 2 indicates that the response of the MRDC scheme has the smaller undershoot and its noise component due to measurement noise is less than that of the CDC scheme. Yet, the disadvantage of the MRDC scheme is slower time response and lower system bandwidth.

(ii) Comparison of frequency responses

The loop gain $PC$ for both schemes are plotted in Fig. 3. The gain margin ($GM$) and phase margin ($PM$) for the CDC scheme are

$$GM = 2.5462 \text{ dB}, \quad PM = 57.7856 \text{ degree}$$

and for the MRDC scheme are

$$GM = 5.2554 \text{ dB}, \quad PM = 79.0216 \text{ degree}.$$ 

Thus both gain margin and phase margin (and so the robustness) are improved under the MRDC scheme.

![Figure 2](image-url)

Figure 2. Comparison of output responses in Example 1. Solid line: $r$, dashed line: $y$, under CDC scheme, dotted line: $y$, under MRDC scheme.
Figure 3. Frequency response of loop gain $PC$. Solid line: frequency response under CDC scheme, dashed line: frequency response under MRDC scheme.

In the following, the effect of the reference model upon the system performance for different plants is investigated. For simplicity, we consider only two factors: the noise rejection capability with respect to both the output disturbance and the measurement noise. Both output disturbance and measurement noise are assumed to be white processes which are mutually independent. Under this situation, to investigate the influence of output disturbance and measurement noise at the plant output it suffices to calculate the $H_2$ norm of sensitivity function $S$ and of complementary sensitivity function $T$. The reference model $M$ is assumed in the following first-order rational function

$$M = \frac{1 + g}{1 + g^2}$$

where $g$ is a free parameter with $g \in (-1, 1)$. Denote $S_{MRDC}(g)$ and $T_{MRDC}(g)$ as the sensitivity and complementary sensitivity functions for $C_{MRDC}$ for some fixed $g$. Denote $T_r(g)$ and $S_r(g)$ as

$$T_r(g) = \frac{\|T_{MRDC}(g)\|_2^2}{\|T_{CDC}\|_2^2}, \quad S_r(g) = \frac{\|S_{MRDC}(g)\|_2^2}{\|S_{CDC}\|_2^2}$$

(37)
where $|•|_2$ denotes the $H_2$ norm. Then $T_r(g)(S_r(g))$ indicates the ratio of the variance of the noise component in plant output due to the measurement noise (output disturbance) under the MRDC scheme, to that under the CDC scheme. Thus, $T_r(g_0) < 1(S_r(g_0) < 1)$ implies that the MRDC scheme attains better noise immunity with respect to measurement noise (output disturbance) for some $g = g_0$.

**Example 2**

Consider the same situations as in Example 1 except that the reference model is given in (36). The graphs of $S_r(g)$ and $T_r(g)$ are displayed in Fig. 4. Note that $S_r(0) = T_r(0) = 1$ as the CDC scheme is a special case of MRDC with $g = 0$. From Fig. 4, we see that the MRDC scheme improves the noise rejection capability with respect to both output disturbance and measurement noise by using a low-pass reference model (corresponding to the model in (36) with $g < 0$). Moreover, under the MRDC scheme, the noise rejection capability increases as $g$ decreases. But we should note that the choice of $g$ has a limit and an appropriate compromise must be made between system bandwidth and noise rejection capability. Yet the compromise can be determined according to the graphs of $T_r(g)$ and $S_r(g)$ in Fig. 4.

**Example 3**

Consider the same conditions as in Example 2 with a minimum-phase, unstable plant.

$$P = \frac{0.1107s^3}{1 - 1.2214s}$$

This plant model is obtained by sampling a time-delayed continuous system (Zhao and Kimura 1986). The functions $T_r(g)$ and $S_r(g)$ are plotted in Fig. 5 and their graphs are almost coincident. Recall that the conventional deadbeat controller coincides with the model reference deadbeat controller under the choice $g = 0$. Note that now $T_r(g)$, $S_r(g) > 1$ for $g \in [-0.9, 0]$. Although $T_r(g)$ and $S_r(g)$ are slightly less than 1 when $g$ is greater than 0.6, a reference model with such $g$ is not acceptable since the bandwidth may be too wide such that the robustness decreases.

![Figure 4. Graphs of $T_r(g)$ and $S_r(g)$ in example 2. Solid line: $T_r(g)$, dashed line: $S_r(g)$.]
It is seen that the noise rejection capability of the conventional deadbeat control system is superior to the MRDC scheme in this case.

6. Conclusions and suggestions

In this paper, the performance issue of a deadbeat control system under an ODF scheme has been addressed. The conventional deadbeat control has been further analysed and it has been noted that its closed-loop property may not meet additional specifications, such as noise immunity and robustness, etc. The model reference deadbeat control was then proposed to remedy such deficiencies. The class of all model reference deadbeat controllers based on the YJB parametrization was characterized by two independent diophantine equations. The system performance of the MRDC scheme for various kinds of plants was also investigated.

Basically, the MRDC scheme is good for adjusting the system performance. However, under the ODF structure, it has the following drawbacks.

(i) In order to attain deadbeat tracking with respect to reference signal, the reference model cannot be arbitrarily chosen. It must be constrained by \( D_0 | \bar{F} \). This will limit the use of the MRDC approach.

(ii) When the plant is minimum-phase and unstable, a reference model with wide bandwidth will give better transient response but perhaps poor robustness. A worse case may happen if the plant is both non-minimum-phase and unstable. Under this circumstance, a satisfactory transient response can hardly be obtained by adjusting the reference model.

It is known from Vidyasagar (1985), that if the control system is under a TDF scheme, the tracking and robustness problems can be individually determined by using separate control parameters and the input–output response is not affected by the poles of the plant. Thus, the MRDC under a TDF scheme may effectively attain good transient response and provide the flexibility to adjust system performance for both unstable and non-minimum-phase plants. On the other hand, we may increase the complexity (the order) of the reference model to provide flexibility so that satisfactory performance can be obtained for general plants. The selection procedure for an adequate reference model to meet various specifications for general plants is a good topic for further research.
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