Minimal Sensitivity Perfect Model Matching Control

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Abstract—This note is concerned with the design of minimal sensitivity perfect model matching control. The key observation is that in addition to the achievement of perfect model matching control, the design parameters still leave a great deal of flexibility which could be fully utilized to reduce the effect of external disturbances. Based on the analysis, a simple and direct design procedure is proposed to design the perfect model matching control system with minimal sensitivity. An example is included for illustration.

I. INTRODUCTION

In the past decades, considerable attention has been paid to the design of a perfect model matching control system so as to adjust the system behavior to exactly follow the prespecified system model which meets the required specifications [1]-[5]. Due to the fact that the desired system model is usually characterized in terms of a transfer function, it is natural to treat the design of perfect model matching control in the frequency domain directly rather than in the time domain. Thus, recent work concerning perfect model matching control is mostly done in the frequency domain; see [6]-[13].

As is known, it is highly desired to design feedback systems with good immunity to disturbances. Recently, numerous papers have discussed this important consideration [16]-[25]; special emphasis is put on robust stability. Although the design of a perfect model matching control system is well developed, rare research is conducted on reducing the effect of external disturbance in addition to the achievement of perfect model matching control. In view of the solution to perfect model matching control [11], [12], the key observation is that the design parameters still leave a great deal of flexibility. This motivates us to investigate how to fully utilize this freedom such that the designed perfect model matching control system is insensitive to disturbances. In this note, an explicit design procedure for achieving perfect model matching control with minimal sensitivity will be proposed.

In Section II, we precisely state the problem of minimal sensitivity perfect model matching control. Section III gives a complete solution and an explicit design procedure. For illustration, an example is included in Section IV. Finally, we end this note with a conclusion.

II. PROBLEM STATEMENT

Consider the perfect model matching control system with external disturbance \( u(t) \), shown in Fig. 1. The plant being controlled is described by

\[
\begin{align*}
\frac{y(s)}{u(s)} &= \frac{B(s)}{A(s)} \\
\end{align*}
\]

(2.1)

where \( u(s) \) is the control input and \( y(s) \) is the output. It is assumed that \( A(s) \) and \( B(s) \) are coprime polynomials with

\[
\deg(A(s)) = \deg(B(s)).
\]

(2.2)

The controller, composed of two inputs and one output, is expressed as

\[
P(s) = -Q(s) + R(s)r(s)
\]

(2.3)

where \( P(s), Q(s), \) and \( R(s) \) are design parameters to be determined such that the closed loop is stable and the transfer function from the command input \( r(t) \) to the output \( y(t) \) is equal to the prespecified stable model

\[
M(s) = \frac{G(s)}{F(s)}.
\]

(2.4)

This perfect model matching controller (2.3) could be thought of as a combination of feedback having the transfer function \( Q(s)/P(s) \) and a feedforward with the transfer function \( R(s)/P(s) \).

It follows from (2.1) and (2.3) that the closed-loop transfer function relating \( y(s) \) to \( r(s) \) is given by

\[
H(s) = \frac{y(s)}{r(s)} = \frac{B(s)R(s)}{A(s)P(s) + B(s)Q(s)}.
\]

(2.5)

To achieve perfect model matching control, the closed-loop transfer function \( H(s) \) should equal \( M(s) \), that is,

\[
\frac{B(s)R(s)}{A(s)P(s) + B(s)Q(s)} = \frac{G(s)}{F(s)}.
\]

(2.6)

Now, factor \( B(s) \) into

\[
B(s) = B^*(s)B^T(s)
\]

(2.7)

where \( B^*(s) \) is the stable part and \( B^T(s) \) is the unstable part. It has been shown by Kucera [12] that the necessary condition for achieving perfect model matching control is that \( B^*(s) \) should be a factor of \( G(s) \). Let \( D(s) \) be the greatest common divisor of \( B^*(s) \) and \( G(s) \), and we write

\[
B(s) = D(s)B_0(s)
\]

(2.8)

and

\[
G(s) = D(s)G_0(s).
\]

(2.9)

Note that \( D(s) \) must contain the factor \( B^T(s) \) in the perfect model matching control, i.e., for some polynomial \( D'(s) \)

\[
D(s) = D'(s)B^T(s).
\]

(2.9a)

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Using (2.8) and (2.9), (2.6) can then be written as

\[ \frac{D(s)B(s)R(s)}{A(s)P(s) + B(s)Q(s)} = \frac{D(s)G_0(s)}{F(s)} \]

which reduces to

\[ \frac{B(s)R(s)}{A(s)P(s) + B(s)Q(s)} = \frac{G_0(s)}{F(s)} \]

Taking into account the fact that the closed-loop system must be stable and \( B(s) \) must be a factor of \( A(s)P(s) + B(s)Q(s) \) for perfect matching, it follows that the set of all solutions \( P(s), Q(s), \) and \( R(s) \) must obey the following relations [9], [11]:

\[ A(s)P(s) + B(s)Q(s) = F(s)B(s)T(s) \]  
(2.10)

\[ R(s) = G_0(s)T(s) \]  
(2.11)

where \( T(s) \) is any stable polynomial in \( s \). It should be noted that the Diophantine equation (2.10) is always solvable since \( A(s) \) and \( B(s) \) are coprime, and it is easily solved by the Algorithm in [7]. More importantly, it is seen that there leaves a great deal of flexibility in selecting design parameters \( P(s), Q(s), \) and \( R(s) \).

In the above solution of perfect model matching control, the effect of disturbance \( u(s) \) on the output \( y(s) \) is not taken into account. In view of Fig. 1, it is found that the relation between \( y(s) \) and \( v(s) \) is given by

\[ A(s) \]

\[ A(s)P(s) + B(s)Q(s) \]

which can be interpreted as a sensitivity function relating output to disturbance. To have perfect model matching control with minimal sensitivity to disturbance \( u(s) \), it is required to find the design parameters \( P(s), Q(s) \) which not only satisfy (2.10) for achieving perfect model matching control, but also minimize the following meaningful measure of sensitivity:

\[ J = \frac{A(s)P(s)}{A(s)P(s) + B(s)Q(s)} = \left| \frac{A(s)P(s)}{F(s)B(s)T(s)} \right| \]  
(2.13)

where \( || \cdot ||_\infty \) is defined as

\[ ||A(jw)||_\infty = \sup_{w} |A(jw)|. \]

(2.14)

More precisely, the problem at hand is to determine \( P(s) \) and \( Q(s) \) such that (2.13) is minimized under the equality constraint (2.10). In the next section, an attempt is made to solve this problem.

III. MAIN RESULTS

This section is devoted to solve the aforementioned problem. As stated previously, the performance index (2.13) is subjected to the constraint (2.10). To get its solution, some manipulation should be made. It is well known that for coprime polynomials \( A(s) \) and \( B(s) \), there exist polynomials \( \hat{X}(s) \) and \( \hat{Y}(s) \) such that the following Bezout identity holds

\[ A(s)\hat{X}(s) + B(s)\hat{Y}(s) = 1. \]

Thus, the solution to (2.10) can be explicitly expressed as

\[ P(s) = \frac{F(s)B(s)T(s)}{A(s)P(s) + B(s)Q(s)} \]  
(3.2a)

\[ Q(s) = \frac{F(s)B(s)T(s)}{A(s)P(s) + B(s)Q(s)} \]  
(3.2b)

where \( K(s) \) is any polynomial in \( s \). And (2.13) becomes

\[ J = \frac{1}{||W(s)||_\infty} \]

by letting

\[ W(s) \triangleq \frac{A(s)P(s)}{F(s)B(s)T(s)}. \]

(3.3)

Equation (3.4) could be represented by

\[ W(s) = A(s) \frac{F(s)B(s)T(s)}{F(s)B(s)T(s)} \frac{A(s)P(s) + B(s)K(s)}{F(s)B(s)T(s)} \]

(3.5a)

\[ \hat{X}(s) + D'(s)B'(s)K(s) \]

(3.5b)

on employing (3.2a), (2.8), and (2.9). So far, the problem being treated is reduced to finding the polynomial \( K(s) \) and the stable polynomial \( T(s) \) such that (3.3) is minimized. Note that \( \hat{X}(s) \) is available from the Bezout identity (3.1).

In perfect model matching control, it can be verified that the unstable part of \( D(s) \) is \( B'(s) \) since \( D(s) \) is the greatest common divisor of \( B(s) \) and \( G(s) \), and \( G(s) \) contains the factor \( B'(s) \). This implies that \( D'(s) \) is a stable polynomial [see 2.9b)). In view of (3.5b), since \( F(s) \) and \( T(s) \) are both stable polynomials, it is obvious that \( B'(s) \) cannot be cancelled by \( F(s) \) or \( T(s) \). Noting that \( A(b_i) \hat{X}(b_i) = 1 \) (see (3.1)), it follows from (3.5b) that \( W(s) \) is subjected to the following interpolation constraints:

\[ W(b_i) = A(b_i) \hat{X}(b_i) = 1 \quad i = 1, 2 \cdots n \]

(3.6)

where \( b_1, b_2, \cdots, b_n \) are the distinct zeros of \( B'(s) \). It should be noted that if \( b_i \) is a zero of \( B'(s) \) with multiplicity \( q_i \), then additional differential interpolation constraints \( W^{(k)}(b_i) = 0 \) where \( k = 1, 2 \cdots q_i - 1 \) must also be imposed. Furthermore, by inspecting (3.5a), it is seen that the function \( W(s) \) itself is also subjected to the next interpolation constraints:

\[ W(a_j) = 0 \quad j = 1, 2 \cdots m \]

(3.7)

where \( a_1, a_2, \cdots, a_m \) are distinct zeros of \( A'(s) \) because \( A'(s) \), the unstable part of \( A(s) \), cannot be cancelled by the stable characteristic polynomial \( F(s)B(s)T(t) \). Similarly, if \( a_i \) is a zero of \( A'(s) \) with multiplicity \( q'_i \), then additional differential interpolation constraints \( W^{(k')}(a_i) = 0 \) where \( k' = 1, 2 \cdots q'_i - 1 \) must also be imposed. Summing up the above analysis, the problem being treated is finally translated into minimizing (3.3) subject to the interpolation constraints (3.6) and (3.7). For this constrained interpolatory \( H^\infty \)-norm optimization problem, it is fortunate that there exists a closed-form solution, as given in the following lemma. Based on this lemma, a theorem concerning the optimal solution to the problem treated is subsequently developed.

Lemma [22]-[25]: The optimal \( W(s) \), which minimizes \( ||W(s)||_\infty \) subject to \( k \) interpolation constraints, is of the all-pass form

\[ W(s) = \rho \frac{h(-s)}{h(s)} \]

(3.8)

where \( h(s) \) is a Hurwitz polynomial of degree at most \( k - 1 \). The constant \( \rho \) and the coefficients of \( h(s) \) are real, and are uniquely determined by the interpolation constraints. Moreover, the minimized \( ||W(s)||_\infty \) is given by

\[ \min ||W(s)||_\infty = ||W(s)||_\infty = |\rho|. \]

(3.9)
Proof: The proof can be found in Sarason [24] as well as Zames and Francis [21].

Remarks:
1) While \( |W(jw)| \) achieves a constant \( \rho \) for all frequency \( w \), the worst effect on output due to external disturbance is minimized.
2) Our design problem is reduced to finding control parameters \( K(s) \) and \( T(s) \) to make \( W(s) \) achieve the all-pass form (3.8) under the constraints (3.6) and (3.7).

Theorem: Let \( P(s) \) and \( T(s) \) be related by (3.2a), \( W(s) \) be given by (3.4), and \( \tilde{X}(s) \) be available from (3.1). Then the stable rational function \( K(s)/T(s) \) which minimizes (3.3) subjected to interpolation constraints (3.6) and (3.7) is of the form

\[
\frac{K(s)}{T(s)} = \frac{Z(s)F(s)}{A^*(-s)h(s)A^*(s)D^*(s)} \tag{3.10}
\]

where \( h(s) \) is a Hurwitz polynomial of degree \( (n-1) \) whose coefficients and the constant \( \rho \) are determined by the following \( n \) simultaneous equations:

\[
A^*(-s)h(\bar{b}) - \rho A^*(s)h(\bar{b}) = 0 \quad \text{for} \quad i = 1, 2 \cdots n \tag{3.11}
\]

and the polynomial \( Z(s) \) is obtained by

\[
Z(s) = \rho \frac{h(s)-A^*(s)h(s)\tilde{X}(s)}{B^*(s)} \tag{3.12}
\]

Proof: In view of the above lemma, since the minimum \( \bar{W}(s) \) is of the all-pass form and must satisfy the interpolation constraint (3.7), it follows that the minimum \( W(s) \) should take the form

\[
W(s) = \rho \frac{A^*(-s)h(s) - \rho A^*(s)h(s)\tilde{X}(s)}{A^*(-s)h(s)} \tag{3.13}
\]

where \( \rho \) and the Hurwitz polynomial \( h(s) \) of degree \( (n-1) \) are to be determined to satisfy that the additional interpolation constraints (3.6). In fact, the interpolation constraints (3.6) could be expressed as

\[
1 - W(h(s)) = 0 \quad i = 1, 2 \cdots n \tag{3.14}
\]

From (3.13), we have

\[
1 - \bar{W}(s) = \frac{A^*(-s)h(s) - \rho A^*(s)h(s)\tilde{X}(s)}{A^*(-s)h(s)} \tag{3.15}
\]

Thus, (3.14) means that

\[
A^*(-s)h(s) - \rho A^*(s)h(s) = Y(s)B^*(s) \tag{3.16}
\]

for some polynomial \( Y(s) \) or, equivalently,

\[
A^*(-b)h(s) - \rho A^*(s)h(s) = 0 \quad i = 1, 2 \cdots n \tag{3.17}
\]

Now, it directly follows from (3.5b) and (3.13) that

\[
A(s) \left( \tilde{X}(s) + D^*(s)B^*(s)K(s)F(s)T(s) \right) = \rho \frac{A^*(-s)h(s)}{A^*(-s)h(s)} \tag{3.18}
\]

which can be rearranged as

\[
D^*(s)B^*(s)K(s)F(s)T(s) = \frac{\rho h(s)-A^*(s)h(s)\tilde{X}(s)}{A^*(-s)h(s)A^*(s)} \tag{3.19}
\]

Note that \( A(s) = A^*(-s)A^*(s) \) has been used to arrive at (3.19). Next, from (3.17), we have

\[
A^*(-b)h(s)A(s)\tilde{X}(s) - \rho A^*(s)h(s) = 0 \quad i = 1, 2 \cdots n \tag{3.20}
\]

by using the fact \( A(b)\tilde{X}(s) = 1 \) [see (3.1)]. Equation (3.20) implies that there exists a polynomial \( Z(s) \) such that

\[
A^*(-s)h(s)A(s)\tilde{X}(s) - \rho A^*(s)h(s) = B^*(s)Z(s) \tag{3.21}
\]

or, equivalently,

\[
A^*(-s)\rho h(s)-A^*(s)A^*(-s)h(s)\tilde{X}(s) = -B^*(s)Z_1(s) \tag{3.22}
\]

Since \( A^*(-s) \) and \( B^*(s) \) are coprime, from (3.22), it is clear that there should exist a polynomial \( Z(s) \) such that

\[
\rho h(s)-A^*(s)A^*(-s)h(s)X(s) = B^*(s)Z(s) \tag{3.23}
\]

Combining (3.23) and (3.19) yields

\[
\frac{D^*(s)B^*(s)K(s)}{F(s)T(s)} = \frac{B^*(s)Z(s)}{A^*(-s)h(s)A^*(s)} \tag{3.24}
\]

or

\[
\frac{K(s)}{T(s)} = \frac{Z(s)F(s)}{A^*(-s)h(s)A^*(s)D^*(s)} \tag{3.25}
\]

Remarks:
1) It is worthwhile to note that all the unknown parameters involved in (3.10) could be easily found. By setting \( h(s) \) in the following parametric form

\[
h(s) = s^{i-1} + k_1 s^{i-2} + \cdots + k_i s + k_0 \tag{3.26}
\]

where \( k_i, i = 0, \cdots, n-2 \) are unknown coefficients and then solving the \( n \) simultaneous equation (3.11), it is easy to determine \( \rho \) and \( k_i, i = 0, \cdots, n-2 \) uniquely. Once \( \rho \) and \( h(s) \) are determined, \( Z(s) \) can be obtained by factoring (3.12), and thus the optimal solution \( K(s)/T(s) \) results.

2) Since the stable rational function \( K(s)/T(s) \) takes the form (3.10), the polynomials \( T(s) \) and \( K(s) \) can also be expressed as

\[
T(s) = A^*(-s)h(s)A^*(s)D^*(s)T_1(s) \tag{3.27}
\]

\[
K(s) = Z(s)F(s)T_1(s) \tag{3.28}
\]

where \( T_1(s) \) is any stable polynomial.

3) This theorem could be extended to the case where a subset of the zeros of \( A^*(-s) \) and/or \( B^*(s) \) are of multiplicities by taking into account some additional differential interpolating constraints:

\[
W_{ki}(s) = 0, \quad k_i = 1, 2 \cdots q_i - 1 \tag{3.29a}
\]

\[
W_{i+1}(s) = 0, \quad k_i = 1, 2 \cdots q_i - 1 \tag{3.29b}
\]

provided that \( h_i \) is a zero of \( B^*(s) \) with multiplicity \( q_i \), and \( a_i \) is a zero of \( A^*(-s) \) with multiplicity \( q_i' \),

So far, the remaining work is to find the controller parameters \( P(s), Q(s), \) and \( R(s) \) from the obtained optimal \( K(s)/T(s) \) which minimizes (3.3) subjected to interpolation constraints (3.6) and (3.7). However, directly substituting \( T(s) \) and \( K(s) \) into (3.2a), (3.2b), and (2.11) to obtain \( P(s), Q(s), \) and \( R(s) \) may result in a design procedure involving too much computational work. Hence, in the following, we will propose a simplified and direct solution.

It follows from (3.13) and (3.4) that

\[
P(s) = \frac{\rho h(s)F(s)B_0(s)}{T(s)A^*(-s)h(s)A^*(s)} \tag{3.30}
\]

Thus, substituting (3.27) into (3.30), one finds the design parameter

\[
P(s) = \rho h(-s)F(s)B_0(s) \tag{3.31}
\]

Moreover, from (2.10) and (3.4), one obtains

\[
1 - W(s) = \frac{D(s)Q(s)}{F(s)T(s)} \tag{3.32a}
\]

Coupling (3.31), (3.15), and (3.16), it yields

\[
\frac{D(s)Q(s)}{F(s)T(s)} = \frac{Y(s)B^*(s)}{A^*(-s)h(s)} \tag{3.32b}
\]
or

\[ Q(s) = \frac{Y(s)F(s)T(s)}{A'(s)h_3(s)D'(s)} = \frac{Y(s)F(s)A'(s)T(s)}{s} \]  

(3.33)

on employing (3.27). And, applying (3.27) to (2.11) gives

\[ R(s) = \frac{G_0(s)A'(s)h_3(s)A'(s)D'(s)T(s)}{s} \]  

(3.34)

Consequently, a simple and direct design algorithm for achieving perfect model matching control with minimal sensitivity can be outlined below.

**Design Algorithm:**

**Step 1:** Choose a stable polynomial \( T(s) \).

**Step 2:** Solve (3.11) to obtain \( \rho \) and \( h_3(s) \) by setting \( h_3(s) \) in the parametric form (3.26).

**Step 3:** Form (3.16) to get \( Y(s) \).

**Step 4:** Substitute the results obtained in Steps 2 and 3 into (3.30), (3.33), and (3.34) to get \( P(s) \), \( Q(s) \), and \( R(s) \); then the controller takes the form (2.3).

With regard to the properness of the designed controller, it is required that

\[ \deg(P(s)) \geq \deg(Q(s)) \]  

(3.35)

and

\[ \deg(P(s)) \geq \deg(R(s)) \]  

(3.36)

Using (2.8), (2.9a), and (2.9b), one can show that (3.35) and (3.36) are, respectively, equivalent to

\[ \deg(h_3(s)) \geq \deg(Y(s)) \geq \deg(A'(s)) - \deg(B'(s)) \]  

(3.37)

and

\[ \deg(F(s)) \geq \deg(G(s)) \geq \deg(A(s)) - \deg(B(s)) \]  

(3.38)

Since the degree of \( A(s) \), \( B(s) \), \( G(s) \), and \( F(s) \) are known exactly, it is observed that if the proper controller is required, then (3.38) could be checked first. Provided (3.38) holds, the design algorithm may result in a proper controller if (3.37) is satisfied.

**IV. ILLUSTRATIVE EXAMPLE**

Consider the perfect model matching control system with disturbance, shown in Fig. 1. Let the plant to be controlled have the transfer function

\[ B(s) = 2(s - 1) \]  

\[ A(s) = s^2 + 2s + 10 \]  

(4.1)

and the desired system model characterized by

\[ M(s) = \frac{G_0(s)}{F'(s)} = \frac{s + 2(s - 2)}{s + 10(s + 12)} \]  

(4.2)

Thus, we have

\[ A'(s) = s + 2, \quad A'(s) = s - 10, \quad B'(s) = -(s - 1)(s + 1), \]\n
\[ D'(s) = G_0(s) - B_0(s) = 1 \]

\[ h_3(s) = s + k_0 \]  

(4.3)

**Step 1:** For convenience, we choose \( T(s) = 1 \).

**Step 2:** By (2.11) and (2.2), \( b_1 = 1, b_2 = 2 \), set \( h_3(s) \) in the parametric form [see (3.26)]

\[ h_3(s) = s + k_0 \]

Then it follows from (3.11) that

\[ A'(-b)h_3(b) = \rho A'(-b)h_3(b) = 0, \quad l = 1, 2; \quad \text{that is} \]

\[ (-11)(s + k_0) - \rho(-9)(s - 1 + k_0) = 0 \]  

(4.4)

\[ (-12)(s + k_0) - \rho(-8)(s + 2 + k_0) = 0 \]  

(4.5)

Hence, solving (4.4) and (4.5), it yields \( \rho = -1.833 \) and \( k_0 = 0.2 \).

**V. CONCLUSION**

The design of a perfect model matching control system with minimal sensitivity is treated in this note. By applying the interpolation theory, the designed controller’s parameters have been determined so that the measure of sensitivity function is minimized. Finally, a simple and direct design algorithm is presented and an illustrative example is given. A natural but nontrivial generation to the multivariable case where the solution to the optimal sensitivity minimization problem is not unique will be considered in a future paper.

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Control of Slowly-Varying Linear Systems

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Abstract—State feedback control of slowly-varying linear continuous-time and discrete-time systems with bounded coefficient matrices is studied in terms of the frozen-time approach. This note centers on pointwise stabilizable systems: that is, systems for which there exists a state feedback gain matrix placing the frozen-time closed-loop eigenvalues to the left of a line \( \Re s = -\gamma < 0 \) in the complex plane (or within a disk of radius \( \rho < 1 \) in the discrete-time case). It is shown that if the entries of a pointwise stabilizing feedback gain matrix are continuously differentiable functions of the entries of the system coefficient matrices, then the closed-loop system is uniformly asymptotically stable if the rate of time variation of the system coefficient matrices is sufficiently small. It is also shown that for pointwise stabilizable systems with a sufficiently slow rate of time variation in the system coefficients, a stabilizing feedback gain matrix can be computed from the positive definite solution of a frozen-time algebraic Riccati equation.

1. INTRODUCTION

Linear time-varying systems are sometimes studied using the frozen-time method in which the time variable in the system coefficients is viewed as a parameter. An example of the power of this approach is Rosenbrock's result [1] that a linear continuous-time system is asymptotically stable if the frozen-time eigenvalues of the system matrix are to the left of a line \( \Re s = -\gamma < 0 \) in the complex plane and if the rate of time variation of the system matrix is sufficiently small. Desoer [2] proved that uniform asymptotic stability can be deduced under the same conditions on the system matrix and gave an explicit bound on the rate of time variation (for results in the nonlinear case, see [3, pp. 218-223]). A corresponding result for linear time-varying discrete-time systems was also proved by Desoer [4]. Recent results on the linear time-varying continuous-time case are given in [5] and [6].

Although the frozen-time method appears to be often utilized in practice in the control of linear time-varying systems, not much is currently known regarding analytical conditions on the given system which guarantee asymptotic stability of the closed-loop system. However, we should note that in [7] sufficient conditions (with the correction given in [8]) are given for the existence of a stabilizing state feedback gain matrix computed from the solution to a frozen-time algebraic Riccati equation. In this note we also consider the application of the frozen-time approach to the construction of a stabilizing state feedback gain matrix. We consider pointwise stabilizable systems for which there exists a state feedback gain matrix placing the frozen-time closed-loop eigenvalues to the left of a line \( \Re s = -\gamma < 0 \) in the complex plane, or within a disk of radius \( \rho < 1 \) in the discrete-time case. Such a feedback is said to be pointwise stabilizing.

In the next section we begin with the continuous-time case. We first consider the question as to when a pointwise stabilizing feedback gain matrix results in a uniformly asymptotically stable closed-loop system, assuming that the rate of variation of the system coefficient matrices is sufficiently small. If the system coefficient matrices are bounded with bounded derivatives, an answer (a sufficient condition) is that the entries of the feedback gain matrix be continuously differentiable functions of the entries of the system coefficient matrices. It follows from Delchamp's lemma [10] that such a feedback can be constructed from the positive definite solution of a frozen-time algebraic Riccati equation. A discrete-time version of the results is presented in Section III, and in Section IV some concluding remarks are given.

II. CONTINUOUS-TIME CASE

With \( R \) equal to the field of real numbers, for any positive integer \( q \) let \( R^q \) denote the space of \( q \)-element column vectors with entries in \( R \). The norm \( \|x\| \) of an element \( x \in R^q \) is defined by

\[
\|x\| = (x^T x)^{1/2}
\]

where \( x^T \) is the transpose of \( x \). Given positive integers \( p, q, a \), a \( p \times q \) matrix \( M \) over \( R \) will be viewed as an element of \( R^{pq} \). The Frobenius norm \( \|M\|_F \) is defined by

\[
\|M\|_F = \left( \sum_{i=1}^p \sum_{j=1}^q |m_{ij}|^2 \right)^{1/2}
\]

where \( m_{ij} \) is the \( i, j \) entry of \( M \). For any \( x \in R^q \), it is easy to verify that \( \|Mx\|_F \leq \|M\|_F \|x\|_F \).

Given positive integers \( m \) and \( n \), consider the \( m \)-input \( n \)-dimensional linear time-varying continuous-time system given by the state equation

\[
x(t) = A(t)x(t) + B(t)u(t)
\]

where \( u(t) \) is the \( m \)-vector input or control applied at time \( t \) and \( A(t) \) is the \( n \times n \) state matrix at time \( t \). The \( n \times n \) coefficient matrix \( A(t) \) and the \( n \times m \) coefficient matrix \( B(t) \) in (3) are assumed to be bounded differentiable functions of \( t \) for \( t \geq 0 \). Thus,

\[A(t) : R_+ \rightarrow O \subset R^{n \times n}
\]

\[B(t) : R_+ \rightarrow R^{m \times n}
\]

where \( R_+ = \{ t \in R : t \geq 0 \} \) and \( O, \Gamma \) are compact subsets. Finally, it is assumed that the derivatives \( \dot{A}(t) \) and \( \dot{B}(t) \) are bounded for \( t \geq 0 \), so that

\[
\sup_{t \geq 0} \|\dot{A}(t)\|_F = \delta_\dot{A} < \infty \quad \text{and} \quad \|\dot{B}(t)\|_F = \delta_\dot{B} < \infty.
\]

The rate of time variation of the coefficient matrices \( A(t) \) and \( B(t) \) is measured by the magnitudes \( \delta_\dot{A} \) and \( \delta_\dot{B} \) defined by (4).

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In this note we consider state feedback control of the system (3) with the control \( u(t) \) given by \( u(t) = -K(t)x(t) \), where \( K(t) \) is an \( m \times n \) time-varying gain matrix. With this control, the resulting closed-loop system is

\[
x(t) = (A(t) - K(t)B(t))x(t)\quad (5).
\]

The particular problem of interest is constructing a feedback gain matrix \( K(t) \) (assuming one exists) which results in uniform asymptotic stability of the closed-loop system (5). We shall approach this problem using the frozen-time method in which the time variable \( t \) in \( A(t) \) and \( B(t) \) is viewed as a parameter \( p(t) \) with \( p(t) \) ranging over all positive numbers. This results in an infinite collection of linear time-invariant systems

\[
x(t) = A(p)x(t) + B(p)u(t), \quad p \geq 0. \quad (6)
\]