Optimal robustness via constrained controller in SISO discrete-time feedback systems

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A new sufficient condition for robust stabilization of SISO discrete-time feedback systems with both non-linear (or linear) time-varying uncertainties and constrained actuators is derived. In this paper, neither the plant nor the controller has to be restricted to the stable or minimum-phase case. The necessary and sufficient condition for solvability of a stabilizer satisfying such a robust constraint is also introduced. The optimal $\mathcal{H}_\infty$-norm theory is employed to achieve the maximum robustness to tolerate the uncertainties via constrained control. Finally, a design algorithm is proposed to specify adequate parameters in the compensator to satisfy the criterion of optimal robustness and an example with computer simulation is given to illustrate the results.

Notation

\[ R[z] \] denotes the ring of proper stable rational functions with real coefficients, i.e. the elements of $R[z]$ are analytic in $|z| \geq 1$

\[ \|A(z)\|_\infty = \sup_{|z|=1} |A(z)| = \sup_{w \in [0,2\pi]} |A \exp(jw)| \]

\[ \bar{m}(z) = z^{n}m(z^{-1}) \], where $m(z)$ denotes a polynomial with degree $n$

\[ \Delta P(\cdot,\cdot,\cdot) \] denotes a non-linear time-varying operator, where

\[ \Delta P(\cdot,\cdot,\cdot)u(k) = \Delta P(u(k), k) \]

\[ N(\cdot) \] denotes the non-linear constrained actuator operator, where

\[ N(\cdot)u(k) = N(u(k)) = \text{sat}(u(k)) \]

\[ A(z) = A_0(z)A_1(z) \] denote the inner–outer factorization of a stable rational function $A(z)$, where $A_0(z), A_1(z) \in R[z]$, all zeros of $A_0(z)$ are in $|z| \leq 1$ and $|A_1 \exp(jw)| = 1$.

1. Introduction

Recently, robust stabilization, i.e., the problem of determining whether the closed-loop system remains stable after being subjected to perturbation, has been studied in detail. Doyle and Stein (1981) and Chen and Desoer (1982) present necessary and sufficient conditions that a controller must satisfy in order to stabilize all plants with a class of linear uncertainties. A considerable number of extensions and applications are also proposed. Since real plants are inherently non-linear and/or time-varying but only linear models are considered for the convenience of control design, some literature (Sandberg 1964, Zames 1966) is devoted to the robustness of a stable plant.

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under non-linear time-varying uncertainties. Those results are also extended to the system with unstable plants by Chen and Chiang (1986).

However, in many practical control cases, the controllers are merely software algorithms. Actuators must be employed in the control loop as shown in Fig. 1. It is evident that there exist certain finite energy limitations in real actuators. Since the robust control law is usually a high gain one in some frequency ranges (Doyle and Stein 1981), it is highly desirable to design a control system with the constraint of actuator saturation in order for the control system synthesis to be workable and reliable from the practical point of view. Stability analysis for systems with a constrained actuator have been studied by Glattfelder and Schaufelberger (1983), Krikelis (1980) and Gutman and Hagander (1985). Unfortunately, those results are essentially based on the circle or Popov's criterion which is not easily applied to analyze the stability of perturbed systems with constrained actuators and/or unstable plants.

![Figure 1. Feedback system with plant uncertainties and constrained actuators.](image)

This paper is concerned with the stability of an actual system with both non-linear (or linear) time-varying uncertainties and constrained actuators. Without loss of generality, the constrained actuator is described by means of a saturation function to simplify the algebraic analysis. A new sufficient condition for robust stabilization of SISO discrete-time feedback systems with both non-linear (or linear) time-varying uncertainties and the constraint of actuator saturation is derived. Here, neither the plant nor the controller has to be restricted to the stable or minimum-phase case. The necessary and sufficient condition for solvability of a stabilizer satisfying such a robust constraint is also introduced by using the Nevanlinna–Pick interpolation theory. From a more practical point of view, a less conservative stability condition, based on the assumption that the operation range of a system actuator is infinite, is discussed in detail. Then the optimal \( H_\infty \)-norm theory is employed to achieve the maximum robustness to tolerate the uncertainties via constrained control. Finally, a design algorithm is proposed to specify adequate parameters in the compensator to achieve the optimal robustness and an example with simulation is presented to illustrate these results. For the sake of clarity, we focus our attention exclusively on the design of a controller for SISO discrete-time systems. However, it is very easy to extend our results to continuous-time and/or multivariable systems.

In § 2, the system model is presented and the design goals formulated. Section 3 gives some preliminaries. The robustness problem is considered and the robustness optimization problem is subsequently solved in § 4 and § 5, respectively. Then we propose an algorithm to synthesize the controller and give an example with simulation in § 6.

2. Problem formulation

The scalar discrete-time feedback system with both plant uncertainties and actuator saturation constraint is shown in Fig. 1. \( r(k) \), \( u(k) \) and \( y(k) \) denote the reference signal, control signal and plant output, respectively. \( C(z) \) is the controller to
be synthesized and the control output is fed into a non-linear constrained actuator \( N(\cdot) \) to generate the plant input, which is described as follows

\[
\bar{u}(k) = N(u(k)) = \text{sat}(u(k))
\]

(2.1)

where the saturation function is defined as

\[
\text{sat}(u(k)) = \begin{cases} 
\bar{u} & u(k) > \bar{u} \\
u(k) & \bar{u} \leq u(k) \leq \bar{u} \\
u & u(k) < \underline{u} 
\end{cases}
\]

(2.2)

and

\[
\bar{P}(\cdot, \cdot) = P(z)[1 + \Delta P(\cdot, \cdot)]
\]

(2.3)

denotes the physical plant with multiplicative non-linear time-varying uncertainties. Notice that the nominal plant \( P(z) \) can be unstable and/or non-minimum phase, and the true plant is not necessarily \( P(z) \), but lies within some domain of uncertainties containing \( P(z) \) (Doyle and Stein 1981, Chen and Desoer 1982). The non-linear time-varying uncertainty \( \Delta P(u(k), k) \) is a real-valued function of the control signal \( u(k) \) and time step \( k \), and satisfies the following assumptions (see Fig. 5 a).

**Assumption 1**

\[ \Delta P(0, k) = 0, \text{ for all } k \geq 0. \]

**Assumption 2**

\[ \Delta P(u(k), k) \text{ is a measurable function whenever } u(k) \text{ is measurable.} \]

**Assumption 3**

There exists a finite constant \( \alpha > 0 \) with the property that \( |\Delta P(u(k), k)| \leq \alpha |u(k)| \), for all \( k \geq 0 \).

The main problems are as follows.

(i) Under what conditions will the stability of the system in Fig. 1 be still ensured when the above non-linear time-varying uncertainties \( \Delta P(u(k), k) \) and the actuator saturation occur?

(ii) How to determine, before the actual design procedure, whether it is possible to find a robust controller satisfying the above robust stability conditions to stabilize the system.

(iii) If the system can be robust stabilized, how do we synthesize a controller so that the system has the maximum robustness to tolerate such uncertainties and the actuator saturation?

These answers will be found in the subsequent sections.

3. Preliminaries

In general, we should not pose any limitation to the value of the control signal \( u(k) \); in other words, the signal out of the controller could possibly be arbitrarily large. Thus we assume \( u(k) \in (-\infty, \infty) \), i.e. the operation range of the saturated actuator is
inside the sector $[0, 1]$ centred at $\frac{1}{2}u(k)$ (see Fig. 2 and Remark 2), i.e.

$$|N(u(k)) - \frac{1}{2}u(k)| \leq \frac{1}{2}|u(k)|$$  \hspace{1cm} (3.1)$$

and construct the linearized nominal feedback control system around the centre $\frac{1}{2}u(k)$.

![Figure 2. Ideal saturated actuator, centred at $\frac{1}{2}u(k)$.](image)

of $N(\cdot)$ as shown in Fig. 3. Then the sensitivity function of this linearized nominal system is

$$S(z) := \left[1 + \frac{1}{2}P(z)C(z)\right]^{-1}$$  \hspace{1cm} (3.2)$$

Let the factorization of the linearized nominal plant be described as

$$\frac{1}{2}P(z) = \frac{B(z)}{A(z)}$$  \hspace{1cm} (3.3)$$

where $B(z), A(z) \in \mathbb{R}[z]$ are stable coprime rational functions. Thus there exist some stable rational functions $X(z)$ and $Y(z)$ that satisfy the following Bezout identity (Vidyasagar 1985)

$$A(z)X(z) + B(z)Y(z) = 1$$  \hspace{1cm} (3.4)$$

Lemma 1 provides the basis for deriving the stability condition in the following section.

![Figure 3. Linearized nominal feedback system.](image)

**Lemma 1 (Vidyasagar 1985)**

(i) The closed-loop linearized nominal system of Fig. 3 is asymptotically stable if and only if

$$S(z)C(z) = \left[Y(z) + A(z)K(z)\right]A(z)$$  \hspace{1cm} (3.5)$$

i.e.

$$1 - S(z) = \left[Y(z) + A(z)K(z)\right]B(z)$$  \hspace{1cm} (3.6)$$

where $K(z) \in \mathbb{R}[z]$ is any rational function analytic in $|z| \geq 1$ and satisfies the
constraint
\[ X(z) - B(z)K(z) \neq 0 \]
(ii) The stabilizing controller associated with a particular choice of \( K(z) \) possesses the transfer function
\[ C(z) = \frac{Y(z) + A(z)K(z)}{X(z) - B(z)K(z)} \quad (3.7) \]
From (3.5) and (3.7), it is seen that
\[ S(z) = [X(z) - B(z)K(z)]A(z) \quad (3.8) \]

Remark 1

\( C(z) \) in (3.7) is a general parameterized stabilizing controller, i.e., for any stabilizing controller of the linearized nominal feedback system in Fig. 3, there is a corresponding \( K(z) \in \mathbb{R}[s] \) in (3.7).

Remark 2

The assumption on the operation sector \([0, 1]\) of constrained actuator simplifies our algebraic analysis. However, as the objective of the stabilizer synthesis is to achieve a stable system, it is thus reasonable to expect that the control signal should be always a finite one. In order to obtain a less conservative result, one may assume that only a finite part of the constrained actuator in actual systems is considered, i.e. the operation range of the actuator is inside a sector \([a, 1]\) centred at \((1 + a)/2\) instead of \([0, 1]\) (see Fig. 4). Hence, we obtain
\[ \left| (N(u(k)) - \frac{1 + a}{2} u(k)) \right| \leq \frac{1}{2} \frac{a}{2} |u(k)|, \quad \text{for} \ 0 \leq a \leq 1 \quad (3.9) \]
If \( a = 1 \), it is the typical linear system without the saturation constraint. If \( a \rightarrow 0 \), the operation sector is then as large as possible.

![Figure 4. Less conservative saturated actuator, centred at \((1 + a)u(k)/2\).](image)

4. Robust stability

The stability of the feedback system with both the constrained actuator and plant uncertainties are now described in detail as follows.

Definition 1 (Desoer and Vidyasagar 1975)

The system shown in Fig. 1 is called \( \text{BIBO} \) stable if there exist finite positive real numbers \( c_1, c_2, c_3 \) and \( c_4 \) such that the control \( u(k) \) and system output \( y(k) \) satisfy the
inequalities

(i) \(|y(k)| \leq c_1 |r(k)| + c_2\)

(ii) \(|u(k)| \leq c_3 |r(k)| + c_4\) for any \(|r(k)| < \infty\).

Then a sufficient condition for the robust stabilization of the compensating system with both non-linear uncertainties and the constrained actuator is given as follows.

**Theorem 4.1: Case of non-linear time-varying uncertainties**

For a system with both non-linear time-varying uncertainties satisfying Assumptions 1–3 and the constrained actuator \(N(\cdot)\) satisfying (2.2), if the stabilizing controller \(C(z)\) is chosen as (3.7), and the following inequality is also held

\[\|(1 + 2x)[1 - S(z)]\|_\infty < 1\] (4.1 a)

i.e.

\[\|(1 + 2x)[B(z)\{Y(z) + K(z)A(z)\}]\|_\infty < 1\] (4.1 b)

for some \(K(z) \in R[z]\), then there exist two positive-finite constants \(c_1\) and \(c_2\) such that

\[|y(k)| \leq c_1 |r(k)|, \quad |u(k)| \leq c_2 |r(k)|\]

for \(|r(k)| < \infty\), i.e. the system with such non-linear uncertainties and the constrained actuator is BIBO stable.

**Proof**

See Appendix A.

**Remark 3**

Basically, Popov's criterion and the circle criterion cannot be directly applied to the case where \(C(z)\) and \(P(z)\) are unstable. Before they are employed to test the stability of a system with unstable plant, the system must be transferred to an equivalent system with stable elements (Zames 1966). Such a transformation is not easy to do in a system that has two non-linear elements in the control loop (see Fig. 1). The importance of this theorem is that it directly gives a criterion for robust stabilization of a system that can have unstable elements, with both non-linear time-varying uncertainties and constrained actuators.

**Remark 4**

For the less conservative case that assumes the operation range of the system to be within \([a, 1]\), an entirely similar robust condition is given as

\[\left\|\frac{1 - a + 2x}{1 + a}[1 - S'(z)]\right\|_\infty < 1\] (4.2 a)

i.e.

\[\left\|\frac{1 - a + 2x}{1 + a}[B(z)\{Y(z) + K(z)A(z)\}]\right\|_\infty < 1\] (4.2 b)

where

\[S'(z) = \left[1 + \frac{1 + a}{2} P(z)C(z)\right]^{-1}\] (4.3)
and

\[
\frac{1 + a}{2} P(z) = \frac{B'(z)}{A'(z)}, \quad A'(z)X'(z) + B'(z)Y'(z) = 1
\]  \hspace{1cm} (4.4)

However, the above modification can only be taken as a design technique. If such a finite operation range of actuator is assumed, it would not be surprising to see that, if a too large control signal \(u(k)\) exceeding this sector \([a, 1]\) occurs in the system, stability would be seriously deteriorated. At this situation, the designer must be responsible for guaranteeing that the system does not violate the constraint of this range during operation. This may be checked by simulation.

**Remark 5**

When stochastic signals are involved, the operation range of the actuator should be carefully determined. At least we have to consider both the mean and variance of control signals. In general, it is better to preserve the sector as large as possible to prevent the signal exceeding the assumed range. A method of determining the maximum tolerable operation range is given in the remainder of this section. However, in this situation, it is highly likely that the ‘conservative case’ would give more reliable results.

**Remark 6**

If the constraint of the control signal is released, i.e. \(a = 1\), the robust stability condition is then reduced as

\[
\|x\{B'(z)[Y'(z) + K(z)A'(z)]\} \|_z < 1
\]  \hspace{1cm} (4.5 a)

which is the robust stability condition for a system with multiplicative non-linear time-varying uncertainties. Sandberg (1964) has proved the inequality (4.5 a) in continuous-time domain under the assumption that \(P\) and \(C\) must be analytic in \(Re s \geq 0\). Here, we have relaxed this restriction.

**Remark 7**

Likewise, if only the saturation of actuators is considered (i.e. \(\Delta P = 0\)), the stability criterion is also valid to treat the stabilization problem of non-linear saturated feedback systems. In this case, by setting \(x = 0\) in inequality (4.1 b), the stability criterion is given as

\[
\|B(z)[Y(z) + K(z)A(z)]\|_z < 1
\]  \hspace{1cm} (4.5 b)

In the case of linear uncertainties, an analogous result is obtained. Let \(\Delta P(z)\) represent the multiplicative linear uncertainties of the plant satisfying the following conditions:

(i) \(\tilde{P}(z)\) and \(P(z)\) have the same number of poles in \(|z| \geq 1\) (i.e. the perturbation introduces no extra unstable poles), where \(\tilde{P}(z) = P(z)(1 + \Delta P(z))\);

(ii) \[|\Delta P \exp(jw)| \leq \|G \exp(jw)|, \quad w \in [0, 2\pi] \]  \hspace{1cm} (4.6)

where \(G \exp(jw)\) is a stable rational function that characterizes the envelope of \(\Delta P \exp(jw)\) (see Fig. 5 b).

Then the robustness condition in Theorem 4.1 is modified as the following.
Corollary 4.1: Case of linear uncertainties

For a plant with both linear uncertainties that are enveloped by a stable function $G(z)$ and the constrained actuator $N(\cdot)$, if the stabilizing controller $C(z)$ is chosen as (3.7) and the following inequality is also held

$$\beta[1 - S(z)]_\infty < 1$$  \hspace{1cm} (4.7a)

i.e.

$$\|\beta \{B(z)[Y(z) + K(z)A(z)]\\|_\infty < 1$$  \hspace{1cm} (4.7b)

for some $K(z) \in R[z]$, where $\beta = 1 + 2\|G(z)\|_\infty$, then the system is BIBO stable.

Proof

Similar to the proof of Theorem 4.1.

Remark 8

For the less conservative case, a similar robust stability condition is given by

$$\|\beta - a \|_1 \{B(z)[Y(z) + K(z)A(z)]\\} \|_\infty < 1$$  \hspace{1cm} (4.8)

Reviewing the above analysis, the design of the robust stabilization is reduced to find an auxiliary stable rational function $K(z) \in R[z]$ such that the above robust stability conditions are satisfied. In the case of stable plant, if we choose

$$K(z) = \frac{1}{A(z)} \left[ \frac{aT(z)}{(1 + 2z)\beta(z)} - Y(z) \right]$$  \hspace{1cm} (4.9)
where \( q \) is a scalar with \( |q| < 1 \) and \( T(z) \in R[z] \) is a stable function where its numerator contains the zeros of \( B(z) \) in \( |z| > 1 \) and \( |T(z)| \leq 1 \). For example, \( T(z) = n(z)/d(z) \), where the polynomial \( n(z) \) contains all zeros of \( B(z) \) in \( |z| > 1 \) and \( |n(z)| \leq |d(z)| \), then

\[
\| (1 + 2z) [Y(z) + A(z)K(z)]B(z) \|_\infty = |q| |T(z)| \leq |q| < 1
\]

i.e. the stability condition (4.1) always holds. It is always possible to find a robust stabilizer in the perturbed system with a constrained actuator. In case of unstable plants, however, it is well known that such a \( K(z) \) may not exist. To check up on the solvability problem for robust stabilization in the perturbed and saturated control systems with unstable plants, the Nevanlinna–Pick interpolation theory is employed.

**Lemma 4.1: Nevanlinna–Pick theory** (Kimura 1984)

Suppose \( z_1, \ldots, z_n \) are distinct complex numbers with \( |z_i| > 1 \) and \( E_1, \ldots, E_n \) are complex matrices with \( |E_i| < 1 \) for all \( i \). Define a Hermitian matrix that is called the Pick matrix

\[
W = \begin{bmatrix}
W_{11} & W_{12} & \cdots & W_{1n} \\
W_{21} & & & \\
\vdots & & & \\
W_{n1} & \cdots & W_{nn}
\end{bmatrix}
\]

(4.10)

where

\[
W_{ij} = \frac{1}{z_i z_j} - (1 - E_i^* E_j)
\]

(4.11)

Then there exists a \( \phi(z) \in H \) such that \( \| \phi(z) \|_\infty < 1 \) and \( \phi(z_i) = E_i \), for \( i = 1, \ldots, n \), if and only if the matrix \( W \) is positive-definite.

Now let us start from the robust stability conditions in Theorem 4.1. Suppose that all the zeros of \( A(z) \) in \( |z| > 1 \) are simple and factorize \( A(z), B(z) \) as

\[
A(z) = A_i(z)A_o(z)
\]

\[
B(z) = B_i(z)B_o(z)
\]

(4.12)

where \( A_i(z), A_o(z) \) and \( B_i(z), B_o(z) \) constitute the inner–outer factorizations of \( A(z) \) and \( B(z) \), respectively. By the fact that (Vidyasagar 1985)

\[
\| B_i(z) \phi(z) \|_\infty = \| \phi(z) \|_\infty
\]

(4.13)

i.e. the multiplication of inner function preserves the value of norm. The robustly stabilizable condition (4.1 b) is equivalent to

\[
\| (1 + 2z) [Y(z) + K(z)A_i(z)A_o(z)]B_i(z)B_o(z) \|_\infty
\]

\[
= \| (1 + 2z) [Y(z)B_o(z) + K(z)A_i(z)A_o(z)B_o(z)] \|_\infty < 1
\]

(4.14)

Let \( z_i, i = 1, 2, \ldots, n \) be the zeros of \( A_i(z) \) in \( |z| > 1 \), i.e. the unstable poles of \( P(z) \), and define

\[
E(z) = (1 + 2z)[Y(z)B_o(z) + K(z)A_i(z)A_o(z)B_o(z)]
\]

(4.15)

then from (4.14) and (4.15), the robust stability condition is equivalent to

\[
\| E(z) \|_\infty < 1
\]

(4.16)
and

\[ E_i = E(z_i) = (1 + 2\alpha)Y(z_i)B_0(z_i), \quad i = 1, 2, \ldots, n \]  \hspace{1cm} (4.17 a)

Now based on the Nevanlinna–Pick theory, the solvability condition for robust stabilizability is described in Corollary 4.2.

**Corollary 4.2: Case of non-linear time-varying uncertainties**

The stable rational function \( K(z) \) that satisfies the robust conditions of Theorem 4.1 is solvable if and only if the matrix \( W \) defined by (4.10), (4.11) and (4.17 a) is positive-definite.

**Remark 9**

For the less conservative case, (4.17 a) should be replaced by

\[ E_i = E(z_i) = \frac{1 - \alpha + 2\alpha}{1 + \alpha} \ Y(z_i)B_0(z_i), \quad i = 1, 2, \ldots, n \]  \hspace{1cm} (4.17 b)

In general, it is hard to decide the operation range of an actuator in advance. Usually, the value of \( \alpha \) has to be determined by some iterative computations and simulations. However, this corollary can also be taken as a more efficient tool for evaluating the value of \( \alpha \). Since \( \alpha \) is given, from this corollary we can find the smallest value of \( \alpha \) which makes the matrix \( W \) positive-definite. As this smallest \( \alpha \) is found, by computer simulation, we can check if the operation range of the system is always inside the sector \([\alpha, 1]\). If it is, then we can use this less conservative robust stabilizer according to the less conservative stability condition (4.2).

Corollary 4.2 only deals with the solvability of \( K(z) \) for plants whose unstable poles are simple. Similar results apply even in the case where the plant has repeated unstable poles (Vidyasagar and Kimura 1986). Similarly, for the linear uncertainty case, suppose we define

\[ E_i(z) = \beta [Y(z)B_0(z) + K(z)A_0(z)A_1(z)B_0(z)] \]  \hspace{1cm} (4.18)

\[ E_i = E_i(z_i) = \beta Y(z_i)B_0(z_i), \quad i = 1, 2, \ldots, n \]  \hspace{1cm} (4.19 a)

where \( \beta = 1 + 2\|G(z)\|_\infty \), then Corollary 4.3 follows.

**Corollary 4.3: Case of linear uncertainties**

The stable rational function \( K(z) \) that satisfies the robust conditions of Corollary 4.1 is solvable if and only if the matrix \( W \) defined by (4.10), (4.11) and (4.19 a) is positive-definite.

**Remark 10**

For the less conservative case, (4.19 a) should be replaced by

\[ E_i = E_i(z_i) = \frac{\beta - \alpha}{1 + \alpha} Y(z_i)B_0(z_i), \quad i = 1, 2, \ldots, n \]  \hspace{1cm} (4.19 b)

If the plant is under additive uncertainties, i.e.

\[ \tilde{P}(\cdot, \cdot) = P(z) + \Delta P(\cdot, \cdot) \]  \hspace{1cm} (4.20)
where $\Delta P$ also satisfies Assumptions 1–3, it is easy to verify that the stabilization condition for such an additive non-linear time-varying perturbed system with constrained controller can be described as follows.

**Lemma 4.2: Case of additive non-linear time-varying uncertainties**

A system with the perturbed plant shown in (4.20) and the constrained controller $N(\cdot)$ is BIBO stable if

(i) the controller is chosen as (3.7), and

(ii) $\|[Y(z) + A(z)K(z)]B(z)\|_\infty + \alpha \|[Y(z) + A(z)K(z)]A(z)\|_\infty < 1$ \hspace{1cm} (4.21)

**Proof**

See Appendix B.

All the other analysis is entirely analogous to the results of this section.

5. **Robustness optimization**

Although the robust stabilizability may be sure, it is still a complicated computation to find out such a stabilizer. Fortunately, there exists a closed form solution to the robustness optimization. Let us define the robustness of the control system as (Salonov and Chen 1982)

$$R := \frac{1}{\|[1 + 2\alpha][Y(z) + K(z)A(z)]B(z)\|_\infty} \hspace{1cm} (5.1)$$

It is seen that if $R > 1$, the perturbed system will be robustly stabilized via the constrained actuator. Define

$$\hat{K} := \max_{K(z)} R = \frac{1}{\min_{K(z)} \|[1 + 2\alpha][Y(z) + K(z)A(z)]B(z)\|_\infty} \hspace{1cm} (5.2)$$

Thus the robustness optimization in the discrete-time feedback control system is equivalent to

$$\min_{K(z)} \|[1 + 2\alpha][Y(z) + K(z)A(z)]B(z)\|_\infty \hspace{1cm} (5.3)$$

The optimal $H_\infty$-norm theory developed by several mathematicians (Sarason 1967, Adamjan et al. 1978) is now employed to treat this problem.

**Lemma 5.1**

(i) The optimal $\hat{F}(z)$ that minimizes $\|F(z)\|_\infty$, is of an all-pass form, i.e.

$$\hat{F}(z) = \rho \frac{m(z)}{\bar{m}(z)} \hspace{1cm} (5.4)$$

where $m(z)$ is a strictly Hurwitz polynomial with all roots in $|z| < 1$.

(ii) $\min \|F(z)\|_\infty = |\hat{F}(z)| = |\rho| \hspace{1cm} (5.5)$

Based on the above analysis, if we can specify a $K(z) \in R[z]$ such that the function
in (5.3) is an all-pass form, the system will have the greatest ability to tolerate non-linear uncertainties and actuator saturation. Thus

$$
\dot{K}(z) = \frac{1}{A(z)} \left[ \frac{-\hat{m}(z)}{(1 + 2\gamma \hat{m}(z)B(z))} - Y(z) \right]
$$

(5.6 a)

or the less conservative case

$$
\dot{K}'(z) = \frac{1}{A'(z)} \left[ \frac{(1 + a)\hat{m}(z)}{(1 - a + 2\gamma \hat{m}(z)B'(z))} - Y'(z) \right]
$$

(5.6 b)

Similarly, for the linear uncertainties case, we have to specify $K(z)$ to make the rational function in (4.7 b) an all-pass form. Thus

$$
\dot{K}(z) = \frac{1}{A(z)} \left[ \frac{\hat{m}(z)}{\hat{m}(z)B(z)} - Y(z) \right]
$$

(5.7 a)

and the less conservative result is

$$
\dot{K}'(z) = \frac{1}{A'(z)} \left[ \frac{(1 + a)\hat{m}(z)}{(1 - a - 2\gamma \hat{m}(z)B'(z))} - Y'(z) \right]
$$

(5.7 b)

Remark 11

(a) The scalar constant $\rho$ and the coefficients of polynomial $m(z)$ with minimum degree are determined by the requirement of the analyticity of $K(z)$ in $|z| \geq 1$. The optimal interpolation technique will be employed to treat this problem and will be illustrated in the following example.

(b) If $K(z)$ is chosen as (5.6) or (5.7), the robustness optimization compensator follows (3.7) immediately.

(c) One of the practical meanings of this optimal design is that it can always guarantee stability in the maximum operation range of the actuator even if a smaller range is assumed. For example, the robust stability condition (4.2 a) is equivalent to

$$
\|1 - S'(z)\|_\infty < \frac{1 + a}{1 - a + 2\gamma}
$$

(5.8)

and our optimal design algorithm always achieves the minimum value of function $\|1 - S'(z)\|_\infty$ regardless of the value of $a$. Thus the minimum value of $a$ may be solved from (5.8) and (5.5), i.e., after the value of $\rho$ of min $\|1 - S'(z)\|_\infty$ is found, we can obtain the smallest value of $a$ from

$$
\frac{1 + a}{1 - a + 2\gamma} = \rho
$$

(5.9)

Then $[a, 1]$ is the largest operation range that this system could be stabilized. It is also seen that there should be a trade-off between the uncertainty bound and the largest sector of control signal. Corollary 4.2 provides the same function of evaluating the value of $a$. However, this lemma also provides the way to synthesize the optimal stabilizer.

6. Algorithm and example

From the above analysis, an algorithm for the synthesis of discrete-time feedback controller with maximum robustness is described as follows.
Step 1. Perform the stable rational coprime factorization of $\frac{1}{2}P(z)$

$$\frac{1}{2}P(z) = \frac{B(z)}{A(z)}$$

and solve (3.4) to obtain stable rational functions $X(z)$ and $Y(z)$.

Step 2. If the plant is stable, select $K(z)$ as (4.9) and go to Step 5.

Step 3. Factorize $A(z) = A_1(z)A_0(z)$, $B(z) = B_1(z)B_0(z)$, and check whether the $W$ in Corollary 4.2 or 4.3 is positive-definite. If yes, go to Step 4. If not, a modification of this problem is necessary, i.e. the operation range of system has to be assumed to be finite. Then from the Corollary 4.2 and 4.3 again, we can find out the smallest value of $a$ with $0 < a < 1$ to make $\tilde{W} > 0$ (if such an $a$ does not exist, the system is not robustly stabilizable via our method and hence there is no need to continue).

Step 4. From Theorem 4.1 and (5.6) for the case of non-linear uncertainties (or Corollary 4.1 and (5.7) for the case of linear uncertainties), we can obtain the corresponding robust optimization function $\tilde{K}(z)$ in which $\rho$ and the coefficients of polynomial $m(z)$ with minimal degree are determined by the analyticity of $\tilde{K}(z)$ in $|z| \geq 1$.

Step 5. Substitute $\tilde{K}(z)$ obtained from Step 4 or Step 2 into (3.7) to obtain the unknown parameters of controller $C(z)$. If the value of $a$ is not zero, simulations are required to verify whether the control signal exceeds this range. If not, the controller is acceptable, and vice versa.

Completing the above steps, the design problem is finished. An example is given to illustrate these results.

**Example**

Consider the system of Fig. 1 with the following nominal plant

$$P(z) = \frac{(2z + 3)}{(5z - 6)} \quad (6.1)$$

Suppose the non-linear uncertainties are

$$\Delta P(u(k), k) = 0.5 u(k)^4 \sin(u(k)) \quad (6.2)$$

and the energy limitation of the actuator is at $\pm 1.2$. We will show here how to synthesize a controller to achieve optimal robustness.

**Step 1**

Since $\frac{1}{2}P(z) = (2z + 3)/[2(5z - 6)]$, we can choose

$$A(z) = \frac{10z - 12}{2z + 1}, \quad B(z) = \frac{2z + 3}{2z + 1} \quad (6.3)$$

and solve (3.4) to obtain

$$X(s) = \frac{1.0494z + 0.8333}{4z^2 + 1}, \quad Y(s) = \frac{1.3765z + 2}{4z^2 + 1} \quad (6.4)$$
Step 3

The perturbation is obviously bounded by

$$|\Delta P(u(k), k)| \leq 0.5|u(k)|$$

i.e. \(x = 0.5\), and factorize \(A(z), B(z)\) as (4.12). thus

$$A_1(z) = \frac{5z - 6}{6z - 5}, \quad A_2(z) = \frac{12z - 10}{2z + 1} \quad (6.5)$$

$$B_1(z) = \frac{2z + 3}{3z + 2}, \quad B_2(z) = \frac{3z + 2}{2z + 1} \quad (6.6)$$

There is only one unstable zero of \(A(z)\) at \(z = 1.2\). According to Corollary 4.2 and (4.17 a), we obtain the Pick matrix as

$$E(1.2) = (1 + 2z) Y(1.2) B_a(1.2) = 2.0741 > 1 \quad (6.7)$$

The system is not robustly stabilizable under the assumption of infinite operation range. Hence, we have to use the less conservative result. Follow Corollary 4.2 and (4.17 b) again to solve the minimal value of \(a\), i.e. solve the following inequality

$$E(1.2) = \frac{1 - a + 2z}{1 + a} Y(1.2) B_a(1.2) < 1 \quad (6.8)$$

which yields

$$a_{\text{min}} \approx 0.5273 \quad (6.9)$$

with the Pick matrix

$$W = \frac{1 - E(1.2)^2}{(1.2)^2 - 1} > 0 \quad (6.10)$$

i.e. if the control signals are always inside the sector \([a_{\text{min}}, 1]\), the system will be robustly stabilizable. Then repeating the factorization of (4.4) with \(a = a_{\text{min}}\), we obtain

$$A'(z) = \frac{6.5476z - 7.8571}{2z + 1}, \quad B'(z) = \frac{2z + 3}{2z + 1} \quad (6.11)$$

and solve (3.4) to obtain

$$X'(s) = \frac{0.8013z + 0.6364}{4z + 1}, \quad Y'(s) = \frac{1.3765z + 2}{4z + 1} \quad (6.12)$$

Step 4

From Lemma 5.1 and (5.6), the corresponding robust optimization function \(\tilde{K}'(z)\) is given as follows

$$\tilde{K}'(z) = \frac{(2z + 1)}{(6.5476z - 7.8571)} \left[ \frac{(1 + a)\tilde{n}(s)(2z + 1)}{(1 - a + 2z)n(s)(2z + 3)} - \frac{1.3765z + 2}{4z + 1} \right] \quad (6.13)$$

Under the requirement of analyticity of \(\tilde{K}'(s)\) in \(|z| \geq 1\), we obtain

$$\tilde{n}(z) = (2z + 3) \quad \text{and} \quad \rho < 1$$

then

$$\tilde{K}'(z) = \frac{(2z + 1)(0.6364z + 0.3771)}{(3z + 2)(4z + 1)} \quad (6.14)$$
Step 5
Substituting (6.14) into (3.7), we obtain the controller as

$$C(z) = \frac{1.0370[(2z + 1)/(3z + 2)]}{0.1414[(2z + 1)/(3z + 2)]} = \frac{22}{3}$$  (6.15)

Since $a \neq 0$, we have to check whether the operation sector is suitable. If a control signal that is greater than $u(k)/a_{\text{min}}$ occurs in the system, the stability may seriously deteriorate. By computer simulation, we find that the maximum control signal $u = 1.9$. Since $1.2/1.9 > a_{\text{min}}$, the controller is acceptable, i.e. by using this controller, the system can tolerate the uncertainties (6.2) via the constrained actuator. These results are illustrated by the computer simulation in Fig. 6.

![Simulation result of the example](image)

Figure 6. Simulation result of the example.

7. Conclusion
From a more practical point of view, a theoretical analysis for the robust stabilization of a discrete-time feedback system with both non-linear time-varying uncertainties and a constrained actuator is described. It is found that if the plant is stable, it is always possible to find a robust stabilizer to tolerate the perturbation under actuator saturation. An algorithm is proposed to give an optimal stabilizer to maximize the robustness to tolerate plant uncertainties and the saturation of the actuator. This algorithm also provides a way to determine the greatest tolerable operation range of the constrained actuator. The result can be easily extended to multivariable systems or the control design for other purposes.

Appendix A
Proof of Theorem 4.1
(i) From the system in Fig. 1, we obtain

$$u(k) = C(z)r(k) - C(z)\bar{P}(\cdot, \cdot)N(\cdot)u(k)$$

$$= C(z)r(k) - C(z)P(z)N(\cdot)u(k) - C(z)P(z)\Delta P(\cdot, \cdot)N(\cdot)u(k)$$
\[= C(z)r(k) - C(z)P(z) \left[ N(\cdot) - \frac{1}{2} \right] u(k) - C(z)P(z) \frac{1}{2} u(k) \]
\[- C(z)P(z) \Delta P(\cdot, \cdot) N(\cdot) u(k) \]  
(A 1)

After rearrangement, we obtain
\[ \left[ 1 + \frac{1}{2} C(z)P(z) \right] u(k) = C(z)r(k) - C(z)P(z) \left[ N(\cdot) - \frac{1}{2} \right] u(k) \]
\[- C(z)P(z) \Delta P(\cdot, \cdot) N(\cdot) u(k) \]  
(A 2)

From Lemma 3.1, since we choose the stabilizing controller as (3.7), the sensitivity function following (3.8) is asymptotically stable, i.e.
\[ S(z) = \left[ 1 + \frac{1}{2} C(z)P(z) \right]^{-1} \]  
(A 3)

exists. Then
\[ u(k) = S(z)C(z)r(k) - S(z)C(z)P(z) \left[ N(\cdot) - \frac{1}{2} \right] u(k) \]
\[- S(z)C(z)P(z) \Delta P(\cdot, \cdot) N(\cdot) u(k) \]  
(A 4)

From (3.5) and (3.6), it is seen that \( S(z)C(z) \) and \( S(z)C(z)P(z) \) are all asymptotically stable. Since \( u(t) = N[u(t)] \) and \( \Delta P[u'(t), t] \) satisfies the Assumptions 2 and 3, we then perform the \( \| \cdot \| \) operation on (A 4), which yields
\[ |u(k)| \leq |S(z)C(z)||r(k)| + |S(z)C(z)P(z)||N(\cdot) - \frac{1}{2}|u(k)| \]
\[ + |S(z)C(z)P(z)||\Delta P(\cdot, \cdot)N(\cdot)u(k)| \]  
(A 5)

Since
\[ |(N(\cdot) - \frac{1}{2})u(k)| \leq \frac{1}{2}|u(k)| \]  
(A 6)
\[ |N(u(k))| \leq |u(k)| \]  
(A 7)

and
\[ |\Delta P(\cdot, \cdot)N(\cdot)u(k)| = |\Delta P(\cdot, \cdot)u'(k)| \leq \alpha |u'(k)| \leq \alpha |u(k)| \]  
(A 8)

(A 5) becomes
\[ |u(k)| \leq |S(z)C(z)||r(k)| + |S(z)C(z)P(z)|\frac{1}{2}|u(k)| \]
\[ + |S(z)C(z)P(z)|\alpha|u(k)| \]
\[ \leq \|S(z)C(z)\|_{\infty} |r(k)| + \frac{1}{2} \|S(z)C(z)P(z)\|_{\infty} |u(k)| \]
\[ + \alpha \|S(z)C(z)P(z)\|_{\omega} |u(k)| \]  
(A 9)
i.e.
\[ \left[ 1 - \frac{1 + 2\alpha}{2} \right] \|S(z)C(z)P(z)\|_{\infty} |u(k)| \leq \|S(z)C(z)\|_{\infty} |r(k)| \]  
(A 10)

where
\[ S(z)C(z)P(z) = 2S(z)C(z)\frac{1}{2}P(z) = 2[1 - S(z)] \]  
(A 11)

Hence (A 10) is equivalent to
\[ \left[ 1 - (1 + 2\alpha)\|1 - S(z)\|_{\infty} \right] |u(k)| \leq \|S(z)C(z)\|_{\infty} |r(k)| \]  
(A 12)

If we assume
\[ (1 + 2\alpha)\|1 - S(z)\|_{\infty} < 1 \]  
(A 13)
then
\[ |u(k)| \leq \frac{\|S(z)C(z)\|_\infty}{1-(1+2z)\|1-S(z)\|_\infty} |r(k)| = c_1 |r(k)| \]  
(A 14)

where
\[ c_1 = \frac{\|S(z)C(z)\|_\infty}{1-(1+2z)\|1-S(z)\|_\infty} \]  
(A 15)

(ii) Similarly
\[ r(k) = r(k) - [1 + \tilde{P}(\cdot, \cdot)N(\cdot)C(z)]^{-1}r(k) \]
\[ = r(k) + [1 + \frac{1}{2}P(z)C(z)]^{-1} [1 + \tilde{P}(\cdot, \cdot)N(\cdot)C(z)] - [1 + \frac{1}{2}P(z)C(z)]^{-1}r(k) \]
\[ \times [1 + \tilde{P}(\cdot, \cdot)N(\cdot)C(z)]^{-1}r(k) - [1 + \frac{1}{2}P(z)C(z)]^{-1}r(k) \]
\[ = r(k) + S(z)[P(z)N(\cdot) - \frac{1}{2}]C(z) + P(z)\Delta P(\cdot, \cdot)N(\cdot)C(z) \]
\[ \times [1 + \tilde{P}(\cdot, \cdot)N(\cdot)C(z)]^{-1}r(k) - S(z)r(k) \]  
(A 16)

From the first identity of (A 1), it is seen that
\[ u(k) = C(z)[1 + \tilde{P}(\cdot, \cdot)N(\cdot)C(z)]^{-1}r(k) \]  
(A 17)

Hence
\[ y(k) = [1 - S(z)]r(k) - S(z)P(z)[N(\cdot) - \frac{1}{2}]u(k) \]
\[ + S(z)P(z)\Delta P(\cdot, \cdot)N(\cdot)u(k) \]  
(A 18)

From (3.7) and (3.8), we can see that both \(1 - S(z)\) and \(S(z)P(z)\) are asymptotically stable. Then perform the norm operation again, which yields
\[ |r(k)| \leq \|[1 - S(z)]||r(k)| + |S(z)P(z)||[N(\cdot) - \frac{1}{2}]u(k)| \]
\[ + |S(z)P(z)||\Delta P(\cdot, \cdot)N(\cdot)u(k)| \]  
(A 19)

Following (A 7), (A 8), (A 9) and the result of (A 14), we have
\[ |y(k)| \leq \|(1 - S(z))||r(k)| + |S(z)P(z)|\frac{1}{2}|u(k)| + |S(z)P(z)||x||u(k)| \]
\[ \leq \|[1 - S(z)]||r(k)| + \frac{1}{2}|S(z)P(z)|c_1 |r(k)| \]
\[ + x|S(z)P(z)|c_1 |r(k)| \]
\[ \leq \|(1 - S(z))\|_\infty |r(k)| + \frac{1 + 2x}{2}c_1 \|S(z)P(z)\|_\infty |r(k)| \]
\[ = c_2 |r(k)| \]  
(A 20)

where
\[ c_2 = \|[1 - S(z)]\|_\infty + \frac{1 + 2x}{2}c_1 \|S(z)P(z)\|_\infty \]  
(A 21)

Hence we conclude that the compensating system is BIBO stable if the assumption (A 13) holds. \qed
Appendix B

Brief steps of the proof of Lemma 4.2

The proof of Lemma 4.2 is quite similar to that of Appendix A except the true plant has an additive representation, i.e.

\[ \tilde{P}(\cdot, \cdot) = P(z) + \Delta P(\cdot, \cdot) \]  

(B 1)

Following steps (A 1)–(A 3), we yield

\[
u(k) = S(z)C(z)r(k) - S(z)C(z)P(z)\left[N(\cdot) - \frac{1}{3}\right]u(k) - S(z)C(z)\Delta P(\cdot, \cdot)N(\cdot)u(k)
\]  

(B 2)

and similar to the derivations of (A 5)–(A 11), we can obtain

\[ \{1 - \left[1 - S(z)\|_{\infty} + \alpha |S(z)C(z)|_{\infty}\right]\} |u(k)| \leq \|S(z)C(z)\|_{\infty} |r(k)| \]  

(B 3)

Now following the other discussion in Appendix A and identities (3.5) and (3.6), it is easy to verify that the stability criterion of the additive uncertainty case is given as Lemma 4.2.

REFERENCES