Robust Linear Controller Design: Time Domain Approach

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Abstract—In this note a new robust linear controller design in the time domain is introduced for multivariable systems with linear or nonlinear time-varying model uncertainties. The Gronwall lemma is employed to investigate the robust stability conditions which are based on the upper norm-bounds of the uncertainties. The parameters of a dynamic controller are selected to satisfy the requirements of robust stability under plant uncertainties.

I. INTRODUCTION

The main theoretical problems of robust controller design are to find a linear time-invariant feedback controller for the incompletely known plant and to develop design strategies for a linear controller in the presence of severe model uncertainties. Over the last few years, several authors [1]-[7] have treated this problem from the frequency domain by means of norm or norm-like (the largest singular value) inequalities.

In this note, a new robust controller design procedure in the time domain is introduced for the multivariable dynamic systems with parametrical as well as structural linear or nonlinear time-varying model uncertainties. The Gronwall lemma is employed to investigate the robust stability conditions which are based on the upper norm-bounds of the uncertainties. The parameters of a dynamic controller are selected to satisfy the requirement of robust stability under plant uncertainties.

II. PROBLEM FORMULATION

Consider the following multivariable dynamic system with parametrical uncertainties:

\[ \begin{align*}
\dot{x} &= Ax + Bu + \Delta A(x) + \Delta B(u) \\
y &= Cx + Du + \Delta C(x) + \Delta D(u), \quad x(0) = x_0
\end{align*} \tag{1} \]

where \( x \in \mathbb{R}^n \) is the state vector, \( u \in \mathbb{R}^m \) is the input vector, \( y \in \mathbb{R}^r \) is the output vector, and \( A, B, C, \) and \( D \) are constant matrices. \( \Delta A(x), \Delta B(u), \Delta C(x), \) and \( \Delta D(u) \) are nonlinear time-varying parametrical uncertainties with the following known upper norm-bounds. (See Fig. 1 as an illustration.)

\[ \begin{align*}
\|\Delta A(x)\| &\leq \beta_1 \|x\|, \\
\|\Delta B(u)\| &\leq \beta_2 \|u\|, \\
\|\Delta C(x)\| &\leq \beta_1 \|x\|, \\
\|\Delta D(u)\| &\leq \beta_2 \|u\|.
\end{align*} \tag{2} \]

The norm of real vector \( x \in \mathbb{R}^n \), denoted by \( \|x\| \), is defined by [9]-[11], [14] as

\[ \|x\| = \sum_{i=1}^{n} |x_i| \tag{3} \]

and the induced norm corresponding to the vector norm is given as

\[ \|A\| = \max_{i=1}^{n} \sum_{j=1}^{n} |A_{ij}| \quad \text{(column sum)} \tag{4} \]

where \( x_i, i = 1, 2, \ldots, n \) denotes the element of the vector \( x \) and \( A_{ij}, i, j = 1, 2, \ldots, n \) denotes the entries of matrix \( A \).

This system (1) yields an approximate dynamic system as follows:

\[ \begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du, \quad x(0) = x_0
\end{align*} \tag{5} \]

Thus, system (5) represents the "approximate model" of the plant. Without loss of generality, we will assume that \((A, B)\) is controllable and \((C, A)\) is observable.

If the plant uncertainties are unknown and cannot be expressed as parametrical uncertainties in (1) but can be described by the input-output form as a structural model error with an upper bound (see Fig. 2), then

\[ \begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du + \Delta h \ast u, \quad x(0) = x_0
\end{align*} \tag{6} \]

where the operator \( \ast \) denotes the convolution while the structural model uncertainty \( \Delta h \) is linear time-varying but bounded by the following inequality:

\[ \|\Delta h(t)\| \leq r \exp(-\beta t). \tag{7} \]

In this note the dynamic controller has the following structure [11]:

\[ \begin{align*}
x_c &= Ax_c + Bu, \\
u &= K_1 x_c + K_2 x, \quad x(0) = x_0
\end{align*} \tag{8} \]

where \( x_c \in \mathbb{R}^{n_2} \) and \( A_c, B_c, K_1, \) and \( K_2 \) have appropriate dimensions. Thus, our problems are formulated as follows.

**Problem (I):** The first design problem is to choose the parameters \( A_c, B_c, K_1, \) and \( K_2 \) in the dynamic controller (8) such that the closed-loop approximate system of (5) and (8) is asymptotically stable and at the same time the closed-loop system with uncertainties of (1) and (8) is also asymptotically stable, i.e., the uncertainties can be tolerated in our design and the controller is a robust controller.

**Problem (II):** The second design problem is to choose the dynamic controller (8) such that the closed-loop approximate system of (5) and (8) is asymptotically stable and at the same time the closed-loop system with uncertainties of (6) and (8) is also asymptotically stable.

III. ROBUST STABILIZATION

In problem (I), the closed-loop system with nonlinear parametrical uncertainties is described by (1) and (8). Combining (1) and (8), we get
Let us define
\[
\begin{bmatrix}
A + BK_2 \\
B, C + B, DK_1
\end{bmatrix}
\begin{bmatrix}
x \\
x_0
\end{bmatrix}
+ \begin{bmatrix}
\Delta A(x) + \Delta B(u) \\
\Delta C(x) + \Delta D(u)
\end{bmatrix}, \tag{9}
\]

\[
y = [C + DK_2, DK_1] \begin{bmatrix}
x \\
x_0
\end{bmatrix} + \{\Delta C(x) + \Delta D(u)\}.
\]

Let us define
\[
x = \begin{bmatrix}
x \\
x_0
\end{bmatrix}, \bar{A} = \begin{bmatrix}
A + BK_2 \\
B, C + B, DK_1
\end{bmatrix}, C = [C + DK_2, DK_1],
\]

\[
\Delta \bar{A}(x) = \begin{bmatrix}
\Delta A(x) + \Delta B(u) \\
\Delta C(x) + \Delta D(u)
\end{bmatrix}, \Delta \bar{C}(x) = \{\Delta C(x) + \Delta D(u)\}. \tag{10}
\]

Then the closed-loop system with parametrical uncertainties can be represented as
\[
\dot{x} = \bar{A}x + \Delta \bar{A}(x), \quad y = \bar{C}x + \Delta \bar{C}(x). \tag{11}
\]

And the approximate closed feedback system is given by
\[
\dot{x} = \bar{Ax}, \quad y = \bar{Cx} \tag{12}
\]

Let us define the transition matrix \(\Phi(t)\) of (12) as
\[
\Phi(t) = \exp \bar{A}t \tag{13}
\]

and suppose
\[
\|\Phi(t)\| \leq m \exp \(-\alpha t\), \quad t \geq 0 \tag{14}
\]

for some constants \(m > 0, \alpha > 0\). In order to satisfy the requirement of (14), we must choose the parameters \(A, B, K_1,\) and \(K_2\) of dynamic controller (8) such that all the eigenvalues of \(\bar{A}\) are distinct and in the l.h.p. It is also seen that \(-\alpha = \max \text{Re}(\lambda_i(\bar{A}))\), where \(\lambda_i(\bar{A})\) and \(i = 1, 2, \cdots, n\) denotes the eigenvalues of \(\bar{A}\). That is, \(-\alpha\) is the real part of the eigenvalue nearest to the imaginary axis.

Using the Gronwall lemma [12], [14], we get the following theorem.

**Theorem 1:** In problem (I), suppose the nonlinear parametrical uncertainties are bounded by (2) and if we choose the control parameters of (8) such that the approximate closed-loop system (12) is asymptotically stable and the following inequality is satisfied:

\[
\alpha > m[\beta_1 + (\beta_1 + \beta_2\|B\|)][(K_2, K_1)] + \beta_2\|B\|] \tag{15}
\]

then the nonlinear parametrical perturbed closed-loop system (11) is also asymptotically stable (i.e., the nonlinear perturbations \(\Delta A(x), \Delta B(u), \Delta C(x),\) and \(\Delta D(u)\) can be tolerated).

**Proof:** See Appendix A.

**Remarks:**
1. In the input-output modeling case (i.e., transfer function), the nonlinear uncertainties are considered to be static in the loop [9], [10]. In this case, the nonlinear uncertainties are in a dynamic model. As a result, the approach under investigation will be more practical in the true control system.
2. From the above analysis, it is seen that robustness margins are given by the eigenvalue closest to the imaginary axis.
3. If \(A = B = K_1 = 0\) and \(K_2 = -FX,\) i.e., the dynamic controller (8) becomes the state feedback case \(u = -FX\), then the robust stability condition (15) is reduced to \(\alpha > m[\beta_1 + \beta_2\|F\|]\).
4. The inequality (15) is only a sufficient condition. That is, even if (15) does not hold, we cannot say that no robust controller exists.
5. When the parametrical uncertainties are linear in problem (I), i.e., \(\Delta A(x), \Delta B(u), \Delta C(x),\) and \(\Delta D(u)\) in (1) are changed to \(\Delta Ax, \Delta Bu,\) \(\Delta Cx,\) and \(\Delta Du\), respectively, and these uncertainties are bounded by the following inequalities:

\[
\|\Delta A\| \leq \beta_1, \quad \|\Delta B\| \leq \beta_2,
\]

\[
\|\Delta C\| \leq \beta_3, \quad \|\Delta D\| \leq \beta_4. \tag{16}
\]

We can also prove that if we choose parameters \(A, B, K_1,\) and \(K_2\) such that the approximate closed-loop system (12) is asymptotically stable and (15) is satisfied, then the linear parametrical perturbed closed-loop system is also asymptotically stable.

For problem (II), combining (6) and (8) leads to the following closed-loop system with linear model uncertainty \(\Delta h\)

\[
\dot{x} = A\bar{x} + \begin{bmatrix}
0 \\
B, \Delta h(u)
\end{bmatrix}, \quad \dot{x}(0) = \begin{bmatrix}
x_0 \\
x_{0,0}
\end{bmatrix}
\]

\[
y = C\bar{x} + \{\Delta h + (K_1x + K_2x)\}. \tag{17}
\]

Then we get the following theorem.

**Theorem 2:** In problem (II), assume the linear model uncertainty \(\Delta h\) is norm-bounded by (7) and if we choose the dynamic controller (8) such that approximate closed-loop system (12) is asymptotically stable and

\[
1 - \frac{m}{\alpha} \|B\| \|K_1\| > 0 \tag{18}
\]

then closed-loop system (17) with model uncertainty is also asymptotically stable.

**Proof:** See Appendix B.

**Remark:** Suppose the model uncertainty \(\Delta h\) is nonlinear and bounded by the following inequality:

\[
\|\Delta h(u(t))\| \leq r\|u(t)\|, \tag{19}
\]

then change (17) into

\[
\dot{x} = A\bar{x} + \begin{bmatrix}
0 \\
B, \Delta h(u)
\end{bmatrix},
\]

\[
y = C\bar{x} + \Delta h(u). \tag{20}
\]

and this problem is reduced to problem (I) with \(\Delta A(x) = \Delta B(u) = \Delta C(x) = 0,\) and \(\Delta D(u) = \Delta h(u).\)

From the above analysis, a controller design procedure is proposed as follows:

1. Check the upper norm-bounds of the system uncertainties and adequately give negative eigenvalues \(\lambda_i, i = 1, 2, \cdots, n\), where \(n = n_1 + n_2\).
2. Choose an adequate matrix \(U\) to compute \(\bar{A} = U^{-1}\text{diag}[\lambda_1, \cdots, \lambda_n]U\) and get \(\alpha\). (Since \(-\alpha\) is determined by the real part of the eigenvalue of \(\bar{A}\) nearest to the imaginary axis.)
3. Find the corresponding \(A, B, K_1, K_2\) by the eigenvalue assignment technique and check if the robust stability condition is satisfied. If so, go to step 5).
4. Shift eigenvalues to the left by \(\beta_1, \beta_2, \cdots, \beta_n\).
5. Obtain the robust controller from (8).

**Example:** Consider the following linear dynamic system:

\[
\dot{x} = \begin{bmatrix}
-4 \\
-2 \\
3
\end{bmatrix} x + \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} u
\]

\[
y = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} x, \quad x_0 = \begin{bmatrix}
1 \\
1
\end{bmatrix}. \tag{21}
\]

Suppose this system has the following nonlinear parametrical perturbations:

\[
\dot{x} = \begin{bmatrix}
-4 \\
-2 \\
3
\end{bmatrix} x + \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} u + \begin{bmatrix}
-\sin(x_1) \\
0.5\sqrt{|x_3|} \\
0.1\sin(u_1) + 0.1\sin(u_2)
\end{bmatrix}, \quad y = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} x, \quad x_0 = \begin{bmatrix}
1 \\
1
\end{bmatrix}. \tag{21}
\]
How do we design a robust controller to stabilize the linear dynamic system (21) under the perturbations such as (22)?

**Solution:** From Theorem 1, we get

\[
\|\Delta A(x)\| \leq 1.5\|x\|, \quad \|\Delta B(u)\| \leq 0.2\|u\|,
\]

i.e.,

\[
\beta_1 = 1.5, \quad \beta_2 = 0.2, \quad \beta_3 = 0, \quad \beta_4 = 0.
\]

From the above design algorithm, we get the eigenvalues of \(A\) as \(-3, -4, -6, -8\) (i.e., \(\alpha = 3\)) and the dynamical controller as

\[
x = \begin{bmatrix} -5 & 0 \\ 1 & -7 \end{bmatrix} x + \begin{bmatrix} -3 \\ 0 \\ -12 \end{bmatrix} y,
\]

\[
u = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} -5 & 2 \\ 1 & -3 \end{bmatrix} x
\]

then robust stability inequality (15) is satisfied and dynamic controller (23) is a robust controller which can stabilize perturbed system (22). The simulation result is shown in Fig. 3.

**IV. CONCLUSION**

In this note, a new robust controller design in the time domain has been introduced for the multivariable system with linear, nonlinear, or time-varying model uncertainties. We have employed the Gronwall inequality to investigate the robust stability condition. Our design work is simplified to choose the dynamical control parameters to satisfy the requirement of robust stability.

**APPENDIX A**

**PROOF OF THEOREM 1**

From (11), we get

\[
\dot{x}(t) = \Phi(t) x_0 + \int_0^t \Phi(t-r) \Delta A(x(r)) \, dr.
\]

Performing the operator \(\|\cdot\|\) to both sides of (A1), we get

\[
\|\dot{x}(t)\| \leq \|\Phi(t)\| \|x_0\| + \int_0^t \|\Phi(t-r)\| \|\Delta A(x(r))\| \, dr.
\]

From (10), we get

\[
\|\Delta A(x(t))\| = \left\| \frac{\Delta A(x(t))+\Delta B(u)}{B_{\text{d}}[\Delta C(x(t))+\Delta D(u)]} \right\|
\leq \|\Delta A(x(t))+\Delta B(u)\| + \|B_{\text{d}}[\Delta C(x(t))+\Delta D(u)]\|
\leq \|\Delta A(x(t))\| + \|\Delta B(u)\| + \|B_{\text{d}}[\Delta A(x(t))+\Delta D(u)]\|
\leq \beta_1 \|x(t)\| + \beta_2 \|u\| + \|B_{\text{d}}[\Delta A(x(t))+\Delta D(u)]\|
\leq \beta_1 \|x(t)\| + \beta_2 \|u\| + \|B_{\text{d}}[\Delta A(x(t))+\Delta D(u)]\|
\]

and

\[
\|\Delta C(\dot{x}(t))\| = \|\Delta C(x(t))+\Delta D(u)\|
\leq \|\Delta C(x(t))\| + \|\Delta D(u)\|
\leq \beta_1 \|x(t)\| + \beta_2 \|u\|
\]
Performing the operator $\| \cdot \|$ to both sides, we get
\[
\| x \| \leq \| \Phi \| \| x_0 \| + \| \Phi \| \left( \sum_{j=k}^\infty B_j \Delta h \ast (K_j x_j + K_2 x) \right)
\]
\[
\leq \| \Phi \| \| x_0 \| + \| \Phi \| B_0 \Delta h \ast (K_0 x_0 + K_2 x).
\]
Integrating both sides, we get
\[
\int_0^m \| x \| \, dt \leq \int_0^m \| \Phi \| \, dt \| x_0 \| + \int_0^m \| \Phi \| \, dt \left( \int_0^m \| B_j \| \left( \int_0^m \| K_j x_j + K_2 x \| \, dt \right) \right)
\]

Since
\[
\int_0^m \| a * b \| \, dt \leq \int_0^m \| a \| \, dt \int_0^m \| b \| \, dt \int_0^m \| c \| \, dt \quad (B2)
\]
and
\[
\int_0^m \| \Delta h \| \, dt \leq \int_0^m r \exp(-\beta t) \, dt = \frac{r}{\beta}
\]
We get
\[
\int_0^m \| x \| \, dt \leq \| \Phi \| \int_0^m \| b \| \, dt \int_0^m \| c \| \, dt
\]
\[
\leq \frac{m}{\alpha} \| x_0 \| + \frac{mr}{\alpha \beta} \| B_m \| \left( \int_0^m \| K_j \| \| K_2 \| \right) \int_0^m \| x \| \, dt
\]
i.e.,
\[
\left( 1 - \frac{mr}{\alpha \beta} \| B_m \| \left( \int_0^m \| K_j \| \| K_2 \| \right) \right) \int_0^m \| x \| \, dt \leq \frac{m}{\alpha} \| x_0 \|
\]
If
\[
\left( 1 - \frac{mr}{\alpha \beta} \| B_m \| \left( \int_0^m \| K_j \| \| K_2 \| \right) \right) > 0
\]
then
\[
\int_0^m \| x \| \, dt \leq \frac{m}{\alpha} \| x_0 \|
\]
\[
1 - \frac{mr}{\alpha \beta} \| B_m \| \left( \int_0^m \| K_j \| \| K_2 \| \right)
\]
Since
\[
y = Cx + [\Delta h \ast (K_x x + K_2 x)]
\]
Performing the operator $\| \cdot \|$ to both sides, we get
\[
\| y \| \leq \| C \| \| x \| + \| \Delta h \ast (K_x x + K_2 x) \|
\]
Integrating both sides, we get
\[
\int_0^m \| y \| \, dt \leq \int_0^m \| C \| \| x \| \, dt + \int_0^m \| \Delta h \ast (K_x x + K_2 x) \| \, dt
\]
\[
\leq \| C \| \int_0^m \| x \| \, dt + \int_0^m \| \Delta h \| \, dt \left( \int_0^m \| K_x x + K_2 x \| \, dt \right)
\]
\[
\leq \| C \| \int_0^m \| x \| \, dt + \left( \int_0^m \| K_x \| \| K_2 \| \right) \int_0^m \| \Delta h \| \, dt \int_0^m \| x \| \, dt
\]
\[
\leq \| C \| \int_0^m \| x \| \, dt + \frac{r}{\beta} \left( \int_0^m \| K_2 \| \| K_2 \| \right) \int_0^m \| x \| \, dt
\]

Assume the output signal is uniformly continuous, so [8]
\[
\| y(t) \| \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty.
\]

REFERENCES


**An Algorithm for Computing the Roots of a Complex Polynomial**

JOHN F. AURAND

Abstract—An algorithm is proposed for determining all the roots of a polynomial with complex coefficients. A description is given of the method and an example application is included. The method offers guaranteed convergence, a straightforward structure, and excellent performance (including multiple root extraction).

I. INTRODUCTION

This note presents an algorithm for determining the roots of a polynomial with complex coefficients, with guaranteed convergence and

Manuscript received June 24, 1986; revised September 24, 1986.

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IEEE Log Number 8611954.