# Lesson 15 Potential Theory Using Complex Analysis (EK 18)

■ Introduction

"Potentials" in physics can simplify the derivation of forces. They are typically described by solutions to Laplace's equation $\nabla^2\Phi=0$. The solutions are called "harmonic functions" if they have continuous 2nd partial derivatives.

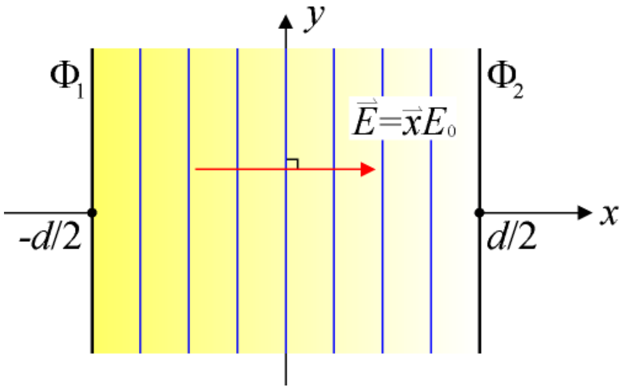
■ Why using complex analysis?

The real and imaginary parts of a complex analytic function $F(z=x+iy)=\Phi(x,y)+i\Psi(x,y)$ are satisfied with 2-D Laplace's equation: $\nabla^2\Phi=\nabla^2\Psi=0$ (proved by CR-conditions, Lesson 9). If $\Phi(x,y)$ represents the potential function, by working with $F(z)$, we can: (1) handle both equipotential lines ($\Phi$=constant) and lines of force ($\Psi$=constant) simultaneously; (2) solve Dirichlet problems with complicated boundary geometry by introducing another analytic transformation function $f(z)$ for conformal mapping.

## Complex Potential

■ Examples of real potentials

1) Parallel plates: The electrostatic potential $\Phi$ between two parallel conducting plates is governed by: Laplace's equation: $\nabla^2\Phi=\Phi''(x)=0$, and BCs: $\Phi\left(-\frac{d}{2}\right)=\Phi_1$, $\Phi\left(\frac{d}{2}\right)=\Phi_2$.



The solution is a linear function:

$$\Phi(x)=ax+b \tag{15.1}$$

where $a=-\dfrac{\Phi_1-\Phi_2}{d}$, $b=\dfrac{\Phi_1+\Phi_2}{2}$. The equipotential line $\Phi=\Phi_0$, is a vertical line $x=x_0$ parallel to the plates. The E-field is $\vec{E}=-\nabla\Phi$, $\Rightarrow$

$$\vec{E}=-a\vec{x} \tag{15.2}$$

which is constant and perpendicular to the plates.

Note: Infinite dimension (along $y$-axis) causes constant potential ($\Phi$ is independent of $y$).

2) Coaxial cylinder: If $\Phi$ is independent of $\theta$, $\nabla^2\Phi= r^2\Phi_{rr}+r\Phi_r=0$, $\Rightarrow$

$$\Phi(r)= a\ln r+ b \tag{15.3}$$

where $a$, $b$ are determined by BCs [$\Phi(r_1)=\Phi_1$, $\Phi(r_2)=\Phi_2$]. The equipotential line $\Phi=\Phi_0$ is a **circle** $r=r_0$. The E-field is $\vec{E}=-\nabla\Phi$, $\Rightarrow$

$$\vec{E}=-\frac{a}{r}\vec{r} \tag{15.4}$$

which is in **radial** direction, perpendicular to the equipotential lines.

3) Angular region: If the region of interest is confined by two plates in the radial directions and with an included angle $\alpha$, it is difficult to directly solve the potential $\Phi$ by traditional methods. Instead, we can borrow the concept of analytic complex functions:

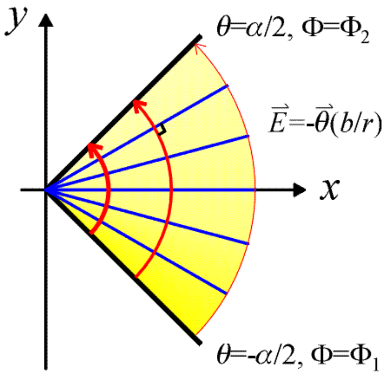
**(1)** To be satisfied with the two BCs: $u\left(\theta=-\dfrac{\alpha}{2}\right)=\Phi_1$, $u\left(\theta=\dfrac{\alpha}{2}\right)=\Phi_2$, $u(x,y)=\tan^{-1}\left(\dfrac{y}{x}\right)$ ($=\theta$ in the polar coordinates) is a choice. **(2)** $u(x,y)$ is also satisfied with 2-D Laplace's equation for it is the imaginary part of an analytic function: $F(z)=\text{Ln}z=\ln|z|+i\cdot\text{Arg}(z)$ [eq. (9.7)]. By (1-2), $\Rightarrow$

$$\Phi=a+b\theta \tag{15.5}$$

where $a=\dfrac{\Phi_1+\Phi_2}{2}$, $b=\dfrac{\Phi_2-\Phi_1}{\alpha}$. The equipotential line $\Phi=\Phi_0$ is a **ray** $\theta=\theta_0$. The E-field is $\vec{E}=-\nabla\Phi$, $\Rightarrow$

$$\vec{E}=-\frac{b}{r}\vec{\theta} \tag{15.6}$$

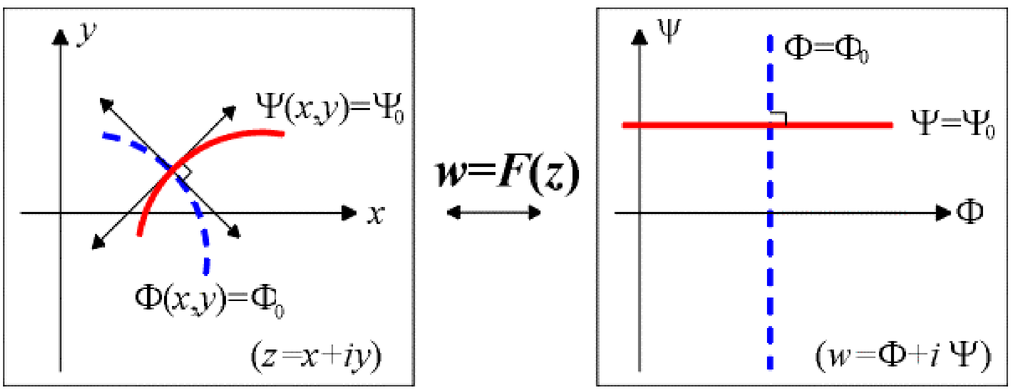
which is in azimuthal (方位角) direction, perpendicular to the equipotential lines.




■ Concept of complex potential

For a given real potential $\Phi(x,y)$, we can uniquely (except for an additive constant) determine a conjugate $\Psi(x,y)$ by CR-equations, such that complex potential $F(z=x+iy)=\Phi(x,y)+i\Psi(x,y)$ is analytic. As a result, $F(z)$ maps curves in the $xy$-plane onto curves in the $\Phi\Psi$-plane "conformally" or vise versa, i.e. included angle is preserved during mapping (Appendix 9A).

Since $\Phi=\Phi_0$ (vertical line) is always perpendicular to $\Psi=\Psi_0$ (horizontal line) in the $\Phi\Psi$-plane, $\Rightarrow$ the corresponding curves in the $xy$-plane: $\Phi(x,y)=\Phi_0$ (equipotential line), and $\Psi(x,y)=\Psi_0$ always make a right angle as well.




Since gradient defines the steepest ascent/descent direction, which is always perpendicular with the equipotential lines (zero-variation direction), $\Rightarrow$ E-field $\vec{E}=-\nabla\Phi$ is perpendicular with $\Phi(x,y)$=constant, $\Rightarrow$ **$\Psi(x,y)=\Psi_0$** stands for a **E-field (force) line**.

■ Examples of deriving lines of force by complex potential

1) Parallel plates: By eq. (15.1), $\Phi(x)=ax+b$. By CR-equations, its conjugate is:

$$\Psi=ay+c \tag{15.7}$$

$\Rightarrow$ complex potential $F(z)=(ax+b)+i(ay+c)=az+d$, which is analytic. The E-field lines are $\Psi=$ constant, $\Rightarrow$ $y$=constant, same as eq. (15.2).

2) Coaxial cylinders: By eq. (15.3), $\Phi=a\ln r+b=a\ln|z|+b$. By CR-equations, its conjugate is:

$$\Psi=a\mathrm{Arg}(z)+c \tag{15.8}$$

$\Rightarrow$ complex potential $F(z)=(a\cdot\ln|z|+b)+i(a\mathrm{Arg}(z)+c)=a\mathrm{Ln}\,z+d$, which is analytic except for 0 and points on the negative real axis. The E-field lines are $\Psi$=constant, $\Rightarrow$ $\mathrm{Arg}(z)=\theta$=constant, same as eq. (15.4).

3) Angular region: By eq. (15.5), $\Phi=a+b\theta=a+b\mathrm{Arg}(z)$, which is the imaginary part of $(c+ia)+b\mathrm{Ln}\,z$, or the real part of $F(z)=(a+id)-ib\mathrm{Ln}\,z$; $\Rightarrow$ $\Psi=\mathrm{Im}[F(z)]$,

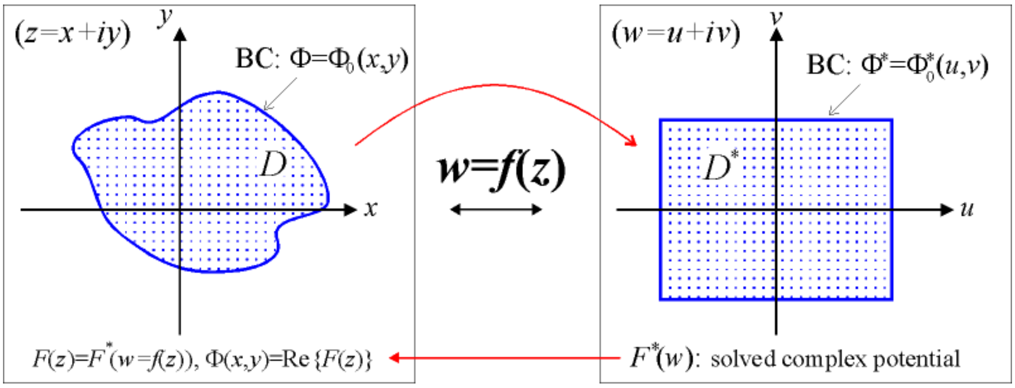
$$\Psi=d-b\ln|z| \tag{15.9}$$

The E-field lines are $\Psi=$ constant, $\Rightarrow$ $|z|=r$=constant, same as eq. (15.6).

## Solving Dirichlet Potential Problems by Conformal Mapping (SJF 47)

■ Concept

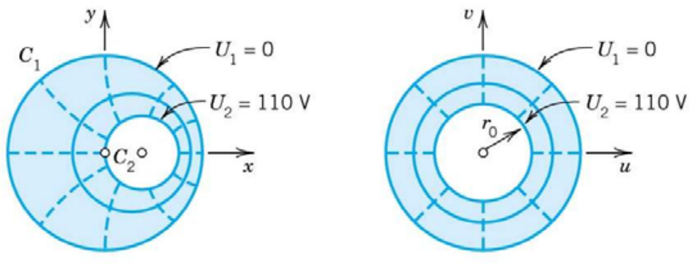
Find an analytic transformation function $f(z)$ to map a complicated domain $D$ (in the $z$-plane) onto a simpler domain $D^*$ (in the $w$-plane), where the complex potential $F^*(w)$ can be easily solved [$\mathrm{Re}\{F^*(w)\}=\Phi^*(w)$ is satisfied with 2-D Laplace's equation and boundary conditions in the $w$-plane]. Then the complex potential in the $z$-plane is derived by inverse transform: $F(z)=F^*(w)|_{w=f(z)}$, from which the real potential is: $\Phi(x,y)=\mathrm{Re}\{F(z=x+iy)\}$.




The strategy works because harmonic functions remain harmonic under conformal mapping.

Proof: $f(z)$ and $F^*(w)$ are analytic (s.t. Re$\{F^*(w)\}$ is harmonic). By the chain rule, $F(z)$ is also analytic: $F'(z)=\dfrac{dF^*}{dw}\cdot f'(z)\big|_{w=f(z)}$ exists. $\Rightarrow$ $\Phi(x,y)$=Re$\{F(z)\}$ is harmonic.

**E.g.** Non-coaxial cylinders: $C_1$: $|z|=1$, $C_2$: $\left|z-\dfrac{2}{5}\right|=\dfrac{2}{5}$; $U_1=0$, $U_2=110$.




(a) z-plane (b) w-plane

Direct solution in the $z$-plane is difficult. By using linear fractional transformation: $w=f(z)=\dfrac{z-1/2}{(z/2)-1}$ (EK 17.2–17.4), domain $D$ is mapped onto $D^*$ in the $w$-plane, consisting of two concentric circles: $C_1^*$ : $|w|=1$, $C_2^*$ : $|w|=\dfrac{1}{2}$; with BCs: $U_1=0$, $U_2=110$.

The complex potential in the $w$-plane is: $F^*(w)=a\cdot\text{Ln}\,w+k$, where $a$, $k$ can be solved by BCs: $\Phi^*\left(|w|=1\right)=0$, $\Phi^*\left(|w|=\dfrac{1}{2}\right)=110$.

The complex potential in the $z$-plane is: $F(z)=F^*(w)|_{w=f(z)}=a\cdot\text{Ln}\left(\dfrac{2z-1}{z-2}\right)$, and the real potential is: $\Phi(x,y)=\text{Re}\{F(z)\}=a\cdot\ln\left(\left|\dfrac{2z-1}{z-2}\right|\right)$.

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The equipotential lines $\Phi(x,y)=\Phi_0$, $\Rightarrow$ $\left|\dfrac{2z-1}{z-2}\right|$=constant, are circles (with different centers) in the $z$-plane (see plot); corresponding to concentric circles in the $w$-plane. The lines of force are circular arcs (see plot), corresponding to rays Arg($w$)=constant in the $w$-plane.