Lesson 14 Integral by Residue Theorem (EK 16)

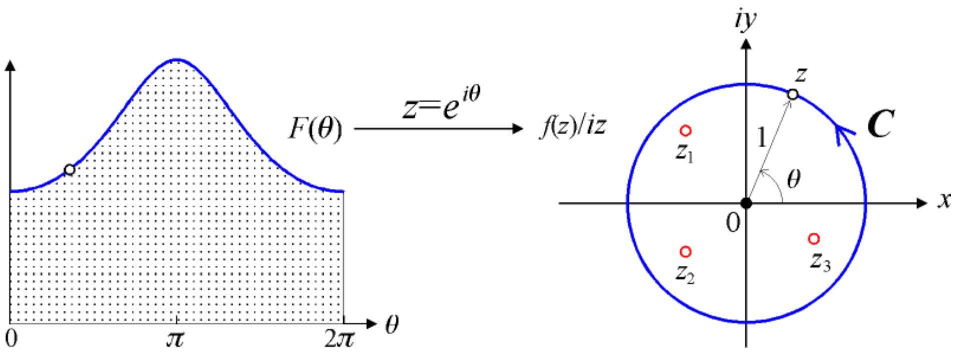
■ Introduction

One of the most important applications of complex variables is evaluating difficult real and complex integrals by residue theorem. Here we simply discuss those with standard formulae.

Integral of rational functions of cosθ & sinθ

$$I=\int_0^{2\pi} F(\cos\theta,\sin\theta)d\theta=\oint_C \frac{f(z)}{iz}dz=2\pi\cdot\sum_{z_k \text{ inside } C}\operatorname{Res}_{z_k}\left[\frac{f(z)}{z}\right] \qquad (14.1)$$

where $f(z)=F\left(\frac{z+z^{-1}}{2},\frac{z-z^{-1}}{2i}\right)$, C: $|z|=1$ in counterclockwise sense.



Proof: by change of variable: $z=e^{i\theta}$, $\Rightarrow$ $\cos\theta=\frac{z+z^{-1}}{2}$, $\sin\theta=\frac{z-z^{-1}}{2i}$, $\frac{dz}{d\theta}=iz$. As θ changes from 0 to 2π, z traverses through the entire unit circle once in counterclockwise sense.

E.g. $I=\int_0^{2\pi}\frac{1}{a+b\cos\theta}d\theta$, where a, $b\in R$, $|a|>|b|$. $\Rightarrow f(z)=\frac{1}{a+b\left[(z+z^{-1})/2\right]}$, $I=\frac{2}{ib}\oint_C g(z)dz$, where $g(z)=\frac{1}{z^2+(2a/b)z+1}$ has two simple poles: $z_1=-\frac{a}{b}+\sqrt{(a/b)^2-1}$, $z_2=-\frac{a}{b}-\sqrt{(a/b)^2-1}$. Since z_2 lies outside the unit circle C, we only calculate $\operatorname{Res}(z_1)=\frac{1}{z_1-z_2}=\frac{b}{2\sqrt{a^2-b^2}}$. $\Rightarrow I=\frac{2}{ib}\cdot 2\pi i\cdot\frac{b}{2\sqrt{a^2-b^2}}=\frac{2\pi}{\sqrt{a^2-b^2}}$.

Improper integral (瑕積分)

■ Definition

Improper integral $I=\int_{-\infty}^{\infty} f(x)dx$ exists, if the two limits: $I_1=\lim_{a\to-\infty}\int_a^0 f(x)dx$ and $I_2=\lim_{b\to\infty}\int_0^b f(x)dx$ exist simultaneously. If either I_1 or I_2 is divergent, we may try to evaluate its Cauchy principal value

$$\text{p.v.}\int_{-\infty}^{\infty} f(x)dx \equiv \lim_{r\to\infty}\int_{-r}^{r} f(x)dx \qquad (14.2)$$

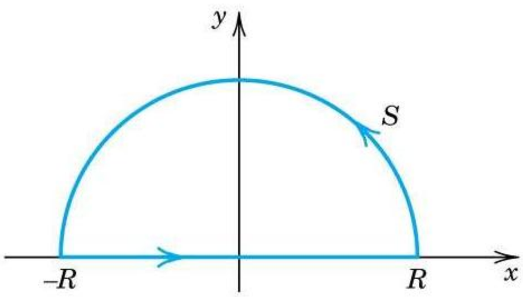
Mostly, the principle value equals $I=\int_{-\infty}^{\infty} f(x)dx$ (making the following formulas useful). In some rare occasions, however, the principle value exists while I does not. **E.g.** p.v. $\int_{-\infty}^{\infty} xdx = \lim_{r\to\infty}\left(x^2/2\right)\Big|_{-r}^{r}=0$, but $I_1=\lim_{a\to-\infty}\int_a^0 xdx$, and $I_2=\lim_{b\to\infty}\int_0^b xdx$ are divergent.

■ Improper integral of rational functions

If $f(x)=p(x)/q(x)$ is satisfied with: (1) $q(x)\neq 0$ for all real x, (2) $\deg[q(x)]-\deg[p(x)]\geq \mathbf{2}$, $\Rightarrow$

$$\text{p.v.}\int_{-\infty}^{\infty} f(x)dx = 2\pi i\sum_{\text{UHP}}\text{Res}[f(z)] \qquad (14.3)$$

where Σ is performed for all residues of singularities located in the upper half-plane (UHP).



Proof: Choose an "upper semi-circle" C as shown above. $\oint_C f(z)dz=\int_{-R}^{R} f(x)dx+\int_S f(z)dz$, by the residue theorem eq. (13.2), $=2\pi i\cdot\left\{\sum_{z\text{ inside }C}\text{Res}[f(z)]\right\}$. As $R\to\infty$:

(1) $\int_{-R}^{R} f(x)dx \to$ p.v.$\int_{-\infty}^{\infty} f(x)dx$. **(2)** Since deg[$q(x)$]−deg[$p(x)$]≥2, $|f(z)|=\left|\frac{p(z)}{q(z)}\right|_{z=Re^{i\theta}} < \frac{M}{R^2}$ for all points on S. By ML-inequality, $\left|\int_S f(z)dz\right| < \frac{M}{R^2}\cdot \pi R = \frac{M\pi}{R} \to 0$. **(3)** $\sum_{z \text{ inside } C} \text{Res}[f(z)] \to \sum_{\text{UHP}} \text{Res}[f(z)]$.

Note the condition $q(x)\neq 0$ guarantees that $f(x)$ has no singularity (residue) on the real axis.

E.g. $I=\int_0^{\infty} \frac{1}{1+x^4}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{1+x^4}dx$. $f(z)=\frac{1}{1+z^4}$ has four simple poles at $z_1=e^{i\frac{\pi}{4}}$, $z_2=e^{i\frac{3\pi}{4}}$, $z_3=e^{i\frac{5\pi}{4}}$, $z_4=e^{i\frac{7\pi}{4}}$; where z_1, z_2 fall on the UHP. By eq. (13.5), ⇒ $\text{Res}(z_1)=\frac{1}{4z_1^3}=\frac{1}{4}e^{-i\frac{3\pi}{4}}$, $\text{Res}(z_2)=\frac{1}{4}e^{-i\frac{\pi}{4}}$, $I=\frac{1}{2}2\pi i\left\{\frac{1}{4}e^{-i\frac{3\pi}{4}}+\frac{1}{4}e^{-i\frac{\pi}{4}}\right\}=\frac{\pi}{2\sqrt{2}}$.

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Since the singularities of $f(z)$, i.e. roots of $q(z)=0$, are always in complex conjugate pairs (虛根成雙), singularities in the lower half-plane (LHP) have the same influence on the integral:

$$\text{p.v.}\int_{-\infty}^{\infty} f(x)dx = -2\pi i \sum_{\text{LHP}} \text{Res}[f(z)] \tag{14.4}$$

which can be proven by a “lower semi-circle” contour.

■ Fourier transform of rational functions

The Fourier transform of $f(x)$ involves: $I=\int_{-\infty}^{\infty} f(x)e^{isx}dx$, If $f(x)= p(x)/q(x)$ is satisfied with: (1) $q(x)\neq 0$ for all real x, (2) deg[$q(x)$]−deg[$p(x)$] ≥**1**, and (3) $s>0$;⇒

$$\text{p.v.}\int_{-\infty}^{\infty} f(x)e^{isx}dx = 2\pi i \sum_{\text{UHP}} \text{Res}\left[f(z)e^{isz}\right] \tag{14.5}$$

Note we evaluate the residues of singularities of $f(z)e^{isz}$ [not $f(z)$] located in the UHP.

Proof: Choose the same "upper semi-circle" C as before. By the residue theorem eq. (13.2),

$$\oint_C f(z)e^{isz}dz=\int_{-R}^{R} f(x)e^{isx}dx+\int_S f(z)e^{isz}dz=2\pi i\cdot\sum_{z\text{ inside }C}\text{Res}\left[f(z)e^{isz}\right].$$

Since $\deg[q(x)]-\deg[p(x)]\geq 1$, $|f(z)|<\dfrac{M}{R}$. For any point z on S: $z=Re^{i\theta}$, $\theta=[0,\pi]$, $\Rightarrow$ $|e^{isz}|=$ $\left|e^{isR(\cos\theta+i\sin\theta)}\right|$ = $e^{-sR\sin\theta}$ = $e^{-\delta R}$, where $\delta=s\cdot\sin\theta\geq 0$ ($s>0$, $\sin\theta\geq 0$). By ML-inequality,

$$\left|\int_S f(z)e^{isz}dz\right|<\frac{M}{R}\cdot e^{-\delta R}\cdot\pi R=M\pi e^{-\delta R}\to 0,\text{ as }R\to\infty.$$

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1) The e^{isz} term in eq. (14.5) provides the integrand with an exponential damping factor $e^{-\delta R}$ (if $\delta>0$) over path S, therefore, we only need $\deg[q(x)]-\deg[p(x)]\geq 1$.
2) Damping factor vanishes at end points $z=\pm R$ ($\theta=0,\pi$; $\Rightarrow$ $\delta=0$), but they do not contribute to the integral as long as they are bounded ($|f(z=\pm R)|<\infty$), which is guaranteed by $q(x)\neq 0$.
3) An alternative to eq. (14.5) is:

$$\text{p.v.}\int_{-\infty}^{\infty} f(x)e^{isx}dx=-2\pi i\sum_{\text{LHP}}\text{Res}\left[f(z)e^{isz}\right],\text{ if }s<0 \tag{14.6}$$

because $e^{-sR\sin\theta}$ acts as a damping factor in the LHP ($s<0$, $\sin\theta\leq 0$, for $\theta=[\pi,2\pi]$).

E.g. Find Fourier transform of $f(x)=\dfrac{1}{x^2+a^2}$.

Ans: $F(\omega)=\displaystyle\int_{-\infty}^{\infty}\frac{e^{-i\omega x}}{x^2+a^2}dx$. $\dfrac{e^{-i\omega z}}{z^2+a^2}$ has two simple poles at $z_1=ia$, $z_2=-ia$. $\text{Res}(z_1)=\dfrac{e^{-i\omega z_1}}{2z_1}$ $=\dfrac{e^{a\omega}}{i2a}$, $\text{Res}(z_2)=\dfrac{e^{-a\omega}}{-i2a}$. **(1)** For $\omega>0$ ($s=-\omega<0$), use eq. (14.6): $F(\omega)=-2\pi i\cdot\text{Res}(z_2)=\dfrac{\pi}{a}e^{-a\omega}$;
(2) For $\omega<0$ ($s>0$), use eq. (14.5): $F(\omega)=2\pi i\cdot\text{Res}(z_1)=\dfrac{\pi}{a}e^{a\omega}$; $\Rightarrow F(\omega)=\dfrac{\pi}{a}e^{-a|\omega|}$.

To evaluate $\displaystyle\int_{-\infty}^{\infty} f(x)\begin{Bmatrix}\cos sx\\ \sin sx\end{Bmatrix}dx$, we can:

1) Find p.v. $\int_{-\infty}^{\infty} f(x)e^{isx}dx$ by eq's (14.5-6), then take real or imaginary part. This method is valid only if $f(x)\in \boldsymbol{R}$.

2) Substitute $\cos(sx)=\dfrac{e^{isx}+e^{-isx}}{2}$, $\sin(sx)=\dfrac{e^{isx}-e^{-isx}}{2i}$, then evaluate residues of singularities on both UHP and LHP. This method is valid even $f(x)\in \boldsymbol{C}$.

E.g. $I=\int_{-\infty}^{\infty}\dfrac{\sin x}{x+i}dx=\dfrac{1}{2i}\left[\int_{-\infty}^{\infty}\dfrac{e^{ix}}{x+i}dx-\int_{-\infty}^{\infty}\dfrac{e^{-ix}}{x+i}dx\right]=\dfrac{1}{2i}(I_1-I_2)$. $I_1=2\pi i\cdot\sum_{\text{UHP}}\text{Res}\left(\dfrac{e^{iz}}{z+i}\right)$ $=0$, $I_2=-2\pi i\cdot\sum_{\text{LHP}}\text{Res}\left(\dfrac{e^{-iz}}{z+i}\right)=-2\pi i\cdot\text{Res}(-i)=\dfrac{-2\pi i}{e}$, $\Rightarrow I=\dfrac{\pi}{e}$.

Indented (避點) integral

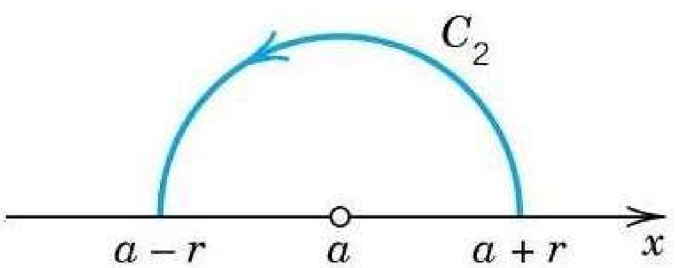
■ What happens if there is singularity on the integral path?

If $f(z)$ has a real singularity $z=a$, we have to redefine the Cauchy principal value of the improper integral as:

$$\text{p.v.}\int_{-\infty}^{\infty} f(x)dx \equiv \lim_{R\to\infty, r\to 0}\left\{\int_{-R}^{a-r} f(x)dx+\int_{a+r}^{R} f(x)dx\right\} \tag{14.7}$$

where there is an infinitesimal interval $x=[a-r,a+r]$ excluded from the integral path.

To evaluate eq. (14.7) by the residue theorem (where a "closed" contour is required), we can insert a semi-circle path C_2: $z=a+re^{i\theta}$ ($\theta=[0,\pi]$) to bridge the gap.



If $z=a\in R$ is a **simple pole** of $f(z)$, its Laurent series becomes: $f(z)=\dfrac{b_1}{z-a}+g(z)$, where $g(z)$ is analytic. $\Rightarrow$ $\int_{C_2} f(z)dz = b_1\int_0^{\pi}\frac{1}{re^{i\theta}}ire^{i\theta}d\theta + \int_{C_2} g(z)dz = b_1\pi i + \int_{C_2} g(z)dz$. Since $|g(z)|<M$ (analytic), $\left|\int_{C_2} g(z)dz\right| \le M\pi r \to 0$, as $r\to 0$. $\Rightarrow$

$$\lim_{r\to 0}\int_{C_2} f(z)dz = \pi i\cdot \mathrm{Res}(a) \qquad (14.8)$$

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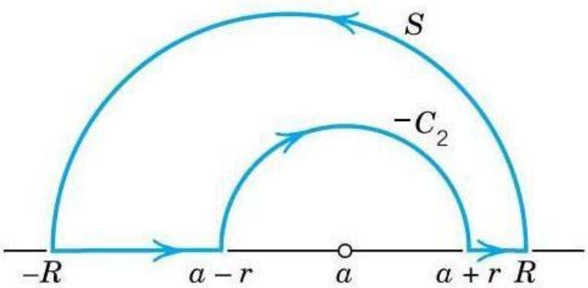
1) It can be generalized to: $\lim_{r\to 0}\int_{C_2} f(z)dz = (\theta_2-\theta_1)i\cdot\mathrm{Res}(a)$, if C_2: $z=a+re^{i\theta}$, $\theta=[\theta_1, \theta_2]$.

2) Eq. (14.8) is normally invalid if $z=a$ is a higher order pole (except for odd functions about a). **E.g.** if $z=a\in R$ is a 2nd-order pole of $f(z)$, its Laurent series contains $b_2(z-a)^{-2}$, contributing to $\int_{C_2} f(z)dz$ by $b_2\int_0^{\pi}\frac{ire^{i\theta}}{r^2e^{i2\theta}}d\theta \to \infty$, as $r\to 0$.

■ Indented integral of rational functions with real simple poles

If $f(x)=\frac{p(x)}{q(x)}$ is satisfied with: (1) $q(x)=0$ for $x=x_1,.., x_N$ ($m=1$), (2) deg[$q(x)$]−deg[$p(x)$]≥**2**, $\Rightarrow$

$$I = \text{p. v.}\int_{-\infty}^{\infty} f(x)dx = 2\pi i\sum_{\text{UHP}}\mathrm{Res}(f(z)) + \pi i\sum_{z=x_i}\mathrm{Res}[f(z)] \qquad (14.9)$$



Proof: $\oint_C f(z)dz = I + \lim_{r\to 0}\int_{-C_2} f(z)dz + \lim_{R\to\infty}\int_S f(z)dz = 2\pi i\sum_{\text{UHP}}\mathrm{Res}[f(z)]$. (i) By eq. (14.8), $\lim_{r\to 0}\int_{C_2} f(z)dz = \pi i\cdot\mathrm{Res}(a)$. (ii) By condition (2), $\lim_{R\to\infty}\int_S f(z)dz = 0$.

E.g. Evaluate $I=\int_{-\infty}^{\infty}\frac{\sin x}{x}dx=\int_{-\infty}^{\infty}(\text{sinc } x)dx$.

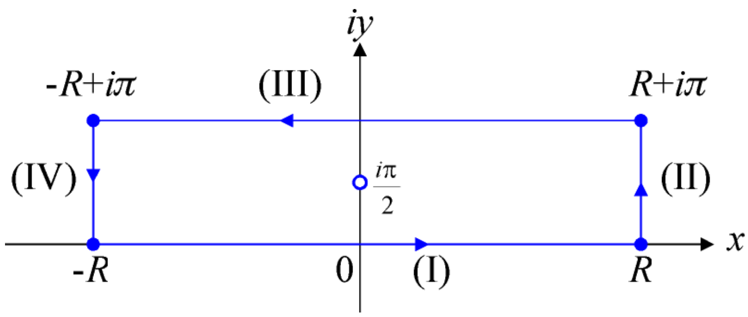
Ans: Choose $f(z)=\frac{e^{iz}}{z}$, which only has a simple pole $z=0$ on the real axis. Res(0)=1,

p.v. $\int_{-\infty}^{\infty}\frac{e^{ix}}{x}dx=\pi i\cdot\text{Res}(0)=\pi i.\Rightarrow I=\text{Im}(\pi i)=\pi$.

■ Indented integral of exponential functions e^x, e^{-x^2}

In evaluating improper integrals by residue theorem, semi-circles are not the only possible integral contours. In the presence of exponential functions, we often use rectangular paths.

E.g. $I=\int_{-\infty}^{\infty}\frac{\cos\alpha x}{e^x+e^{-x}}dx=\int_{-\infty}^{\infty}\frac{2\cos\alpha x}{\cosh x}dx$. $f(z)=\frac{e^{i\alpha z}}{e^z+e^{-z}}$ has simple poles at $z=\frac{i(2n+1)\pi}{2}$, $n\in N$ (roots of $e^z+e^{-z}=0$). Choose a rectangular contour C enclosing single pole $z=i\frac{\pi}{2}$:



$\oint_C f(z)dz=\int_{-R}^{R}f(x)dx+\int_{R}^{R+i\pi}f(z)dz+\int_{R+i\pi}^{-R+i\pi}f(z)dz+\int_{-R+i\pi}^{-R}f(z)dz=$ (I)+(II)+(III)+(IV)= $2\pi i\cdot\text{Res}\left(i\frac{\pi}{2}\right)$. As $R\to\infty$:

(II): $z=R+it$, $t=[0,\pi]$, $\int_{R}^{R+i\pi}f(z)dz=\int_0^{\pi}f(R+it)idt$ (transform into real integral) $=ie^{i\alpha R}\cdot\int_0^{\pi}\frac{e^{-\alpha t}}{e^Re^{it}+e^{-R}e^{-it}}dt\to 0$ (the denominator of the integrand has $e^R\to 0$). **(IV)**: Similar with (II), $\int_{-R+i\pi}^{-R}f(z)dz\to 0$.

(III): $z=x+i\pi$, $x=R\to -R$. $\int_{R+i\pi}^{-R+i\pi}f(z)dz=\int_{R}^{-R}f(x+i\pi)dx=e^{-\alpha\pi}\left(\int_{-R}^{R}\frac{e^{i\alpha x}}{e^x+e^{-x}}dx\right)=$

$e^{-\alpha\pi}\cdot$(I). $\Rightarrow$ (I)+(II)+(III)+(IV)$=(1+e^{-\alpha\pi})\cdot$(I) $\rightarrow$ $(1+e^{-\alpha\pi})\left(\text{p.v.}\int_{-\infty}^{\infty} f(x)dx\right)$;

By eq. (13.5), $\text{Res}\left(i\frac{\pi}{2}\right) = \lim_{z\to(i\pi/2)}\left[\frac{e^{i\alpha z}}{\left(e^{z}+e^{-z}\right)'}\right] = \frac{e^{\frac{-\alpha\pi}{2}}}{2i}$. $\Rightarrow$ $(1+e^{-\alpha\pi})\left(\text{p.v.}\int_{-\infty}^{\infty} f(x)dx\right)=$

$\pi e^{-\frac{\alpha\pi}{2}}$, $\left(\text{p.v.}\int_{-\infty}^{\infty} f(x)dx\right)=\frac{\pi e^{-\frac{\alpha\pi}{2}}}{1+e^{-\alpha\pi}}=\frac{2\pi}{\cosh(\alpha\pi/2)}$. $\Rightarrow I=\text{Re}\left[\text{p.v.}\int_{-\infty}^{\infty} f(x)dx\right]=\frac{2\pi}{\cosh(\alpha\pi/2)}$.

(*) Inverse Laplace Transform

■ Theorem:

Let $F(s)=L\{f(t)\}\equiv\int_0^{\infty} f(p)e^{-sp}dp$, for all $s\in$ region of convergence [ROC, Re(s)>r_0] $\Rightarrow$

$$f(t)=L^{-1}\{F(s)\}=\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}F(s)e^{st}ds\text{, for } t>0,\ r>r_0. \tag{14.10}$$

Proof: Let $I=\frac{1}{2\pi i}\lim_{T\to\infty}\int_{r-iT}^{r+iT}F(s)e^{st}ds=\frac{1}{2\pi i}\lim_{T\to\infty}\int_{r-iT}^{r+iT}e^{st}\left[\int_0^{\infty} f(p)e^{-sp}dp\right]ds$ (if $s\in$ROC). On the path, $s=r+iy$, $I=\frac{1}{2\pi i}\lim_{T\to\infty}\int_{-T}^{T}e^{(r+iy)t}\left[\int_0^{\infty} f(p)e^{-(r+iy)p}dp\right]idy=\frac{e^{rt}}{2\pi}\int_{-\infty}^{\infty}e^{iyt}\left[\int_0^{\infty}e^{-rp}f(p)e^{-iyp}dp\right]dy=$ $\frac{e^{rt}}{2\pi}\left[\int_{-\infty}^{\infty}e^{iyt}G(y)dy\right]=e^{rt}\cdot g(t)$, where $G(y)=\int_0^{\infty}e^{-rp}f(p)e^{-iyp}dp=F\{e^{-rt}f(t)u(t)\}$, $u(t)$ is a unit-step function. $\Rightarrow g(t)=e^{-rt}f(t)u(t)$, $I=e^{rt}g(t)=f(t)u(t)=f(t)$, for $t>0$.

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1) Inverse Fourier transform, $F^{-1}\{F(\omega)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{i\omega t}d\omega$, only involves with a complex function of real variable ω. Therefore, no need to employ complex integration.

2) ROC of $F(s)$ is the point set where the improper integral $\int_0^{\infty} f(p)e^{-sp}dp$ gives a finite number, $\Rightarrow$ all singularities $\{s_k\}$ of $F(s)$ are located outside the ROC, i.e. Re(s_k)<r_0.

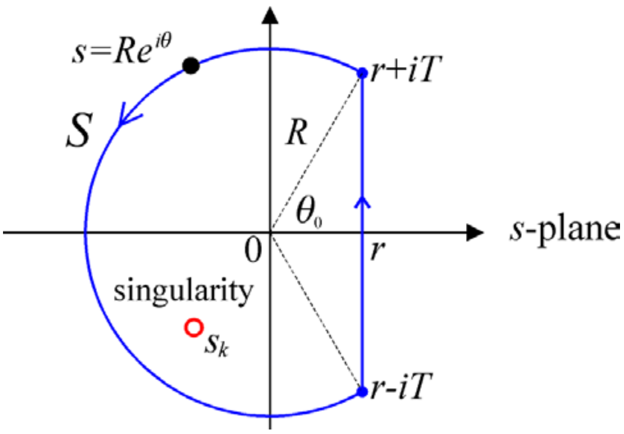
3) Eq. (14.10) is valid if the integral path is within the ROC (i.e. $r>r_0$), $\Rightarrow$ all singularities are located to the left of a valid path, i.e. Re(s_k)<r.

■ Evaluate $\mathscr{L}^{-1}\{\}$ by residue theorem

If $F(s)=L\{f(t)\}=\dfrac{p(s)}{q(s)}$ for Re(s)>r_0, and deg[$q(s)$]−deg[$p(s)$]≥1, ⇒

$$f(t)=\mathscr{L}^{-1}\{F(s)\}=\sum_{\text{all singu.}}\operatorname{Res}_{s_k}\left[e^{st}F(s)\right] \qquad (14.11)$$

where Σ is performed for all singularities of $e^{st}F(s)$.



Proof: Choose a specific closed path C (Bromwich contour), $\oint_C F(s)e^{st}ds=\int_{r-iT}^{r+iT}F(s)e^{st}ds$ $+\int_S F(s)e^{st}ds=2\pi i\cdot\sum_{s_k\text{ inside }C}\operatorname{Res}_{s_k}\left[e^{st}F(s)\right]$. As $T\to\infty$ (i.e. $R\to\infty$, $\theta_0\to\pi/2$):

(1) by eq. (14.10), $\int_{r-iT}^{r+iT}F(s)e^{st}ds\to 2\pi i\cdot f(t)$; **(2)** On the path S, $|F(s)|\le\dfrac{M}{R}$; $s=Re^{i\theta}$, $\theta=[\theta_0,2\pi-\theta_0]$, ⇒ $\left|e^{st}\right|=e^{Rt\cos\theta}=e^{-\delta R}$, where $\delta=-t\cos\theta>0$ ($t>0$, $\cos\theta\le 0$). By ML-inequality, $\left|\int_S F(s)e^{st}ds\right|<\dfrac{M}{R}\cdot e^{-\delta R}\cdot\pi R=M\pi e^{-\delta R}\to 0$. **(3)** $\sum_{s_k\text{ inside }C}\operatorname{Res}_{s_k}\left[e^{st}F(s)\right]\to\sum_{\operatorname{Re}(s_k)<r}\operatorname{Res}_{s_k}\left[e^{st}F(s)\right]=$ $\sum_{\text{all singu.}}\operatorname{Res}_{s_k}\left[e^{st}F(s)\right]$.

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The condition deg[$q(s)$]−deg[$p(s)$]≥1 is sufficient but not necessary for eq. (14.11).

E.g. $f(t)=L^{-1}\left\{\dfrac{1}{(s+1)(s-2)^2}\right\}=?$ $\operatorname{Res}_{s=-1}\left[\dfrac{e^{st}}{(s+1)(s-2)^2}\right]=\dfrac{e^{-t}}{9}$, $\operatorname{Res}_{s=2}\left[\dfrac{e^{st}}{(s+1)(s-2)^2}\right]=\dfrac{te^{2t}}{3}-\dfrac{e^{2t}}{9}$,

$\Rightarrow f(t)=\dfrac{e^{-t}}{9}+\dfrac{te^{2t}}{3}-\dfrac{e^{2t}}{9}$.