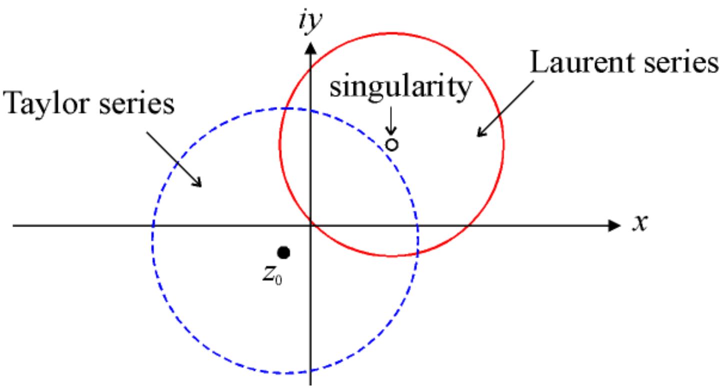
# Lesson 13 Laurent Series and Residue (EK 16)

## Laurent Series (EK 16.1)

■ Motivation

Singularities are routine for complex functions, for bounded entire functions must be constant (Liouville's theorem, Lesson 10). We have to represent complex functions by Laurent series if there is singularity in the region of interest.



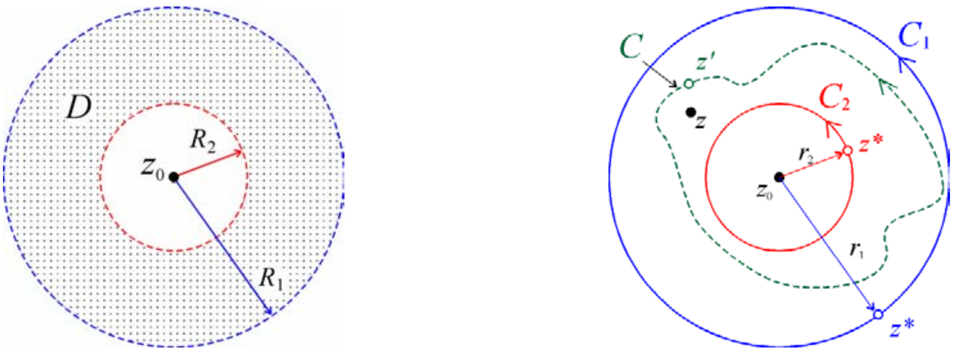

■ Laurent theorem

If $f(z)$ is analytic in an open **annulus** $D$: $R_2<|z-z_0|<R_1$,⇒

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$a_n=\frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}}dz', \qquad b_n=\frac{1}{2\pi i}\oint_C (z'-z_0)^{n-1} f(z')dz' \tag{13.1}$$

for every point $z\in D$, where $C\subset D$ is a simple closed path.




Edited by: Shang-Da Yang

Proof: Let $C_1$, $C_2 \subset D$ are two concentric circles of radii $r_1$, $r_2$. By Cauchy's integral formula for doubly connected domain [eq. (10.10)]: $f(z)=\frac{1}{2\pi i}\oint_{C_1}\frac{f(z^*)}{z^*-z}dz^*-\frac{1}{2\pi i}\oint_{C_2}\frac{f(z^*)}{z^*-z}dz^* \equiv$ $g(z)+h(z)$ = (analytic part)+(principal part), for any $z\in D$.

**(1)** Since $z$ lies **within** $C_1$, and $f(z)$ is **analytic on** $C_1$, by the proof of Taylor series in Lesson 12: $g(z)=\frac{1}{2\pi i}\oint_{C_1}\frac{f(z^*)}{z^*-z}dz^*=\sum_{n=0}^{\infty}a_n(z-z_0)^n$, where $a_n=\frac{1}{2\pi i}\oint_{C_1}\frac{f(z^*)}{(z^*-z_0)^{n+1}}dz^*$.

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The proof of Taylor series consists of two steps: (i) $f(z)=\frac{1}{2\pi i}\oint_{C}\frac{f(z^*)}{z^*-z}dz^*$, which requires $f(z)$ is analytic everywhere inside $C$. (ii) $\oint_{C}\frac{f(z^*)}{z^*-z}dz^* = \sum_{n=0}^{\infty}\left(\oint_{C}\frac{f(z')}{(z'-z_0)^{n+1}}dz'\right)(z-z_0)^n$, which only requires $f(z)$ is analytic on $C$. Here we simply use (ii).

**(2)** Since $z$ lies **outside** $C_2$, $\frac{1}{z^*-z}=\frac{-1}{(z-z_0)\left(1-\frac{z^*-z_0}{z-z_0}\right)}=\frac{-p}{z-z_0}$, $p\equiv\frac{1}{1-q}$, $q=\frac{z^*-z_0}{z-z_0}$, $|q|<1$.

$\frac{1}{1-q}=(1+q+\ldots+q^n)+\left(\frac{q^{n+1}}{1-q}\right)$, $\Rightarrow p=\left[1+\frac{z^*-z_0}{z-z_0}+\ldots+\left(\frac{z^*-z_0}{z-z_0}\right)^n\right]+\frac{(z^*-z_0)^{n+1}}{(z-z^*)(z-z_0)^n}$.

$$\Rightarrow\ 2\pi i\cdot h(z)=\oint_{C_2}f(z^*)\left(\frac{p}{z-z_0}\right)dz^* = \left[\frac{1}{z-z_0}\oint_{C_2}f(z^*)dz^* + \frac{1}{(z-z_0)^2}\oint_{C_2}(z^*-z_0)f(z^*)dz^*\right.$$

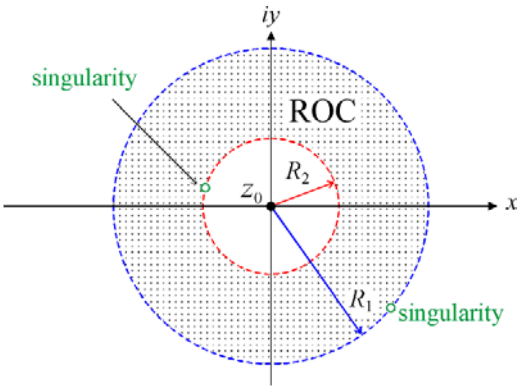
$$\left.+\ldots+\frac{1}{(z-z_0)^{n+1}}\oint_{C_2}(z^*-z_0)^n f(z^*)dz^*\right]+\left(\frac{1}{(z-z_0)^{n+1}}\oint_{C_2}\frac{(z^*-z_0)^{n+1}f(z^*)}{(z-z^*)}dz^*\right).$$

$\Rightarrow\ h(z)=\sum_{m=1}^{n+1}\frac{b_m}{(z-z_0)^m}+R_n^*(z)$, where $b_m=\frac{1}{2\pi i}\oint_{C_2}(z^*-z_0)^{m-1}f(z^*)dz^*$, and $R_n^*(z)=\frac{1}{2\pi i(z-z_0)^{n+1}}\oint_{C_2}\frac{(z^*-z_0)^{n+1}f(z^*)}{(z-z^*)}dz^*$. (We still need to prove $b_m$ exists, and $|R_n^*(z)|\to 0$.)

By the *ML*-inequality and the same procedure in proving Taylor theorem eq's (12.1-2), we derive $\lim_{n\to\infty}\left|R_n^*(z)\right|=0$, $\Rightarrow$ $h(z)=\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$, where $b_n=\frac{1}{2\pi i}\oint_{C_2}(z^*-z_0)^{n-1}f(z^*)dz^*$.

**(3)** By Cauchy's integral theorem 4 [eq. (10.8)], $a_n=\frac{1}{2\pi i}\oint_{C_1}['']dz^*=\frac{1}{2\pi i}\oint_{C}['']dz'$, $b_n=\frac{1}{2\pi i}\oint_{C_2}['']dz^*=\frac{1}{2\pi i}\oint_{C}['']dz'$, for any contour $C\subset D$.

Eq. (13.1) remains valid if we continuously move $C_1$ outward and $C_2$ inward until they reach some singularity. $\Rightarrow$ ROC of Laurent series is the open annulus $D$: $R_2<|z-z_0|<R_1$.

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1) A function may have different power series because of different ROCs or different centers.

   **E.g.** $\frac{1}{1-z}=$ (i) $\sum_{n=0}^{\infty}z^n$, if $|z|<1$; (ii) $\frac{-1}{z(1-z^{-1})}=\frac{-1}{z}\cdot\sum_{n=0}^{\infty}z^{-n}=-\sum_{n=0}^{\infty}z^{-(n+1)}$, if $|z|>1$; (iii) $\frac{-1}{z-1}$, if $0<|z-1|<\infty$.

2) A power series only has one closed function form. **E.g.** $f(z)=-\sum_{n=0}^{\infty}z^{-(n+1)}$. The series will diverge if we substitute $z=\frac{1}{2}$ into it to evaluate $f\left(\frac{1}{2}\right)$. $\Rightarrow$ Deriving the closed form of $f(z)$ first. $f(z)=-(z^{-1})\sum_{n=0}^{\infty}(z^{-1})^n=\frac{-z^{-1}}{1-z^{-1}}=\frac{1}{1-z}$, $\Rightarrow$ $f\left(\frac{1}{2}\right)=2$.

## Singularities and Zeros (EK 16.2)

■ Definitions

1) A function $f(z)$ has a singularity at $z=z_0$ if $f(z)$ is not analytic at $z=z_0$.

2) $z=z_0$ is an isolated singularity if it has a neighborhood without any other singularity.

**E.g.** $\tan z$ has isolated singularities at $z=\pm\dfrac{2n+1}{2}\pi$.

**E.g.** $\tan\left(z^{-1}\right)$ is not continuous/analytic for $\dfrac{1}{z}=\pm\dfrac{2n+1}{2}\pi$, $\Rightarrow$ $z=\pm\dfrac{2}{(2n+1)\pi}=\pm\dfrac{2}{\pi}$, $\pm\dfrac{2}{3\pi}$, …. For an arbitrarily small disk $|z|<\varepsilon$, you can always find infinitely many singularities within it, $\Rightarrow$ $z=0$ is a non-isolated singularity (other sigularities are isolated).

3) If the Laurent series of $f(z)$ centered at $z=z_0$ has nonzero coefficient(s) up to $m$-th order ($b_m\neq 0$), $\Rightarrow$ $z_0$ is a pole of order $m$ of $f(z)$, which can be removed by multiplying $(z-z_0)^m$. It is called a simple pole if $m=1$.

**E.g.** $\dfrac{1}{\sin z}=\dfrac{1}{z-(z^3/3!)+(z^5/5!)-\ldots}$, by long division, $=\dfrac{1}{z}+\dfrac{z}{6}+\dfrac{7z^3}{360}+\ldots$ has a simple pole at $z=0$.

**E.g.** To find the order of pole of $\tan z$ at $z=\dfrac{\pi}{2}$, we change the variable: $u=z-\dfrac{\pi}{2}$: $f(z)=\tan\left(u+\dfrac{\pi}{2}\right)=-\tan u=-\left(\dfrac{1}{u}-\dfrac{u}{3}-\dfrac{u^3}{45}-\ldots\right)$, which has a simple pole at $u=0$, i.e. $z=\dfrac{\pi}{2}$.

4) If the Laurent series of $f(z)$ centered at $z=z_0$ has infinitely many nonzero coefficients, $z_0$ is an essential singularity of $f(z)$.

**E.g.** $e^{1/z}=\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!z^n}$ has an isolated essential singularity at $z=0$.

**E.g.** $\tan\left(z^{-1}\right)=\dfrac{\sin(1/z)}{\cos(1/z)}=\dfrac{1}{z}+\dfrac{1}{3z^3}+\ldots$ has a non-isolated essential singularity at $z=0$.

5) If an analytic function $f(z)$ has a property of $f(z_0)=f'(z_0)=\ldots=f^{(n-1)}(z_0)=0$ (i.e. the first $n$ Taylor coefficients $a_0=a_1=\ldots=a_{n-1}=0$), $\Rightarrow$ $z=z_0$ is a zero of order $n$ of $f(z)$.

**E.g.** $\sin z$ has simple zeros ($n=1$) at $z=0$, $\pm\pi$, $\pm 2\pi$, …; $\sin^2 z$ has second-order zeros at these points.

■ Properties

1) If $f(z)$ is analytic except for a pole at $z=z_0$, $\Rightarrow$ $|f(z)|\to\infty$ as $z\to z_0$ in any manner.

2) Picard's theorem: If $f(z)$ is analytic except for an essential singularity at $z=z_0$, $\Rightarrow$ $f(z)$ can be equal to any complex number in an arbitrarily small neighborhood of $z_0$.

**E.g.** $f(z)=e^{1/z}$: (i) if $z=x=(+\infty\to 0^+)$, $f(z)\to\infty$; (ii) if $z=-x=(-\infty\to 0^-)$, $f(z)\to\mathbf{0}$; (iii) For any given complex number $c=c_0\cdot e^{i\alpha}$, we can find some $z=r\cdot e^{i\theta}$, s.t. $f(z)=\exp\left[\frac{\cos\theta - i\sin\theta}{r}\right]=c$,

$\Rightarrow$ $\left\{e^{\frac{\cos\theta}{r}}=c_0,\ -\frac{\sin\theta}{r}=\alpha+2n\pi\right\}$ ; $\Rightarrow$ $r=\left[(\ln c_0)^2+(\alpha+2n\pi)^2\right]^{-\frac{1}{2}}$ , $\tan\theta=\frac{-(\alpha+2n\pi)}{\ln c_0}$. The solution $z$ can be made arbitrarily close to 0 ($r\to 0$) by increasing $n$.

3) If $f(z)$ is analytic and has an $n$th-order zero at $z_0$, $\Rightarrow$ $\frac{1}{f(z)}$ has an $n$th-order pole at $z=z_0$.

4) The properties of $f(z)$ at large $|z|$ can be investigated by: set $z=1/w$, investigate $g(w)=f(1/w)=f(z)$ in the neighborhood of $w=0$.

**E.g.** $e^z$ has an isolated essential singularity at $\infty$, for $e^{1/w}$ has that at $w=0$.

## Residue Integration (EK 16.3)

■ Evaluate contour integral by residue

To evaluate $I=\oint_C f(z)dz$ for some arbitrary contour $C$:

1) If $f(z)$ is analytic for every point on and within $C$, $\Rightarrow$ $I=0$ by Cauchy's integral theorem 1.

2) If $f(z)$ has only one singularity at $z=z_0$ inside $C$, its "closest" Laurent series with ROC: $\{\mathbf{0}<|z-z_0|<R\}$ is: $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$, $\Rightarrow$

$$I=2\pi i\cdot \operatorname*{Res}_{z=z_0} f(z) \tag{13.2}$$

where $\underset{z=z_0}{\text{Res}} f(z)=b_1$ is called the **residue** of $f(z)$ at $z_0$. [eq. (10.9) is a special case.]

Proof: by $\oint_C (z-z_0)^m dz = \begin{cases} 2\pi i, \text{if } m=-1 \\ 0, \text{otherwise} \end{cases}$, $\oint_C f(z)dz = b_1 \oint_C \frac{1}{z-z_0} dz = 2\pi i b_1$.

**E.g.** $I=\oint_C f(z)dz$, where $f(z)=\frac{1}{z(1-z)}$, $C$: $|z|=\frac{1}{2}$, in counterclockwise sense.

Since only one singularity $z=0$ lies inside $C$, $\Rightarrow$ find the Laurent series centered at $z=0$: (1) $f(z)=\frac{1}{z}+1+z^2+\ldots$, for $0<|z|<1$, $\Rightarrow b_1=1$. (2) $f(z)= -\left[\frac{1}{z^2}+\frac{1}{z^3}+\ldots\right]$, for $|z|>1$, $\Rightarrow b_1=0$. $\Rightarrow$ Choose series-1: $b_1=1$, and $I=2\pi i$.

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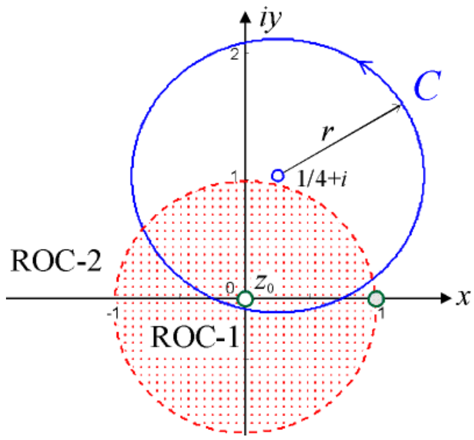
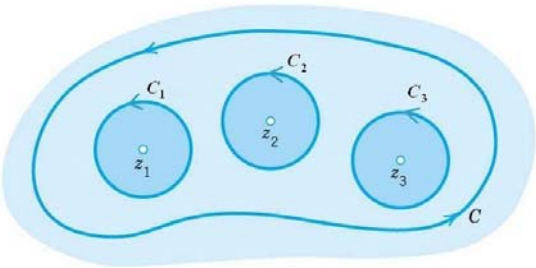
We always choose the closest Laurent series to evaluate residue, even when part of the integral path $C$ falls **outside** its ROC. **E.g.** Let $f(z)=\frac{1}{z(1-z)}$, $C$: $z=\left(\frac{1}{4}+i\right)+re^{i\theta}$, $\theta\in[0,2\pi]$, which passes through ROCs of the two Laurent series and only encloses one singularity $z=0$ if $\sqrt{17}/4<r<5/4$. $\Rightarrow I=2\pi i$ for arbitrary $\sqrt{17}/4<r<5/4$.




3) If $f(z)$ has finitely many isolated singularities $\{z_i;\, i=1-N\}$ inside $C$, $\Rightarrow$

$$I=2\pi i\cdot\sum_{k=1}^{N}\underset{z=z_k}{\text{Res}}\left[f(z)\right] \tag{13.3}$$




Proof: By Cauchy's integral theorem 4 [eq. (10.6)], $\Rightarrow \oint_C f(z)dz = \sum_{k=1}^{N}\oint_{C_k} f(z)dz$, where $C_k$ is a circle that only encloses one singularity $z_k$, and is separated from all the other circles. By eq. (13.2), $\oint_{C_k} f(z)dz = 2\pi i \cdot \operatorname*{Res}_{z=z_k}[f(z)]$, $\Rightarrow$ ….

■ How to evaluate the residue of a singularity

1) If $z=z_0$ is (i) a pole of unknown order, (ii) essential singularity, (iii) non-isolated singularity of $f(z)$, $\Rightarrow$ try to derive the (partial) Laurent series.

2) If $z=z_0$ is a simple pole of $f(z)$, $\Rightarrow$

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0}(z-z_0)f(z) \tag{13.4}$$

Proof: $f(z)=\dfrac{b_1}{z-z_0}+a_0+a_1(z-z_0)+a_2(z-z_0)^2+\ldots$ , $(z-z_0)f(z)=b_1+a_0(z-z_0)+a_1(z-z_0)^2+\ldots$, $= b_1+(z-z_0)[a_0+a_1(z-z_0)+\ldots]$.

4) If $z=z_0$ is a simple pole of $\dfrac{p(z)}{q(z)}$, where $p(z_0)\neq 0$, $q(z)$ has a simple zero at $z_0$, $\Rightarrow$

$$\operatorname*{Res}_{z=z_0}\frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)} \tag{13.5}$$

Proof: If $z=z_0$ is a simple zero of $q(z)$, $q(z)=q'(z_0)(z-z_0)+\dfrac{q''(z_0)}{2!}(z-z_0)^2+\ldots$, by eq. (13.4):

$$\operatorname*{Res}_{z=z_0}\frac{p(z)}{q(z)} = \lim_{z\to z_0}(z-z_0)\frac{p(z)}{q(z)} = \lim_{z=z_0}\frac{(z-z_0)p(z)}{(z-z_0)[q'(z_0)+(z-z_0)q''(z_0)/2!+\ldots]} = \frac{p(z_0)}{q'(z_0)}.$$

5) If $z=z_0$ is ***m*th order pole** of $f(z)$, $\Rightarrow$

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]_{z=z_0} \tag{13.6}$$

Proof: $f(z)=\left[\frac{b_m}{(z-z_0)^m}+\ldots+\frac{b_1}{z-z_0}\right]+\sum_{n=0}^{\infty}a_n(z-z_0)^n$, $g(z)\equiv(z-z_0)^m f(z)= b_m+ b_{m-1}(z-z_0)$ $+ \ldots +\boldsymbol{b_1(z-z_0)^{m-1}}+\sum_{n=0}^{\infty}a_n(z-z_0)^{n+m}$; $\Rightarrow b_1$ is the Taylor coefficient of the power $(z-z_0)^{m-1}$ of $g(z)$, $\Rightarrow b_1=\frac{g^{(m-1)}(z_0)}{(m-1)!}$.

**E.g.** Find the residues of $f(z)=\frac{1}{z(z+2)^3}$ at all poles. **(1)** $z=0$ is a simple pole. $\operatorname*{Res}_{z=0} f(z)=\lim_{z\to 0} zf(z)=\lim_{z\to 0}(z+2)^{-3}=\frac{1}{8}$. **(2)** $z=-2$ is a 3rd order pole, by eq. (13.6), $\operatorname*{Res}_{z=-2} f(z)=\frac{1}{2!}\frac{d^2}{dz^2}\left[(z+2)^3 f(z)\right]_{z=-2}=-\frac{1}{8}$.

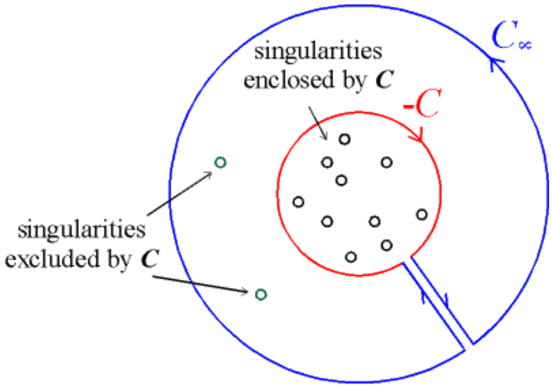
Note: Partial fractions reveal all the residues: $f(z)=\frac{1/8}{z}+\frac{-1/8}{z+2}+\frac{-1/4}{(z+2)^2}+\frac{-1/2}{(z+2)^3}$.

(*) **E.g.** Find the residue of $f(z)=(\cot z)$ at $z=\pi$.

Ans: Find the order first. Let $u=z-\pi$, $\cot z=\cot u$, $\Rightarrow$ {order of pole $z=\pi$ of $\cot z$}={order of pole $u=0$ of $\cot u$}. $\cot u=\frac{\cos u}{\sin u}=\frac{1-(u^2/2!)+(u^4/4!)-\ldots}{u-(u^3/3!)+(u^5/5!)-\ldots}=\frac{1}{u}-\frac{u}{3}+\ldots$, $\Rightarrow u=0$ is a simple pole, $\Rightarrow z=\pi$ is a simple pole of $\cot z$. By eq. (13.4), $\operatorname*{Res}_{z=\pi}\frac{\cos(z)}{\sin(z)}=\frac{\cos(\pi)}{\sin'(\pi)}=1$.

## Appendix 13A − Residue at Infinity

If the contour $C$ of $\oint_C f(z)dz$ encloses a non-isolated singularity or a large number of isolated singularities of $f(z)$, evaluating the "interior" residues becomes inefficient. Instead, we can derive $\oint_C f(z)dz$ by evaluating "exterior" residues of isolated singularities $z_1$, $z_2$, , $z_N$ outside $C$, and perhaps a residue at infinity:




Creating an infinite contour $C_\infty$. By eq. (13.3): $\oint_{C_\infty} f(z)dz - \oint_C f(z)dz = 2\pi i \cdot \sum_{k=1}^{N} \operatorname*{Res}_{z=z_k}[f(z)]$, $\Rightarrow$ $\oint_C f(z)dz = \oint_{C_\infty} f(z)dz - 2\pi i \cdot \sum_{k=1}^{N} \operatorname*{Res}_{z=z_k}[f(z)]$. To evaluate $\oint_{C_\infty} f(z)dz$, let $C_\infty$: $z=re^{i\theta}$, $r\to\infty$, $\theta=[0,2\pi]$ (counterclockwise). By change of variable $\mathbf{w=1/z}$, $C_\infty$ in the $z$-plane becomes $C'_\infty$ in the $w$-plane: $w\equiv\rho e^{i\phi}=(1/r)e^{-i\theta}$, $\rho\to0$, $\phi=[0,-2\pi]$ (clockwise), $f(z)=f(1/w)$, $\Rightarrow$ $\oint_{C_\infty} f(z)dz =$ $\oint_{C'_\infty} f(1/w)\frac{-dw}{w^2} = \oint_{-C'_\infty} \frac{f(1/w)}{w^2}dw$, since $-C'_\infty$ only encloses one point $w=0$ in a counterclockwise sense, $=2\pi i\cdot \operatorname*{Res}_{w=0}\left[\frac{f(1/w)}{w^2}\right]$, $\Rightarrow$

$$\oint_C f(z)dz = 2\pi i\left\{\operatorname*{Res}_{w=0}\left[\frac{f(1/w)}{w^2}\right] - \sum_{z_k \text{ outside } C} \operatorname{Res}[f(z)]\right\} \tag{13A.1}$$

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$\operatorname*{Res}_{w=0}\frac{f(1/w)}{w^2} = \operatorname*{Res}_{z=\infty} f(z)$. <u>Proof</u>: Let $\operatorname*{Res}_{z=\infty} f(z)=b_1$, $\Rightarrow$ the Laurent series of $f(z)$ centered at

$z$=∞ as: $f(z)$={…+$\dfrac{b_1}{z-\infty}+a_0+a_1(z-\infty)+...$} ⇒. By change of variable: $w$=$\dfrac{1}{z}$, $f(z)$= $f(1/w)$= $\left\{...+b_1w+a_0+\dfrac{a_1}{w}+...\right\}$, ⇒ $\dfrac{f(1/w)}{w^2}=\left\{...+\dfrac{b_1}{w}+\dfrac{a_0}{w^2}+\dfrac{a_1}{w^3}+...\right\}$, where $b_1$ is the residue at $w$=0.

**E.g.** Evaluate $\oint_C \dfrac{z^{99}e^{1/z}}{z^{100}+1}dz$, $C$: $|z|$=3.

Ans: There are 101 singularities within $C$ ($z$=0, and 100 points fall on the unit circle), and none outside $C$, ⇒ use residue at infinity. $\operatorname*{Res}_{z=\infty}\dfrac{z^{99}e^{1/z}}{z^{100}+1}=\operatorname*{Res}_{w=0}\dfrac{w^{-99}e^{w}}{w^2(w^{-100}+1)}=\operatorname*{Res}_{w=0}\dfrac{e^{w}}{w(1+w^{100})}$, since $w$=0 is a simple pole, $=\lim_{w\to 0}\dfrac{e^w}{1+w^{100}}$=1. ⇒ $\oint_C f(z)dz=2\pi i\cdot\operatorname*{Res}_{z=\infty}f(z)=2\pi i$.