Lesson 12 Taylor Series (EK 15.4)

■ Taylor theorem

Let $f(z)$ is analytic in a domain D, and $z_0 \in D$; ⇒ **(1)** $f(z)$ can be represented by a power series:

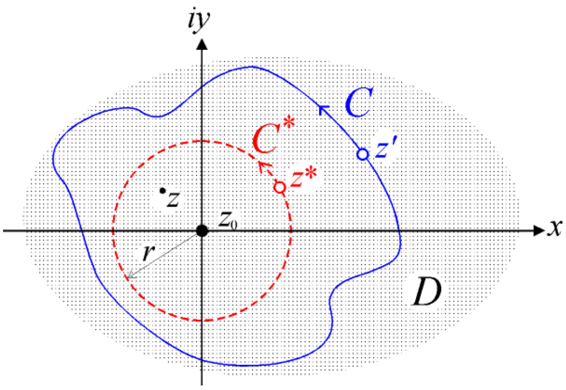
$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n \text{, where } a_n=\frac{f^{(n)}(z_0)}{n!}=\frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}}dz' \qquad (12.1)$$

for every point z in the largest open disk $|z-z_0|<R$ within D, and $C \subset D$ is an arbitrary simple closed path enclosing z_0. **(2)** The remainder $R_n(z) \equiv f(z) - \sum_{m=0}^{n} a_m (z-z_0)^m$ is:

$$R_n(z)=\frac{(z-z_0)^{n+1}}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}(z'-z)}dz' \qquad (12.2)$$

Proof: Since $f(z)$ is analytic in D, by Cauchy's integral formula [eq. (10.9)]: $f(z)=\frac{1}{2\pi i}\oint_C \frac{f(z')}{z'-z}dz'$, for every z lies within $C(\subset D)$.

Choose a circle C^*: $|z^*-z_0|=r>|z-z_0|$. Since the integrand $\frac{f(z')}{z'-z}$ is analytic in a doubly connected domain bounded by C and C^*, by Cauchy's integral theorem 4 [eq. (10.8)]: $\oint_C \frac{f(z')}{z'-z}dz' = \oint_{C^*} \frac{f(z^*)}{z^*-z}dz^* = 2\pi i \cdot f(z)$.



$$\frac{1}{z^*-z}=\frac{1}{(z^*-z_0)\left(1-\frac{z-z_0}{z^*-z_0}\right)} \equiv \frac{p}{(z^*-z_0)}, \text{ where } p \equiv \frac{1}{1-q},\ q=\frac{z-z_0}{z^*-z_0},\ |q|<1.$$

$$p=\frac{1}{1-q}=\sum_{m=0}^{\infty} q^m = (1+q+q^2+\ldots+q^n)+\left(\frac{q^{n+1}}{1-q}\right);$$

$$=\left[1+\frac{z-z_0}{z^*-z_0}+\left(\frac{z-z_0}{z^*-z_0}\right)^2+\ldots+\left(\frac{z-z_0}{z^*-z_0}\right)^n\right]+\frac{(z-z_0)^{n+1}}{(z^*-z)(z^*-z_0)^n}.$$

$$\Rightarrow \oint_{C^*}\frac{f(z^*)}{z^*-z}dz^* = \oint_{C^*}f(z^*)\left(\frac{p}{z^*-z_0}\right)dz^* = \left[\oint_{C^*}\frac{f(z^*)}{z^*-z_0}dz^* + (z-z_0)\oint_{C^*}\frac{f(z^*)}{(z^*-z_0)^2}dz^* +\ldots+\right.$$

$$\left.(z-z_0)^n\oint_{C^*}\frac{f(z^*)}{(z^*-z_0)^{n+1}}dz^*\right]+\left((z-z_0)^{n+1}\oint_{C^*}\frac{f(z^*)}{(z^*-z)(z^*-z_0)^{n+1}}dz^*\right)=2\pi i\cdot f(z).$$

$\Rightarrow f(z)=\sum_{m=0}^{n}a_m(z-z_0)^m+R_n(z)$, where $\boldsymbol{a_m}=\frac{1}{2\pi i}\oint_{C^*}\frac{f(z^*)}{(z^*-z_0)^{m+1}}dz^*=\frac{f^{(m)}(z_0)}{m!}$ [eq. (10.11)], and

$\boldsymbol{R_n(z)}=\frac{(z-z_0)^{n+1}}{2\pi i}\oint_{C^*}\frac{f(z^*)}{(z^*-z)(z^*-z_0)^{n+1}}dz^*$. (We still need to prove a_m exists, and $|R_n(z)|\to 0$.)

Since analytic functions have derivatives of all orders, $a_m=\frac{f^{(m)}(z_0)}{m!}$ exists for all m.

$|R_n(z)|=\frac{|z-z_0|^{n+1}}{2\pi}\cdot\left|\oint_{C^*}\frac{f(z^*)}{(z^*-z)(z^*-z_0)^{n+1}}dz^*\right|\le\frac{|z-z_0|^{n+1}}{2\pi r^{n+1}}\cdot\oint_{C^*}\left|\frac{f(z^*)}{z^*-z}\right|dz^*$. Since $f(z)$ is analytic on C^*, and $z\neq z^*$ $\Rightarrow$ $\left|\frac{f(z^*)}{z-z^*}\right|\le M$ on C^*. By ML-inequality, $|R_n(z)|\le\frac{|z-z_0|^{n+1}}{2\pi r^{n+1}}\cdot M\cdot 2\pi r=$

$Mr\left|\frac{z-z_0}{r}\right|^{n+1}\to 0$, as $n\to\infty$. $\Rightarrow$ Taylor series does converge to $f(z)$.

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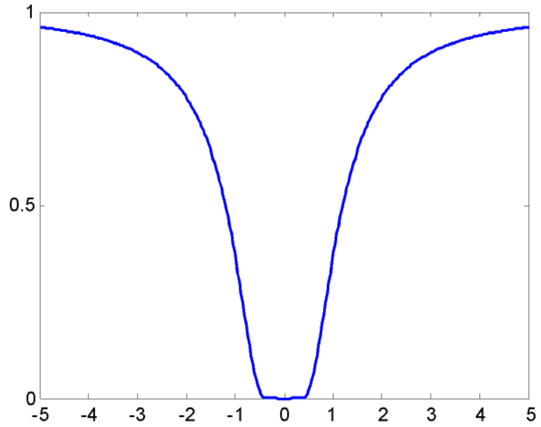
1) Although we only need "$f(z)$ is analytic **on** C^*" in proving $\oint_{C^*}\frac{f(z^*)}{z^*-z}dz^*=2\pi i\cdot\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$, eq. (12.1) is true only if "$f(z)$ is analytic **within** C^*" such that Cauchy's integral formula $\oint_{C^*}\frac{f(z^*)}{z^*-z}dz^*=2\pi i\cdot f(z)$ is valid.

2) There are real functions that are differentiable for all orders but cannot be represented by Taylor series. **E.g.** $f(x)=\begin{cases}\exp(-1/x^2), \text{if } x\neq 0\\ 0, \text{if } x=0\end{cases}$ is differentiable at $x=0$, but $a_n=0$ [$f^{(n)}(0)=0$]

for all n, $\Rightarrow$ no Taylor series.



■ Taylor series of basic functions

1) Geometric series: $\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n$ =1+z+z^2+…, for $|z|<1$.

Proof: $f^{(n)}(z)=\frac{n!}{(1-z)^{n+1}}$, for z_0=0, $a_n=\frac{f^{(n)}(z_0)}{n!}$=1. The distance from the center z_0=0 to the nearest singularity z=1 is 1, $\Rightarrow$ ROC: $\{|z|<1\}$.

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For points located on or outside unit circle ($|z|$=1), $\sum_{n=0}^{\infty} z^n$ cannot represent $\frac{1}{1-z}$ (divergent). **E.g.** $z=-1$, $\Rightarrow$ $\sum_{n=0}^{\infty}(-1)^n$ is undetermined, while $\frac{1}{1-z}=\frac{1}{2}$. We have to find some Taylor series with different center, whose ROC contains the point of interest. **E.g.** $\boldsymbol{z_0=-1}$, $\Rightarrow$ $a_n=\frac{f^{(n)}(z_0)}{n!}=\frac{1}{2^{n+1}}$, $f(z)=\sum_{n=0}^{\infty}\frac{(z+1)^n}{2^{n+1}}$, with ROC: $\{|z+1|<2\}$. $f(-1)=\sum_{n=0}^{\infty}\frac{0^n}{2^{n+1}}=\frac{1}{2}$.

2) Exponential function: $e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$=1+$z$+$\frac{z^2}{2!}$+…, for all z. Proof: by eq. (12.1).

3) Trigonometric functions: $\cos z=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\ldots$; $\sin z=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$ $=z-\frac{z^3}{3!}+\frac{z^5}{5!}-\ldots$; for all z. Proof: by definitions of cos, sin, and 2).

4) Hyperbolic functions: **cosh** z $=\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}$ $=1+\frac{z^2}{2!}+\frac{z^4}{4!}+\ldots$; **sinh** z $=\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}=$ $z+\frac{z^3}{3!}+\frac{z^5}{5!}+\ldots$; for all z. [Relation between cos() and cosh() becomes clear in series.]

5) Logarithm: **Ln(1+z)** $=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n}=z-\frac{z^2}{2}+\frac{z^3}{3}-\ldots$; Replacing z by $-z$, $\Rightarrow$ **−Ln(1−z)** $=\sum_{n=1}^{\infty}\frac{z^n}{n}=z+\frac{z^2}{2}+\frac{z^3}{3}+\ldots$; for $|z|<1$.

■ How to obtain Taylor series

1) Evaluate nth-order derivatives [eq. (12.1)].

2) Rearrange $f(z)$ to use geometric series. **E.g.** $\frac{1}{1+z^2}=\frac{1}{1-(-z^2)}=\sum_{n=0}^{\infty}(-z^2)^n=\ldots$, for $|z|<1$.

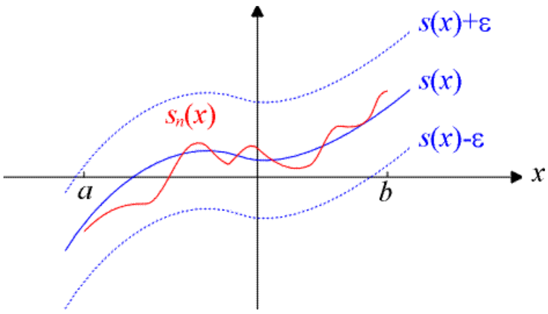
3) Termwise integration: **E.g.** $f(z)=\tan^{-1}z$, $f'(z)=(1+z^2)^{-1}=..$, $f(z)=z-\frac{z^3}{3}+\frac{z^5}{5}-\ldots$; for $|z|<1$.

4) Partial fractions + binomial series. **E.g.** $\frac{2z^2+9z+5}{z^3+z^2-8z-12}=\frac{1}{(z+2)^2}+\frac{2}{z-3}$, for $z_0=1$, $=\frac{1}{9[1+(z-1)/3]^2}-\frac{1}{1-(z-1)/2}$, by the binomial series: $\frac{1}{(1+z)^m}=\sum_{n=0}^{\infty}\binom{-m}{n}z^n=$ $1-\binom{m}{1}z+\binom{m}{2}z^2-\ldots$, $=\ldots$ Note the ROC is the overlapped ones of the two series: $|z-1|<2$.

Appendix 12A − Uniform Convergence (EK 15.5)

■ Definition

Let $s(z)=\sum_{m=0}^{\infty} f_m(z)=\lim_{n\to\infty} s_n(z)$, where $s_n(z)\equiv\sum_{m=0}^{n} f_m(z)$. We say $s(z)$ is uniformly convergent in a region G, if for every $\varepsilon>0$, we can find an N (independent of z), s.t. $|s(z)-s_n(z)|<\varepsilon$, for all $n>N$ and all $z\in G$.



E.g. Geometric series $s(z)=\sum_{m=0}^{\infty} z^m$ is **(1)** uniformly convergent for G: $|z|\le r<1$, but **(2)** not uniformly convergent for G: $|z|<1$.

Proof: **(1)** $|s(z)-s_n(z)| = \left|\dfrac{1}{1-z} - \dfrac{1-z^{n+1}}{1-z}\right| = \dfrac{|z|^{n+1}}{|1-z|} \le \dfrac{r^{n+1}}{1-r}$. By letting $N>\dfrac{\ln\varepsilon(1-r)}{\ln r}-1$ (independent of z), $|s(z)-s_n(z)|<\varepsilon$ for all $n>N$. **(2)** As z gets closer to 1, we need larger N such that $|s(z)-s_n(z)|=\dfrac{|z|^{n+1}}{|1-z|}<\varepsilon$, for all $n>N$. Since $|1-z|$ can be infinitely small, no fixed N.

■ Power series is uniformly convergent

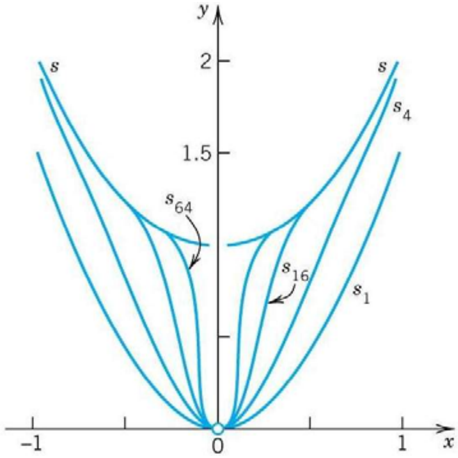
$\sum_{m=0}^{\infty} a_m(z-z_0)^m$ with radius of convergence $R>0$ is uniformly (and absolutely) convergent for all $|z-z_0|\le r<R$.

Proof: by Cauchy's convergence principle (EK 15.1).

■ Properties of uniformly convergent series

1) Continuity: if $F(z)=\sum_{m=0}^{\infty} f_m(z)$ is uniformly convergent in a region G, and each term $f_m(z)$ is continuous at $z_1 \in G$, $\Rightarrow F(z)$ is continuous at z_1.

E.g. $f_m(x)=\dfrac{x^2}{(1+x^2)^m}$ is continuous at x=0 for all m, but $F(x)=\begin{cases} 1+x^2, \text{if } x \neq 0 \\ 0, \text{if } x=0 \end{cases}$ is discontinuous at x=0, $\Rightarrow$ The series is not uniformly convergent.



2) Termwise integration: If $F(z)=\sum_{m=0}^{\infty} f_m(z)$ is uniformly convergent in a region G, and C is some path in G. $\Rightarrow$ $\int_C F(z)dz=\sum_{m=0}^{\infty}\int_C f_m(z)dz$. $\Rightarrow$ Exchange the order of ∫ and Σ is valid.

E.g. $u_m(x) \equiv mxe^{-mx^2}$, $f_m(x) \equiv u_m(x) - u_{m-1}(x)$, $s_n(x)=\sum_{m=1}^{n} f_m(z)=u_n(x)$, $\Rightarrow$ $F(x)=\lim_{n\to\infty} s_n(x)=0$,

For G=[0,1], $\int_0^1 F(x)dx$ =**0**, $\int_0^1 f_m(x)dx = \dfrac{e^{-m+1}-e^{-m}}{2}$, $\Rightarrow$ $\sum_{m=1}^{\infty}\left(\int_0^1 f_m(x)dx\right)$ = $\dfrac{(e^0-e^{-1})+(e^{-1}-e^{-2})+....}{2}=\mathbf{\dfrac{1}{2}}\neq 0$, $\Rightarrow$ $\sum_{m=0}^{\infty} f_m(z)$ is not uniformly convergent.

3) Termwise differentiation: If $F(z)=\sum_{m=0}^{\infty} f_m(z)$ is convergent, $f_m(z)$ is continuous, and $\sum_{m=0}^{\infty} f'_m(z)$ is uniformly convergent in a region G, $\Rightarrow$ $F'(z)=\sum_{m=0}^{\infty} f'_m(z)$. $\Rightarrow$ Exchange the order of (d/dz) and Σ is valid.

4) Absolutely convergent series are not necessarily uniformly convergent, while uniformly convergent series are also not necessarily absolutely convergent.