Lesson 11 Complex Power Series (EK 15)

■ Importance

Every analytic function can be represented by a Taylor (power) series.

Complex Sequence and Series (EK 15.1)

■ Sequences

Definition: an ordered set of complex numbers $\{z_n\}$.

Convergence: $\lim_{n\to\infty} z_n = c$ ($\{z_n\}\to c$), if for every $\varepsilon>0$, we can find N, s.t. $|z_n-c|<\varepsilon$ for any $n>N$.

Let $\{z_n\}=\{x_n+iy_n\}$, $c=a+ib$, then $\{z_n\}\to c \Leftrightarrow \{x_n\}\to a$, and $\{y_n\}\to b$.

■ Series

Definition: $\sum_{n=1}^{\infty} z_n \equiv \lim_{n\to\infty} S_n$, where $S_n=\sum_{m=1}^{n} z_m$ is the n-th partial sum. Series is convergent if its partial sum is convergent.

Let $\{z_n\}=\{x_n+iy_n\}$, $s=u+iv$, then $\sum_{n=1}^{\infty} z_n = s \Leftrightarrow \{\sum_{n=1}^{\infty} x_n = u$, and $\sum_{n=1}^{\infty} y_n = v\}$.

If $\sum_{n=1}^{\infty} |z_n|$ converges, $\Rightarrow \sum_{n=1}^{\infty} z_n$ is absolutely convergent. If $\sum_{n=1}^{\infty} |z_n|$ diverges, while $\sum_{n=1}^{\infty} z_n$ converges, $\Rightarrow \sum_{n=1}^{\infty} z_n$ is simply conditionally convergent.

E.g. Series $S=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ is convergent, but $\sum_{n=1}^{\infty}\frac{1}{n}$ (harmonic series) is divergent. Therefore, $\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n}$ is conditionally convergent.

(*) Series Convergence Tests (EK 15.1)

■ Divergence test

If $\lim_{n\to\infty} z_n \neq 0$, $\sum_{n=1}^{\infty} z_n$ diverges. But $\lim_{n\to\infty} z_n = 0$ does not guarantee the convergence of $\lim_{n\to\infty} z_n$.

E.g. Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, though $\lim_{n\to\infty} \frac{1}{n} = 0$.

■ Comparison test

If we can find a convergent series $\sum_{n=1}^{\infty} b_n$, such that $b_n \geq |z_n|$, $\Rightarrow$ $\sum_{n=1}^{\infty} z_n$ is absolutely convergent.

■ Ratio test

Test 1: For a series $\sum_{n=1}^{\infty} z_n$, if $\left|\frac{z_{n+1}}{z_n}\right| \leq q < 1$ for all $n>N$, $\Rightarrow$ $\sum_{n=1}^{\infty} z_n$ is absolutely convergent.

E.g. $\sum_{n=1}^{\infty} \frac{1}{n}$ has $\left|\frac{z_{n+1}}{z_n}\right| = \frac{n}{n+1} < 1$, but $\frac{n}{n+1}$ can exceed any real number less than 1 if n is sufficiently large, $\Rightarrow$ no fixed upper bound $q<1$, ratio test fails.

Test 2: If $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = L$, then the series $\sum_{n=1}^{\infty} z_n$ is:

(1) absolutely convergent, if $L<1$; (2) divergent, if $L>1$; (3) undetermined, if $L=1$.

E.g. $\sum_{n=1}^{\infty} \frac{1}{n}$ has $L = \lim_{n\to\infty} \frac{n}{n+1} = 1$, which is divergent. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ has $L = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$, which converges to $\frac{\pi^2}{6}$ [by Riemann Zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, with $s=2$].

■ Root test

Similar with ratio test except for replacing $\left|\frac{z_{n+1}}{z_n}\right|$ by $\sqrt[n]{|z_n|}$.

E.g. $\sum_{n=1}^{\infty}\frac{1}{n}$ has $L=\lim_{n\to\infty}\sqrt[n]{1/n}=\lim_{n\to\infty}n^{-1/n}=\lim_{n\to\infty}e^{-(\ln n/n)}$ (variable in exponent only) =1, which is divergent. $\sum_{n=1}^{\infty}\frac{1}{n^2}$ has $L=\lim_{n\to\infty}\sqrt[n]{1/n^2}=\lim_{n\to\infty}n^{-2/n}=\lim_{n\to\infty}e^{-(2\ln n/n)}$ =1, which converges to $\pi^2/6$.

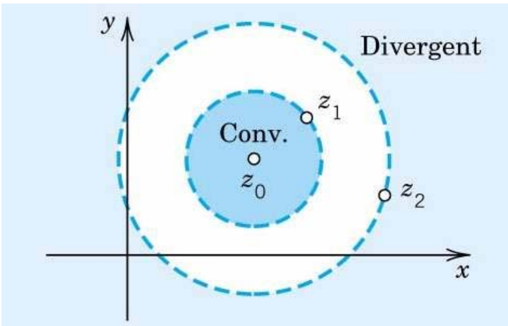
Power Series and its Convergence (EK 15.2)

■ Definition

A series of the form: $\sum_{n=0}^{\infty}a_n(z-z_0)^n$, where variable z, center z_0, and coefficients $\{a_n\}$ are generally complex.

■ Convergence theorem

(1) If power series $\sum_{n=0}^{\infty}a_n(z-z_0)^n$ converges at a point $z=z_1$, ⇒ it is absolutely convergent for every "closer point" $\{z, |z-z_0|<|z_1-z_0|\}$. **(2)** If power series $\sum_{n=0}^{\infty}a_n(z-z_0)^n$ diverges at a point $z=z_2$, ⇒ it is divergent for every "farther point" $\{z, |z-z_0|>|z_2-z_0|\}$.



(*) Proof: **(1)** By divergence test: $\sum_{n=0}^{\infty}a_n(z_1-z_0)^n$ converges, ⇒ $a_n(z_1-z_0)^n\to 0$, i.e. $|a_n(z_1-z_0)^n|<M$ for all n. For a "closer point" z, $|a_n(z-z_0)^n|=\left|a_n(z_1-z_0)^n\left(\frac{z-z_0}{z_1-z_0}\right)^n\right|<Mr^n\equiv b_n$,

where $r=\left|\frac{z-z_0}{z_1-z_0}\right|<1$. Since $\sum_{n=0}^{\infty} b_n = M\left(\sum_{n=0}^{\infty} r^n\right)$ is convergent, by comparison test, $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is also convergent. **(2)** The divergence part can be proved by contradiction (歸謬法).

■ Region of convergence (ROC)

The ROC of a power series is: $\{z$, where $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is convergent$\}$. By the convergence theorem, ROC always has a circular boundary: $|z-z_0|=R$, on which the convergence of series is undetermined.

E.g. The ROCs of $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$, $\sum_{n=0}^{\infty} z^n$, $\sum_{n=1}^{\infty} \frac{z^n}{n}$ have same boundary C: $|z|=1$. $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$, converges, $\sum_{n=0}^{\infty} z^n$ diverges everywhere on C, while $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges at $z=-1$ but diverges at $z=1$.

■ Radius of convergence (Cauchy-Hadamard formula)

The radius of convergence R of $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is evaluated by:

$$R=\frac{1}{L^*}=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \tag{11.1}$$

(*) <u>Proof</u>: By ratio test, $L\equiv\lim_{n\to\infty}\left|\frac{a_{n+1}(z-z_0)^{n+1}}{a_n(z-z_0)^n}\right|=L^*|z-z_0|$. **(1)** $L^*\neq 0, \infty$: series converges for $L<1$ ($|z-z_0|<1/L^*$), and diverges for $L>1$ ($|z-z_0|>1/L^*$). $\Rightarrow R=1/L^*$. **(2)** $L^*=0$: $L=0$ for all z (note that $|z-z_0|$ can approach, but never equal to ∞), series converges everywhere. $\Rightarrow R=1/L^*=\infty$. **(3)** $L^*=\infty$: $L=\infty$ for all z except for $z=z_0$, series diverges everywhere. $\Rightarrow R=1/L^*=0$.

Power Series Representation of Complex Functions (EK 15.3)

■ Power series represent analytic functions

A power series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence R>0 represents an analytic function $f(z)$ for all $|z|<R$, where $f'(z)=\sum_{n=1}^{\infty} na_n z^{n-1} \equiv f_1(z)$ (let z_0=0 without loss of generality).

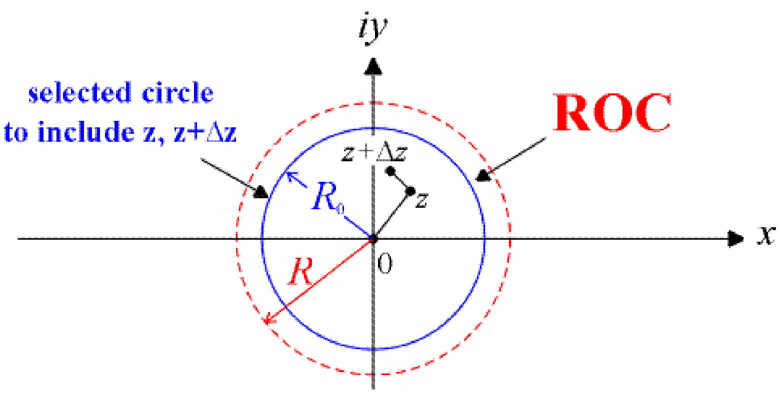
Proof: Strategy: let $f(z)=\sum_{n=0}^{\infty} a_n z^n$, show that $f'(z) \equiv \lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} = f_1(z)$, i.e. $\lim_{\Delta z\to 0}|q|=0$, if $q\equiv\frac{f(z+\Delta z)-f(z)}{\Delta z}-f_1(z)$.

(*) By definition, $q=\sum_{n=2}^{\infty} a_n\left[\frac{(z+\Delta z)^n - z^n}{\Delta z} - nz^{n-1}\right] \equiv \sum_{n=2}^{\infty} a_n t_n$. Let $z\equiv a$, $z+\Delta z\equiv b$, $\Rightarrow t_n = \frac{b^n-a^n}{b-a}-na^{n-1}$, which is a binomial of order $n-1$ (i.e. sum of terms $a^k b^l$, where $k+l=n-1$).

Let $t_n\equiv(b-a)A_n$, A_n is a binomial of order $n-2$, $\Rightarrow A_n=\sum_{m=0}^{n-2} c_m a^m b^{n-2-m}$.

To find coefficients $\{c_m\}$, we use: $(b-a)t_n=(b-a)^2 A_n$, $b^n-a^n-(b-a)na^{n-1}= b^n+(n-1)a^n-(nb)a^{n-1}$ $=(b-a)^2\left(\sum_{m=0}^{n-2} c_m a^m b^{n-m-2}\right)= \{c_0\, b^n+ (c_1-2c_0)\, b^{n-1}a+ (c_2-2c_1+c_0)\, b^{n-2}a^2 + \ldots+ (-2c_{n-2}+c_{n-3})\, ba^{n-1} + c_{n-2}\, a^n\}$. By comparing the corresponding coefficients: $\{c_0=1,\ c_1=2c_0=2,\ \ldots,\ c_m= m+1,\ \ldots,\ c_{n-2}= n-1\}$. $\Rightarrow c_m=m+1$.

Consequently, $q=\sum_{n=2}^{\infty} a_n(b-a)A_n = \sum_{n=2}^{\infty}\left[a_n\Delta z\left(\sum_{m=0}^{n-2}(m+1)z^m(z+\Delta z)^{n-m-2}\right)\right]$, by the triangular inequality $\left(\left|\sum z_n\right| \le \sum |z_n|\right)$, $|q|\le |\Delta z|\cdot\sum_{n=2}^{\infty}\left[|a_n|\cdot\left(\sum_{m=0}^{n-2}(m+1)|z|^m\cdot|z+\Delta z|^{n-m-2}\right)\right]$.



For any point z in the ROC ($|z|<R$), we can choose some $R_0<R$, such that $\{|z|, |z+\Delta z|\}\le R_0$, $\Rightarrow$

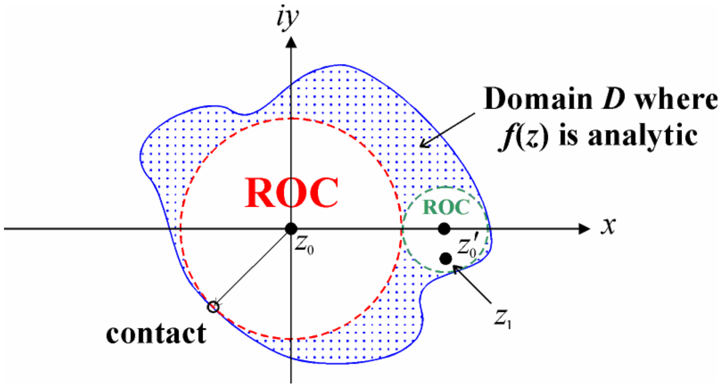
$|z|^m\cdot|z+\Delta z|^{n-m-2}\le R_0^{n-2}$, $\Rightarrow$ $|q| \le |\Delta z|\cdot\sum_{n=2}^{\infty}\left[|a_n|\cdot R_0^{n-2}\left(\sum_{m=0}^{n-2}(m+1)\right)\right]=|\Delta z|\cdot\sum_{n=2}^{\infty}|a_n|\cdot R_0^{n-2}\dfrac{n(n-1)}{2} \le$

$|\Delta z|\cdot\left[\sum_{n=2}^{\infty}n(n-1)|a_n|R_0^{n-2}\right]$.

Since the series $\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}=\sum_{n=0}^{\infty}(a_n z^n)''$, and $\sum_{n=0}^{\infty}a_n z^n$ is absolutely convergent for $|z|=R_0<R$, (i.e. $\sum_{n=0}^{\infty}|a_n|R_0^n$ is convergent), by operation 3 (see below), $\Rightarrow$ $\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}$ is also absolutely convergent for $|z|=R_0$, (i.e. $\sum_{n=2}^{\infty}n(n-1)|a_n|R_0^{n-2}=K$). $\Rightarrow$ $|q|\le|\Delta z|\cdot K$, $\lim_{\Delta z\to 0}|q|=0$.

<Comment>

As will be proved in the Taylor theorem (EK 15.4), for every point z in "the domain $\boldsymbol{D}$ where function $f(z)$ is analytic", there is a unique power series $\sum_{n=0}^{\infty}a_n(z-z_0)^n$, such that $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$ ($z\in$**ROC** of the power series). However, for different points in D, the corresponding power series could be **different**. **E.g.** A point $z_1\in D$ lying outside the ROC of $\sum_{n=0}^{\infty}a_n(z-z_0)^n$ (i.e. $\sum_{n=0}^{\infty}a_n(z_1-z_0)^n \ne f(z_1)$) should correspond to another series $\sum_{n=0}^{\infty}b_n(z-z_0')^n$ with different center z_0'.



■ Operations on power series

1) If $f(z)=\sum_{n=0}^{\infty} a_n z^n$, for $|z|<R_a$, $g(z)=\sum_{n=0}^{\infty} b_n z^n$, for $|z|<R_b$, $\Rightarrow$ $f(z)\pm g(z)=\sum_{n=0}^{\infty} a_n z^n \pm \sum_{n=0}^{\infty} b_n z^n =$ $\sum_{n=0}^{\infty}(a_n \pm b_n)z^n$, where the new radius of convergence $R \geq \min\{R_a, R_b\}$.

2) If $f(z)=\sum_{n=0}^{\infty} a_n z^n$, for $|z|<R_a$, $g(z)=\sum_{n=0}^{\infty} b_n z^n$, for $|z|<R_b$, $\Rightarrow$ $f(z)\cdot g(z)=\sum_{n=0}^{\infty} a_n z^n \times \sum_{n=0}^{\infty} b_n z^n =$ $\sum_{n=0}^{\infty} c_n z^n$, for $|z|<R$, where

$$c_n=\sum_{k=0}^{n} a_{n-k} b_k \tag{11.2}$$

is the Cauchy product (convolution) of $\{a_n\}$, $\{b_n\}$; and $R \geq \min\{R_a, R_b\}$.

3) If $f(z)=\sum_{n=0}^{\infty} a_n z^n$, for $|z|<R$, $\Rightarrow$

$$f'(z)=\sum_{n=1}^{\infty} n a_n z^{n-1} \tag{11.3}$$

(derived series), which is valid for $|z|<R$.

E.g. $f(z)=\sum_{n=2}^{\infty}\binom{n}{2} z^n = \frac{z^2}{2} g''(z)$, where $g(z)=\sum_{n=0}^{\infty} z^n$ $\Rightarrow$ By 3), $f(z)$, $g(z)$ have the same R=1.

You can arrive at the same R by Cauchy-Hadamard formula.

4) Termwise integration of $\sum_{n=0}^{\infty} a_n z^n$ is $\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}$, which has the same radius of convergence R.