Lesson 10 Complex Integration (EK 14)

■ Usefulness

1) Evaluate some difficult real integrals $\int_{x_0}^{x_1} f(x)dx$.

2) Represent derivatives of analytic functions [eq. (10.9)].

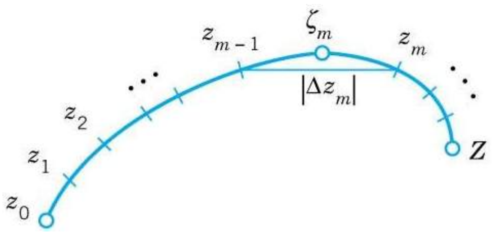
Line Integral (EK 14.1)

■ Definition

The line integral over curve C: $\{z(t)=x(t)+iy(t)\}$ in the complex plane is defined as the limit of partial sum (部分和):

$$\int_C f(z)dz \equiv \lim_{n\to\infty} S_n\,, \quad S_n=\sum_{m=1}^{n} f(\zeta_m)\Delta z_m \tag{10.1}$$

where z_m (m=0, 1, …, n) are partition points of curve C, $z_{m-1}<\zeta_m<z_m$, $\Delta z_m = z_m - z_{m-1}$.



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Work done by a force $\vec{F}=(F_x(x,y), F_y(x,y))$ along a path C is: $W=\int_C \vec{F}\cdot d\vec{x}$, which can also be evaluated by the limit of partial sum. In this case, $\vec{F}(\vec{\xi}_m)\cdot\Delta\vec{x}_m$ (inner product) is a real number, while $f(\zeta_m)\cdot\Delta z_m$ (complex product) is a complex number.

■ Methods of evaluating line integrals

1) Partial integration: By eq. (10.1), if $f(z=x+iy)=u(x,y)+iv(x,y)$ is continuous (not necessarily

analytic) along a piecewise smooth path C, $\Rightarrow$

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (udy + vdx) \tag{10.2}$$

Note: $\int_C udx = \int_{x_0}^{x_1} u(x, \boldsymbol{y(x)})dx$, where x_0 , x_1 are the real parts of end points of C: $y(x)$.

2) Using parametric representation of the path: If C: $\{z(t),\ a\le t\le b\}$ is piecewise smooth, and $f(z)$ is continuous on C, $\Rightarrow$

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt \tag{10.3}$$

<u>Proof</u>: $\int_a^b f(z(t))z'(t)dt = \int_a^b [u(t)+iv(t)]\cdot[x'(t)+iy'(t)]dt = \int_C [u+iv]\cdot[dx+idy] =$ $\int_C [udx - vdy] + i\int_C [udy+vdx] =$ eq. (10.2) $= \int_C f(z)dz$.

E.g. If C is a counterclockwise circle of radius ρ centered at z_0: $\{z(t)=z_0+\rho e^{it},\ 0\le t\le 2\pi\}$,

$I=\oint_C (z-z_0)^m dz = \int_0^{2\pi} \rho^m e^{imt}\cdot i\rho e^{it}dt = i\rho^{(m+1)}\int_0^{2\pi} e^{i(m+1)t}dt = \begin{cases} 2\pi i, \text{if } m=-1 \\ 0, \text{otherwise} \end{cases}$. $\Rightarrow$

$$\oint_C (z-z_0)^m dz = \begin{cases} 2\pi i, \text{if } m=-1 \\ 0, \text{otherwise} \end{cases} \tag{10.4}$$

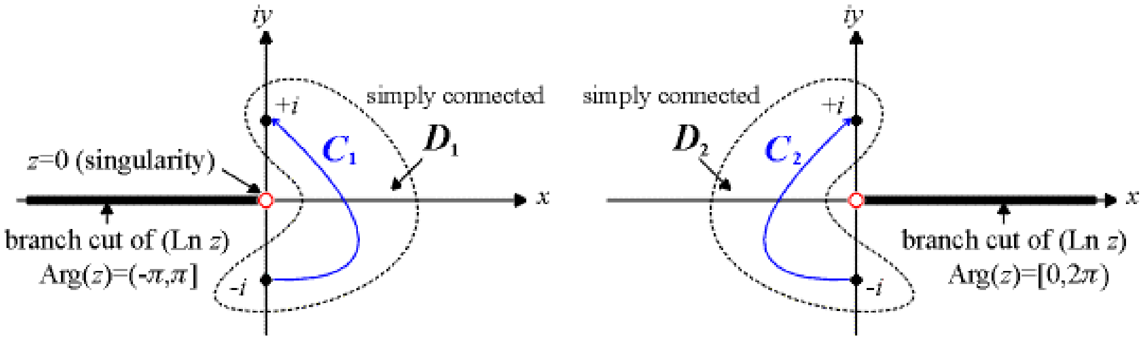
3) By antiderivative: If $f(z)$ is **analytic** in a simply connected domain D (EK14.2), there exists an analytic antiderivative function $F(z)$ (反導函數), such that $F'(z) = f(z)$, and

$$\int_C f(z)dz = \int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0) \tag{10.5}$$

for any integral path C within D. Eq. (10.5) depends only on end points z_0, z_1 of the path C (will be proved by Cauchy's integral theorem in EK 14.2).

E.g. $I=\int_0^{1+i} z^2 dz = \left.\frac{z^3}{3}\right|_0^{1+i} = \frac{(1+i)^3}{3} = \frac{-2}{3} + i\frac{2}{3}$, for z^2 is analytic everywhere.

E.g. $I=\int_{-i}^{+i} z^{-1}dz$ along two paths C_1, C_2 (see below).



Since z^{-1} is analytic except for $\boldsymbol{z=0}$ (check by CR equations), we can find a simply connected domain D_j containing path C_j (j=1, 2) in which z^{-1} is analytic everywhere. By eq. (10.5), $I=\ln(z)\big|_{-i}^{+i}=\ln(i)-\ln(-i)$.

However, ln(z) is multi-valued [eq. (9.6)], and so is $\ln(z)\big|_{-i}^{+i}$ (problematic). If principle value Ln(z) is used [eq. (9.7)], $I=\text{Ln}(z)\big|_{-i}^{+i}=i[\text{Arg}(i)-\text{Arg}(-i)]$. One has to properly define the range of argument such that path C_j does not cross the branch cut [i.e. Ln z = ln(|z|)+iArg(z) experiences no "jump" along C_j].

For an arbitrary path C_1 in D_1, define $-\pi<\text{Arg}(z)\le\pi$ (branch cut is negative real axis), ⇒ $I_1=i\dfrac{\pi}{2}-\left(-i\dfrac{\pi}{2}\right)=i\pi$. For an arbitrary path C_2 in D_2, define $0\le\text{Arg}(z)\le 2\pi$ (branch cut is positive real axis), ⇒ $I_2=i\dfrac{\pi}{2}-i\dfrac{3\pi}{2}=-i\pi$.

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1) $I_1\ne I_2$ for there is no simply connected domain D containing both C_1 and C_2.
2) A singular point has profound impact on complex integral even the path does not pass through it. ⇒ Singularity is the protagonist of complex functions.

■ *ML*-inequality

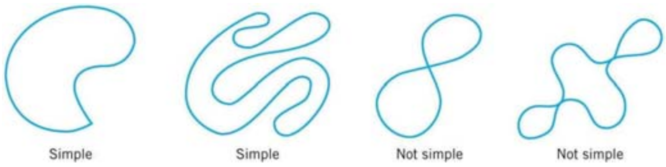
If $|f(z)|\le M$ everywhere on a path C of length L, ⇒

$$\left|\int_C f(z)dz\right|\le ML \qquad (10.6)$$

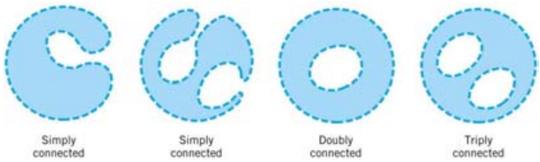
It is useful in proving integral theorems.

Cauchy's Integral Theorem (EK14.2)

■ Concepts



Closed path



Simply and multiply connected domains

■ Theorem 1

If $f(z)$ is analytic in a simply connected domain D, $\Rightarrow$

$$\oint_C f(z)dz = 0 \qquad (10.7)$$

for any simple closed path C in D.

Intuition: Analytic in D $\Rightarrow$ antiderivative approach eq. (10.5) is valid (will be proven in Theorem 3): $\int_C f(z)dz = F(z_1) - F(z_0)$. Closed path C means $z_1 = z_0$, $\Rightarrow$ integral=0.

(*) Proof: **(1)** By eq. (10.2), $I = \oint_C f(z)dz = \oint_C (udx - vdy) + i\oint_C (udy + vdx)$. **(2)** By Green's theorem in the plane (EK 10.4, i.e. a special case of Stoke's theorem in EK 10.9): $\mathrm{Re}\{I\} = \oint_C (udx - vdy) = \iint_R (-v_x - u_y)dxdy$, where v_x, u_y are continuous for $f'(z)$ is continuous in D. **(3)** By CR equations: $u_y = -v_x$, $\mathrm{Re}\{I\} = \iint_R (0)dxdy = 0$. Similarly, $\mathrm{Im}\{I\} = 0$.

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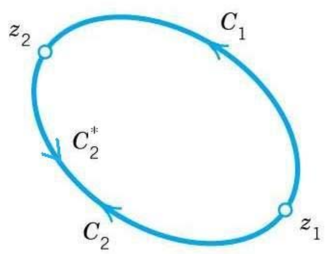
1) Inverse of Theorem 1 is not true. **E.g.** eq. (10.4).

2) (*) If $f(z)$ is continuous in a simply connected domain D, and $\oint_C f(z)dz = 0$ for any

simple closed path C in D, $\Rightarrow f(z)$ is analytic (Morera theorem).

■ Theorem 2

If $f(z)$ is analytic in a simply connected domain D, $\Rightarrow$ $\int_C f(z)dz$ is independent of path C but its end points.



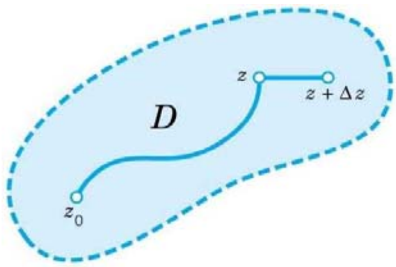
Proof: For two arbitrary paths C_1, C_2 with common endpoints z_1, z_2, Theorem 1 gives $\oint_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2^*} f(z)dz = 0$, $\Rightarrow$ $\int_{C_1} f(z)dz = -\int_{C_2^*} f(z)dz = \int_{C2} f(z)dz$.

■ Theorem 3 [enable eq. (10.5)]

If $f(z)$ is analytic in a simply connected domain D, $\Rightarrow$ (1) antiderivative $F(z) \equiv \int_{z_0}^{z} f(z')dz'$ exists; (2) $F'(z) = f(z)$; (3) $F(z)$ is also analytic in D.

Note: Theorem 3 and Theorem 2 prove eq. (10.5).

(*) Proof: **(1)** By Theorem 2, line integral from fixed z_0 to arbitrary z (in D) is independent of path, therefore, can be uniquely determined, $\Rightarrow F(z) \equiv \int_{z_0}^{z} f(z')dz'$ exists.



(2) By definition, $F'(z) = \lim_{\Delta z\to 0}\frac{F(z+\Delta z)-F(z)}{\Delta z} = \lim_{\Delta z\to 0}\frac{1}{\Delta z}\left[\int_{z_0}^{z+\Delta z} f(z')dz' - \int_{z_0}^{z} f(z')dz'\right] =$ $\lim_{\Delta z\to 0}\frac{1}{\Delta z}\left[\int_{z}^{z+\Delta z} f(z')dz'\right]$. Represent $f(z)$ by integral form: $f(z)= \lim_{\Delta z\to 0}\frac{1}{\Delta z}\left[\int_{z}^{z+\Delta z} f(z)dz'\right]$; $\Rightarrow$

$\left|F'(z)-f(z)\right| = \lim_{\Delta z\to 0}\frac{1}{|\Delta z|}\cdot\left|\int_{z}^{z+\Delta z}[f(z')-f(z)]dz'\right|$, by ML-inequality, $\le \lim_{\Delta z\to 0}\left(\frac{1}{|\Delta z|}\cdot M\cdot|\Delta z|\right) =$

$\lim_{\Delta z\to 0} M$, where M is defined as: $|f(z')-f(z)|\le M$ along the infinitesimal path $z\to z+\Delta z$.

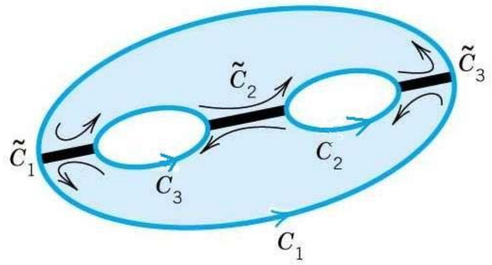
Since $f(z)$ is analytic, it must be continuous; $\Rightarrow$ for any given $\varepsilon>0$, we can always find δ, such that $|f(z')-f(z)|\le\varepsilon$ for all $|z'-z|<\delta$. By choosing $|\Delta z|<\delta$, we have $M=\varepsilon\to 0$, $\Rightarrow$ $|F'(z)-f(z)|\to 0$, and $F'(z)=f(z)$.

(3) For every point z in D, $f(z)$ exists, $\Rightarrow$ $F'(z)=f(z)$ exists, $\Rightarrow$ $F(z)$ is also analytic.

■ Theorem 4

If $f(z)$ is analytic in a multiply connected domain D defined by an outer contour C_1 and multiple inner contours $\{C_i, i=2,3,\dots,n\}$ (all are in counterclockwise sense), $\Rightarrow$

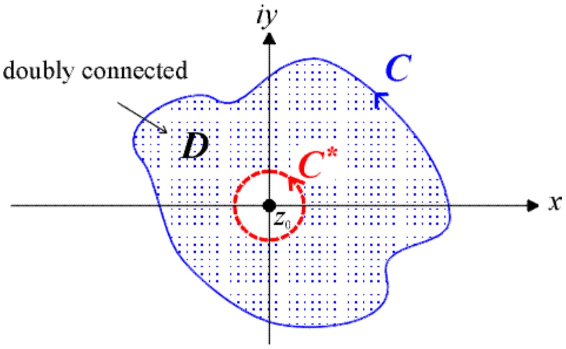
$$\oint_{C_1} f(z)dz = \sum_{i=2}^{n}\oint_{C_i} f(z)dz \tag{10.8}$$



<u>Proof</u>: Introducing three inner cuts $\hat{C}_1$, $\hat{C}_2$, $\hat{C}_3$ to divide the domain D into two simply connected domains. Apply Theorem 1 to them; integral over cuts will be canceled, …

E.g. $\oint_C (z-z_0)^m dz = \begin{cases} 2\pi i, \text{if } m=-1 \\ 0, \text{otherwise}\end{cases}$, for **arbitrary** simple closed path C in counterclockwise sense, which encircles z_0.

$(z-z_0)^m$ is analytic in a **doubly connected** domain D bounded by C and C^* (a circle of sufficiently small radius ρ centered at z_0). By eq. (10.8), $\oint_C (z-z_0)^m dz = \oint_{C^*} (z-z_0)^m dz =$ eq. (10.4).



Cauchy's Integral Formula (EK 14.3)

■ Formula

Let $f(z)$ be analytic in a simply connected domain D. For **any** simple closed path C (not just circles) in D that encloses a point z_0, ⇒

$$\oint_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0) \tag{10.9}$$

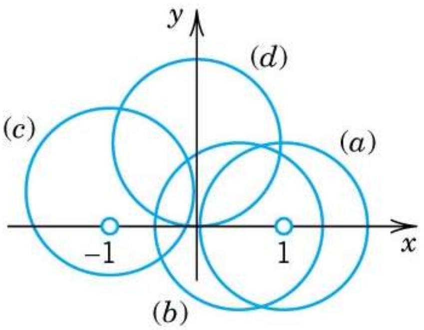
Note: eq. (10.9) is a special case of residue integration formula [eq. (13.2)].

Proof: $f(z)=f(z_0)+[f(z)-f(z_0)]$ ⇒ $\oint_C \frac{f(z)}{z-z_0} dz = f(z_0)\left(\oint_C \frac{1}{z-z_0} dz\right) + \oint_C \frac{f(z)-f(z_0)}{z-z_0} dz =$ $2\pi i f(z_0)+p$. Whether $p\to 0$ (i.e. $|p|\to 0$)?

(*) By eq. (10.8), $p=\oint_C \frac{f(z)-f(z_0)}{z-z_0} dz = \oint_{C^*} \frac{f(z)-f(z_0)}{z-z_0} dz$, where C^*: $|z-z_0|=\rho$ is a circle enclosed by C (see above figure). By ML-inequality, $|p|\le M\cdot 2\pi\rho$, where $\left|\frac{f(z)-f(z_0)}{z-z_0}\right| \le M$.

Since $f(z)$ is analytic, ⇒ continuous, $\lim_{z\to z_0} f(z) = f(z_0)$, i.e. $|z-z_0|<\delta$, ⇒ $|f(z)-f(z_0)|<\varepsilon$. For an arbitrarily small ε, we can choose $\rho<\delta$, such that $\left|\dfrac{f(z)-f(z_0)}{z-z_0}\right| \le \dfrac{\varepsilon}{\rho}=M$ on C^*. $\Rightarrow$ $|p|\le M\cdot 2\pi\rho$ $=\dfrac{\varepsilon}{\rho}\cdot 2\pi\rho=2\pi\varepsilon\to 0$, as $\varepsilon\to 0$.

E.g. Evaluate $I=\oint_C \dfrac{z^2+1}{z^2-1}dz$ for the four contours shown below (in counterclockwise sense).



$\dfrac{z^2+1}{z^2-1}=\dfrac{z^2+1}{(z+1)(z-1)}$. Circles (*a*-*b*) only enclose $z=z_0=1$, $\Rightarrow$ $f(z)=\dfrac{z^2+1}{z+1}$, $I=2\pi i\cdot f(1)=2\pi i$. Circle (*c*) only encloses $z=z_0=-1$, $\Rightarrow$ $f(z)=\dfrac{z^2+1}{z-1}$, $I=2\pi i\cdot f(-1)=-2\pi i$. Circle (*d*) encloses no singularity, $\Rightarrow$ $\dfrac{z^2+1}{(z+1)(z-1)}$ is analytic inside circle (*d*), $I=0$.

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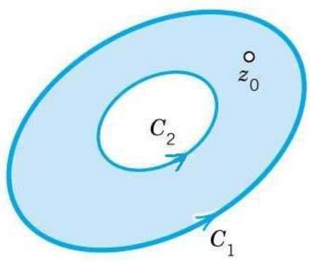
If $f(z)$ is analytic in a doubly connected domain D bounded by two counterclockwise contours C_1, C_2, $\Rightarrow$

$$f(z_0)=\frac{1}{2\pi i}\left[\oint_{C_1}\frac{f(z)}{z-z_0}dz-\oint_{C_2}\frac{f(z)}{z-z_0}dz\right] \qquad (10.10)$$

<u>Proof</u>: Introducing two inner cuts. Used in proving Laurent's theorem (EK 16.1).



Derivatives of Analytic Functions (EK 14.4)

■ Analytic ⇒ differentiable for all orders

If $f(z)$ is analytic in a simply connected domain D, its n-th order derivative $f^{(n)}(z_0)$ exists and can be evaluated by a complex line integral over a simple closed path C in D that encloses z_0:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz \tag{10.11}$$

<u>Proof</u>: For n=1, $f'(z_0) = \lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$; represent $f(z_0+\Delta z)$, $f(z_0)$ by eq. (10.9):

$$f'(z_0) = \lim_{\Delta z\to 0}\frac{1}{2\pi i\Delta z}\left[\oint_C \frac{f(z)}{z-(z_0+\Delta z)}dz - \oint_C \frac{f(z)}{z-z_0}dz\right] = \lim_{\Delta z\to 0}\frac{1}{2\pi i}\left[\oint_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}dz\right];$$

by ML-inequality, $= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz$. For n>1, prove by induction.

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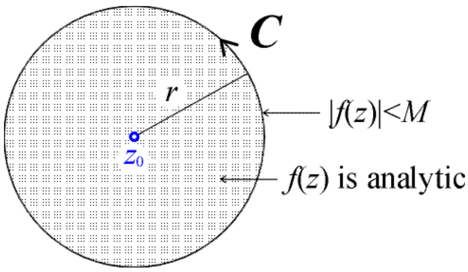
1) Evaluation of $f^{(n)}(x_0)$ involves with real function values in the vicinity of x_0, while evaluation of $f^{(n)}(z_0)$ can involve with complex function values far from z_0.

2) The differentiability of a real function $f(x)$ implies nothing about the differentiability of $f'(x)$, $f''(z)$, …etc. **E.g.** $f(x)= x^{1/3}$ is differentiable for all $x\in R$, but $f'(x) = \frac{1}{3}x^{-2/3}$ is singular at x=0.

■ (*) Cauchy's inequality

If $f(z)$ is analytic on and within a circle C of radius r and center z_0, and $|f(z)|\le M$ on C, ⇒

$$\left|f^{(n)}(z_0)\right| \le \frac{n!M}{r^n} \tag{10.12}$$



Proof: By eq. (10.11), $\left|f^{(n)}(z_0)\right| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right|$, by ML-inequality, $\le \frac{n!}{2\pi}\frac{M}{r^{n+1}}2\pi r \ldots$

Cauchy's inequality will be used to set an upper bound for the coefficients of Taylor series representation of a complex function (EK 15.4).

■ Liouville's theorem

If $f(z)$ is analytic and $|f(z)|<K$ (bounded) in the entire complex plane,⇒ $f(z)$ is constant.

Proof: By eq. (10.12), $\left|f'(z_0)\right| \le \frac{K}{r}$ for arbitrary z_0 and r. By letting $r\to\infty$, $\left|f'(z_0)\right| \le 0$, $\left|f'(z_0)\right| = 0$, i.e. $f(z)$ is a constant.

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1) Bounded, differentiable real functions are not necessarily constant. **E.g.** (sin x).
2) Slight deviation of an analytic function from constant implies the existence of singularity somewhere in the complex plane, ⇒ singularity is almost inevitable!

■ (*) Fundamental theory of algebra

If $p(z)=a_n z^n + a_{n-1}z^{n-1}+\ldots+a_1 z+a_0$, $n\ge 1$, $a_n\ne 0$ (polynomial of order n) ⇒ $p(z)=0$ has at least one root (actually n roots in total).

Proof: By contradiction. $p(z)$ is unbounded and entire (i.e. analytic everywhere). Assume

$p(z) \neq 0$ for all z (no root), $\Rightarrow f(z) \equiv 1/p(z)$ is entire but bounded. By Liouville's theorem, $\Rightarrow$ $f(z)$=constant, $\Rightarrow p(z)$=constant, violating $n \geq 1$, $a_n \neq 0$.

■ (*) Gauss mean-value property

Let $f(z)$ is analytic in a simply connected domain D. If we take a circular contour $C(\subset D)$: $z=z_0+re^{i\theta}$, $\theta=[0,2\pi]$, by eq. (10.9), $f(z_0)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})d\theta$, $\Rightarrow f(z_0)$ is the mean-value of $f(z)$ on circle C with arbitrary radius r (as long as $C \subset D$).

■ (*) Maximum/minimum modulus principle

If $f(z)$ is analytic on and within a simple closed path C, $\Rightarrow$ the maximum and minimum of $|f(z)|$ for the region R (union of C and its interior) must occur on C.

Proof: By Gauss mean-value principle.