Lesson 09 Complex Numbers and Functions (EK 13)

■ Overview

Contents: complex numbers, analytic functions, complex series, complex integral.

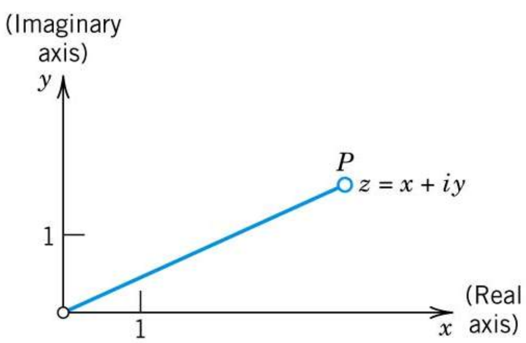
Applications:

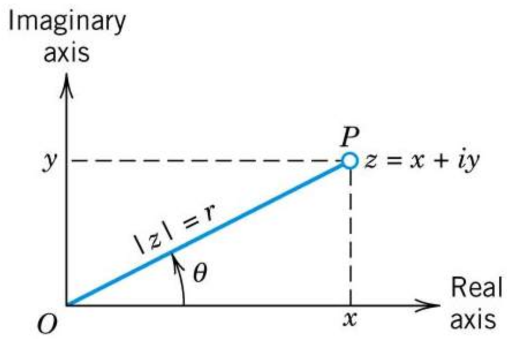
1) Evaluate complicated real and complex integrals (EK 16).
2) Derive Fourier and Laplace transforms in closed form.
3) Solve 2-D Laplace's equation (EK 17, 18).

Complex Numbers (EK13.1-2)

■ Representations

1) Cartesian form: $z=x+iy$, where $x=\text{Re}\{z\}$, $y=\text{Im}\{z\}$, $i=\sqrt{-1}$; a point in the complex plane.
2) Polar form: $z=re^{i\theta}$, where $r=|z|=\sqrt{x^2+y^2}$ = **modulus** of z, $\theta=\text{Arg}(z)=\tan^{-1}\left(\frac{y}{x}\right)$ = **argument** (幅角) of z. Since $\tan^{-1}$ gives values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (I and IV quadrants), $\tan^{-1}\left(\frac{y}{x}\right)$ cannot tell the arguments of z and $-z$, $\Rightarrow$ $\theta=\tan^{-1}\left(\frac{y}{x}\right)+\pi$, if $x<0$.





■ Properties

1) Addition/subtraction: Cartesian form is preferable. $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$.

2) Multiplication/division: Polar form is preferable. $z_1 z_2 = (r_1 r_2)\, e^{i(\theta_1+\theta_2)}$, $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\, e^{i(\theta_1-\theta_2)}$.

3) Complex conjugate: $z^* = (x - iy) = r e^{-i\theta}$. The conjugate pairs z and z^* can be used to express parameters in Cartesian and polar forms:

$$x = \frac{z+z^*}{2},\ y = \frac{z-z^*}{2i},\ r = \sqrt{z\cdot z^*},\ \theta = \frac{1}{2}\,\mathrm{Arg}\left(\frac{z}{z^*}\right).$$

4) Integral powers: $z^n = r^n e^{in\theta}$ (proved by induction).

Applications: expressing $\cos(n\theta)$, $\sin(n\theta)$ in terms of powers of $\cos\theta$, $\sin\theta$.

E.g. $\cos(2\theta) = \mathrm{Re}\{e^{i2\theta}\} = \mathrm{Re}\{(\cos\theta + i\sin\theta)^2\} = \cos^2\theta - \sin^2\theta$.

5) Integral roots: $\sqrt[n]{z} = \sqrt[n]{r}\ \exp\left[i\dfrac{\theta+2k\pi}{n}\right]$, $k=0, 1, \ldots, n-1$ (multivalued function).

Proof: Let $z = re^{i\theta}$, $w = Re^{i\phi} = \sqrt[n]{z}$ $\Rightarrow w^n = z$, $R^n e^{in\phi} = re^{i\theta}$, $R = \sqrt[n]{r}$, $\phi = \dfrac{\theta+2k\pi}{n}$, $k=0, \ldots, n\text{-}1$.

Analytic Functions (EK13.3)

■ Sets in the complex plane

1) Neighborhood of a: $\{z, |z-a|<\rho\}$ (open circular disk).

2) Open set S: every point of S has a neighborhood only consisting of points belonging to S.

E.g. $|z|<1$ is open, $|z|\le 1$ is not open.

3) Connected set S: any two of its points can be joined by a broken line (linear segments) within S. **E.g.** $\{|z|<1$ and $|z-3|<1\}$ is NOT connected.

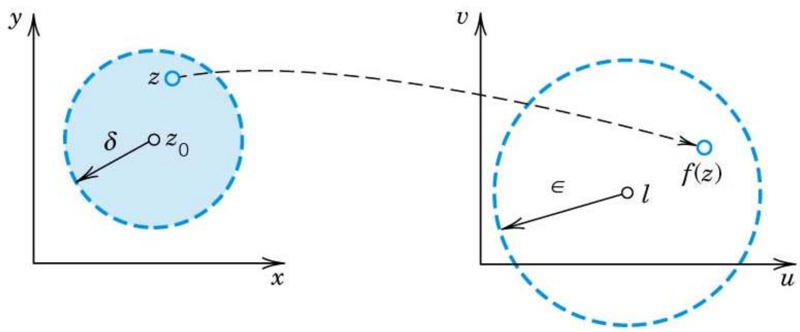
4) **Domain**: an open connected set.

■ Complex functions

$w = f(z = x+iy) = u(x,y) + iv(x,y)$. A complex function w of single complex variable z is equivalent to a pair of real functions $u(x,y)$ and $v(x,y)$, each depending on two real variables x, y.

■ Limit

$\lim_{z\to z_0} f(z)=l$: for every real $\varepsilon>0$, we can find a real $\delta>0$, such that $|f(z)-l|<\varepsilon$ if $0<|z-z_0|<\delta$.



Unlike the limit of real functions: $\lim_{x\to x_0} f(x)=l$, where x can only approach x_0 from left and right hand sides (1-D); z can approach z_0 from infinitely many directions in the complex plane (2-D, draw a plot). The limit exists only if $f(z)$ approaches the same l from all possible paths.

■ Continuous

A function $f(z)$ is continuous at $z=z_0$, if $\lim_{z\to z_0} f(z)=f(z_0)$.

■ Derivative

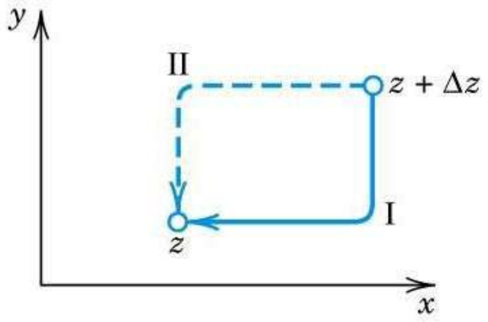
$f(z)$ is **differentiable** at z_0, if the derivative (limit of difference quotient):

$$f'(z_0)=\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} \tag{9.1}$$

approaches the same value as $\Delta z\to 0$ along all paths.

E.g. $f(z)=z^2$. $f'(z)=\lim_{\Delta z\to 0}\frac{(z+\Delta z)^2-z^2}{\Delta z}=\lim_{\Delta z\to 0}\frac{2z(\Delta z)+(\Delta z)^2}{\Delta z}=2z$, for all $z\in C$.

E.g. $f(z)=z^*$. $f'(z)=\lim_{\Delta z\to 0}\frac{(z+\Delta z)^*-z^*}{\Delta z}=\lim_{\Delta z\to 0}\frac{(\Delta z)^*}{\Delta z}=\lim_{\Delta z\to 0}\frac{\Delta x-i\Delta y}{\Delta x+i\Delta y}$, whose value depends on the path (see figure below). $\Rightarrow$ The derivative does not exist for **all** $z \in C$. Unlike real functions, differentiability is a rather strict requirement for complex functions.



For path I, $\Delta y=0$, $\Delta z=\Delta x\to 0$, causing a limit of +1. For path II, $\Delta x=0$, $\Delta z=i\Delta y\to 0$, causing a limit of -1.

Note we cannot let $\Delta x=\Delta y=0$ simultaneously, which violates the definition of "limit".

$f(z)$ is **analytic** in a domain D, if $f(z)$ is **differentiable** at all points of D. $f(z)$ is analytic at a point z_0, if $f(z)$ is analytic in a neighborhood of z_0.

Cauchy-Riemann (CR) Equations (EK13.4)

■ $f(z=x+iy)=u(x,y)+iv(x,y)$ is analytic $\Leftrightarrow$ CR equations are satisfied:

$$\{u_x = v_y, \quad u_y = -v_x\} \tag{9.2}$$

Proof ($\Rightarrow$): Analytic $\Rightarrow$ differentiable, $f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$ has the same value for all possible paths $\Delta z\to 0$. We examine the two special paths I and II illustrated above:

(i) For path I, $\Delta z=\Delta x\to 0$, $f'(z)=\lim_{\Delta x\to 0}\frac{[u(x+\Delta x,y)+iv(x+\Delta x,y)]-[u(x,y)+iv(x,y)]}{\Delta x}=$ u_x+iv_x. (ii) For path II, $\Delta z=i\Delta y\to 0$, $\Rightarrow$ $f'(z)=v_y-iu_y$. Equating their real and imaginary parts gives rise to eq. (9.2).

E.g. $f(z)=z^*=(x-iy)$, $\{u=x, v=-y\}$; $u_x=1\neq v_y=-1$, $\Rightarrow f(z)=z^*$ is not analytic.

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The relations derived during the proof of CR equations:

$$f'(z)=u_x+iv_x=v_y-iu_y \tag{9.3}$$

can be used to evaluate complex derivative.

■ Real and imaginary parts of an analytic function satisfy 2-D Laplace's equation.

Proof: $f(z)=u(x,y)+iv(x,y)$ is analytic, by eq. (9.2), $\Rightarrow \{u_x=v_y,\ u_y=-v_x\}$. $\Rightarrow \{u_{xx}=v_{yx},\ u_{yy}=-v_{xy}\}$;

$u_{xx}+u_{yy}=\nabla^2 u=0$; Similarly, $\nabla^2 v=0$.

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$\nabla^2 u=0$, $\nabla^2 v=0$ does not guarantee that $f(z)=u(x,y)+iv(x,y)$ is analytic.

Exponential Function (EK13.5)

■ Definition:

$$e^z \equiv e^x(\cos y + i\sin y),\ \text{or}\ \sum_{n=0}^{\infty} z^n/n! \tag{9.4}$$

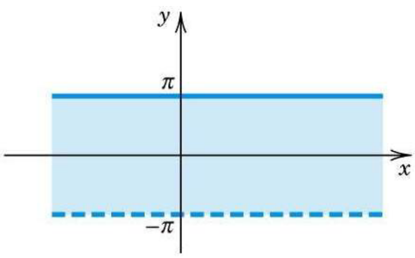
■ Properties:

1) e^z is analytic for all z. Proof: $u=e^x\cdot\cos(y)$, $v=e^x\cdot\sin(y)$, by eq. (9.2), ….

2) $\frac{d}{dz}e^z=e^z$. Proof: by eq. (9.3), $\frac{d}{dz}e^z=u_x+iv_x=u+iv=e^z$.

3) $e^{z_1+z_2}=e^{z_1}\cdot e^{z_2}$. Proof: let $z_1=x_1+iy_1$, $z_2=x_2+iy_2$, apply $e^{x_1+x_2}=e^{x_1}e^{x_2}$ and trigonometric equalities, ….

4) $e^{z+i2n\pi}=e^z$, hence the fundamental strip $(x, -\pi<y\le\pi)$ is mapped onto the entire w-plane.



5) The left half-plane $\{x\le 0\}$ is mapped onto the unit circle in the w-plane.

Trigonometric Functions (EK13.6)

■ Definition:

$$\cos z \equiv \frac{e^{iz}+e^{-iz}}{2},\quad \sin z \equiv \frac{e^{iz}-e^{-iz}}{2i},\quad \tan z \equiv \frac{\sin z}{\cos z} \tag{9.5}$$

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Real functions $\sin x$, $\cos x$ are unrelated to real function e^x.

■ Properties:

1) $\cos z$, $\sin z$ are analytic for all z, but $(\tan z)$ is not wherever $\cos z = 0$.
2) $\cos z = (\cos x\cdot \cosh y) - i(\sin x\cdot \sinh y)$, $\sin z = (\sin x\cdot\cosh y) + i(\cos x\cdot\sinh y)$. <u>Proof</u>: by $\cos z = \frac{1}{2}\left[e^{i(x+iy)} + e^{-i(x+iy)}\right] = \ldots$
3) $|\cos z|^2 = \cos^2 x + \sinh^2 y$, $|\sin z|^2 = \sin^2 x + \sinh^2 y$, not bounded by 1.

   <u>Proof</u>: by $|\cos z|^2 = (\cos x\cdot\cosh y)^2 + (\sin x\cdot\sinh y)^2$; $\cosh^2 y = 1 + \sinh^2 y$, $\Rightarrow$ $|\cos z|^2 = \cos^2 x(1+\sinh^2 y) + \sin^2 x\cdot\sinh^2 y = \ldots$
4) $\cos z = 0$, for $z = \frac{2n+1}{2}\pi$; $\sin z = 0$, for $z = n\pi$. <u>Proof</u>: by property 2.
5) $\frac{d}{dz}\cos z = -\sin z$, $\frac{d}{dz}\sin z = \cos z$. <u>Proof</u>: by eq. (9.5) and $\frac{d}{dz}e^z = e^z$.
6) General formulas of real trigonometric functions remain valid for complex counterparts.

Logarithm Function (EK13.7)

■ Definition:

Inverse of exponential function. $w = \ln z \Leftrightarrow z = e^w$. Let $z = re^{i\theta}$, $w = u + iv$, $\Rightarrow re^{i\theta} = e^{u+iv} = e^u e^{iv}$, $\Rightarrow$ $e^u = r$, $v = \theta + 2n\pi$. $\Rightarrow$

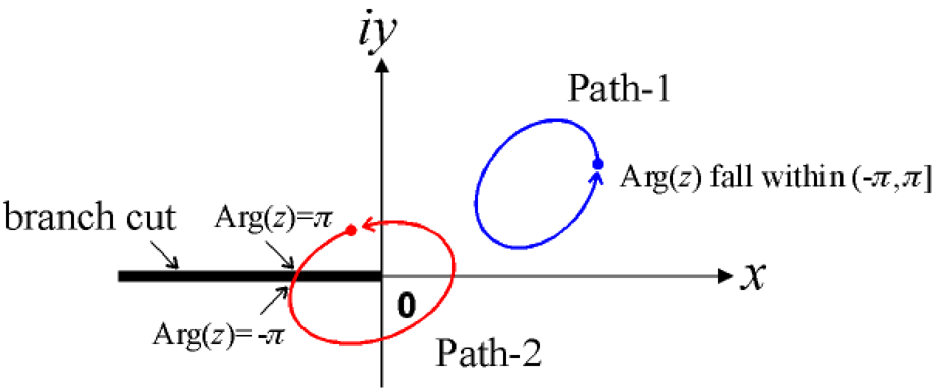
$$\ln z = \ln r + i(\theta + 2n\pi) \tag{9.6}$$

which is a **multi-valued** "function" [i.e. $re^{i\theta}$ and $re^{i(\theta+2n\pi)}$ represent the same point in the z-plane, but are mapped onto different points in the w-plane. Strictly speaking, it is not a function]. **E.g.** $\ln(i)=\ln(1)+i(0.5+2n)\pi$.

The principal value of $\ln z$ is defined as:

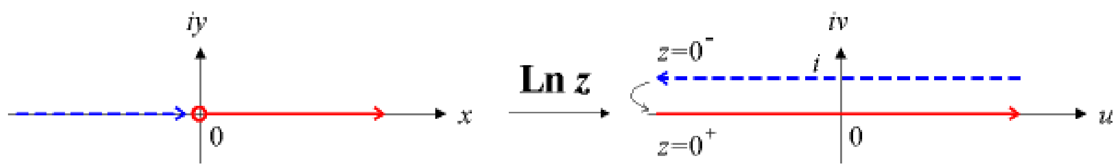
$$\mathrm{Ln}\, z = \ln|z| + i\,\mathrm{Arg}(z), \quad -\pi<\mathrm{Arg}(z)\le\pi \tag{9.7}$$

Though (Ln z) is single-valued, the restriction of argument [$-\pi<\mathrm{Arg}(z)\le\pi$] makes the function **discontinuous** (and non-analytic) when the variable z passes through a branch cut (negative real axis).



■ Properties:

1) If z is negatively real (where real logarithm is undefined), $\mathrm{Ln}\, z = \ln |z|+ i\pi$.



2) The branch cut of (Ln z) depends on the choice of argument range.

   E.g. if $0\le\mathrm{Arg}(z)<2\pi$ is used, branch cut becomes the positive real axis. In this case, (Ln z) is continuous and analytic on the negative real axis.

3) $\ln(z_1 z_2) = \ln z_1 + \ln z_2$, $\ln\left(\dfrac{z_1}{z_2}\right) = \ln z_1 - \ln z_2$. Proof: by $z_1 = r_1 e^{i\theta_1}$, …

4) $\dfrac{d}{dz}\mathrm{Ln}\, z = \dfrac{1}{z}$, except for points on the branch cut. Proof: by $u=\mathrm{Ln}|z|=\dfrac{1}{2}\ln\left(x^2+y^2\right)$,

$v=\mathrm{Arg}\{z\} = \begin{cases} \tan^{-1}(y/x), \text{if } x \ge 0 \\ \tan^{-1}(y/x)+\pi, \text{if } x < 0 \end{cases}$; by eq. (9.3), $\dfrac{d}{dz}\mathrm{Ln}\, z = u_x + i v_x = \dfrac{x-iy}{(x^2+y^2)} = \dfrac{1}{z}$.

5) General powers:

$$z^c = e^{c \ln z}; \quad a^z = e^{z \ln a} \tag{9.8}$$

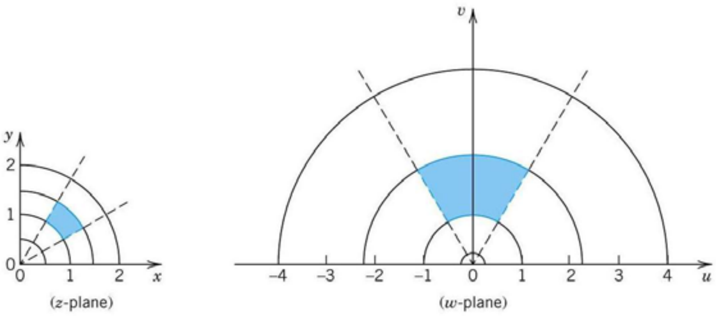
E.g. $i^i = e^{i \ln i} = e^{i\,[0+i(\pi/2+2n\pi)]} = e^{-(\pi/2+2n\pi)} \in R$.

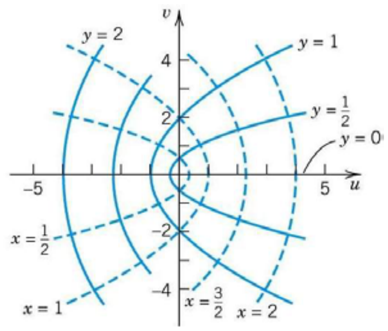
Appendix 9A − Conformal Mapping (EK 17)

■ Mapping

For each point z in the domain of definition D, function $f(z)$ assigns a point $f(z)$ (image of z) in the w-plane, $\Rightarrow f$ defines a mapping of D in the z-plane onto the w-plane.

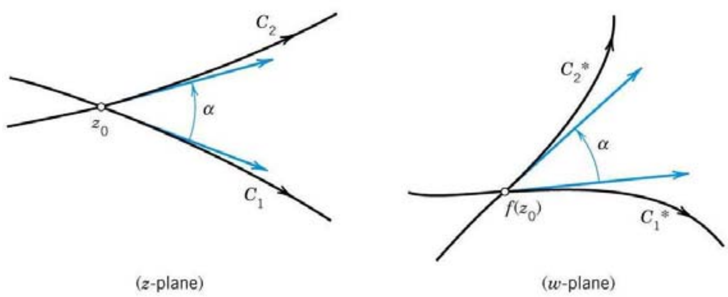
E.g. $w=z^2$. (i) $Re^{i\phi}=(re^{i\theta})^2=(r^2e^{i2\theta})$, $\Rightarrow$ $\{R=r^2,\ \phi=2\theta\}$ (lower left figure). (ii) $u+iv=(x+iy)^2\Rightarrow$ $\{u=x^2-y^2,\ v=2xy\}$. A vertical line in the xy-plane ($x=x_0$) is mapped onto a parabola in the uv-plane: $u=x_0^2-(v^2/4x_0^2)$. A horizontal line ($y=y_0$) is mapped onto a parabola: $u=(v^2/4y_0^2)-y_0^2$ (lower right figure).





■ Conformal mapping

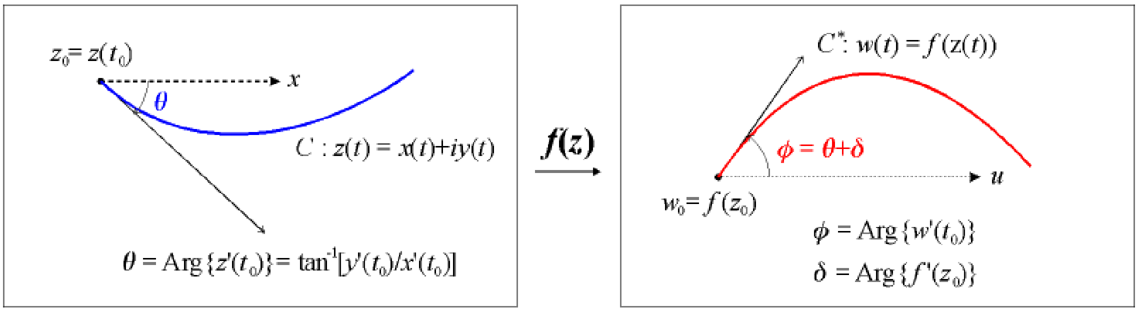
Mapping defined by an analytic function preserves the included angle (夾角, magnitude and sense) between any oriented curves.



<u>Proof</u>: Let curve C in the z-plane have a parametric representation: $z(t)=x(t)+iy(t)$. The tangential at $z=z_0=z(t_0)$ makes an angle θ with respect to the x-axis, where $\tan\theta$

$$=\lim_{\Delta t\to 0}\frac{\Delta y = y(t_0+\Delta t)-y(t_0)}{\Delta x = x(t_0+\Delta t)-x(t_0)}=\lim_{\Delta t\to 0}\frac{\left[y(t_0+\Delta t)-y(t_0)/\Delta t\right]}{\left[x(t_0+\Delta t)-x(t_0)/\Delta t\right]}=\frac{y'(t_0)}{x'(t_0)}\text{, i.e.}$$

$$\theta=\text{Arg}\left\{x'(t_0)+iy'(t_0)\right\}=\text{Arg}\left\{z'(t_0)\right\} \tag{9A.1}$$



Analytic function $f(z)$ maps curve C onto its image C^*: $w(t)=f(z(t))=u(t)+iv(t)$ in the w-plane.

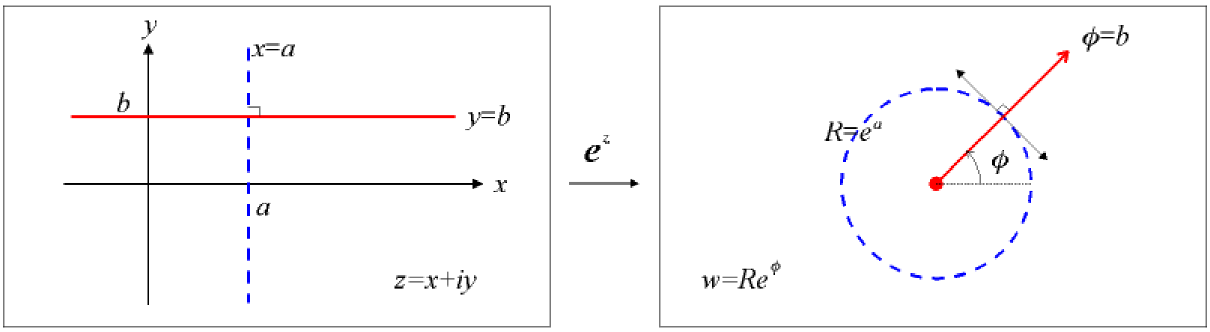
The tangential at $w_0=f(z_0)$ makes an angle ϕ with respect to the u-axis, where $\phi=\text{Arg}\left\{w'(t_0)\right\}$.

By the chain rule, $w'(t_0)=f'(z_0)\cdot z'(t_0) \Rightarrow \text{Arg}\left\{w'(t_0)\right\}=\text{Arg}\left\{f'(z_0)\right\}+\text{Arg}\left\{z'(t_0)\right\}$,

$$\phi=\delta+\theta,\quad \delta=\text{Arg}\left\{f'(z_0)\right\} \tag{9A.2}$$

Since δ is only determined by the mapping function, $f(z)$ rotates any oriented curve by the same angle δ, as long as $f'(z_0)\neq 0$ (i.e. $\text{Arg}\left\{f'(z_0)\right\}$ exists).

E.g. $f(z)=e^z$, $z=x+iy$; $w=Re^{i\phi}$, $=f(z)=e^{x+iy}=e^x e^{iy}$, $\Rightarrow$ $\{R=e^x,\ \phi=y\}$. Vertical line $x=a$ in the z-plane is mapped onto a circle of radius e^a in the w-plane. Horizontal line $y=b$ is mapped onto a ray of angle $\phi=b$. The included angles of $\{x=a, y=b\}$ and $\{R=e^a,\ \phi=b\}$ are the same (90°).



Conformal mapping is historically important in solving 2-D Laplace's equation. However, it relies on symmetry of the problem, and gives way to the more powerful numerical methods.