Lesson 07 Laplace's Equation

■ Overview

Laplace's equation describes the "potential" in gravitation, electrostatics, and steady-state behavior of various physical phenomena. Its solutions are called harmonic functions.

Physical meaning (SJF 31): Laplacian operator ∇^2 is a multi-dimensional generalization of 2nd-order derivative $\frac{d^2}{dx^2}$. Its difference quotient representation, as implied by eq. (1.2), is:

$$u_{xx}+u_{yy}=\lim_{\Delta\to 0}\left\{\frac{u(x+\Delta,y)-2u(x,y)+u(x-\Delta,y)}{\Delta^2}+\frac{u(x,y+\Delta)-2u(x,y)+u(x,y-\Delta)}{\Delta^2}\right\}$$

$$=\lim_{\Delta\to 0}\frac{-4}{\Delta^2}[u(x,y)-\bar{u}(x,y)] \qquad (7.1)$$

where $\bar{u}(x,y)\equiv\frac{u(x-\Delta,y)+u(x+\Delta x,y)+u(x,y-\Delta)+u(x,y+\Delta)}{4}$ represents the average of neighboring points (2D). As a result, $\nabla^2 u=0$ implies that the function value at any point is equal to the average of its neighboring values (dynamic equilibrium, or steady-state).

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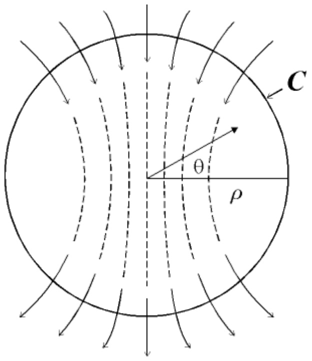
1) $\nabla^2 u=0$ does not necessarily mean u_{xx}=0 and u_{yy}=0.
2) Not all continuous functions satisfy $\nabla^2 u$=0. **E.g.** $u=x^2y$, $\Rightarrow \nabla^2 u=2y\neq 0$.

■ **(*)** Three types of BCs for Laplace's equation (similar with those in Lesson 3):

1) Dirichlet: u is specified on the boundary surface S (curve C). **E.g.** Find the electrostatic potential within/outside a circle where the potential on the circular rim is specified.
2) Neumann: outward normal derivative $u_n=\frac{\partial u}{\partial n}$ (physically, inward flux) is specified on $S(C)$. **E.g.** Find steady-state temperature within a circle if the heat inflow varies around

the boundary C according to: $\dfrac{\partial u}{\partial r}$=sin$\theta$.



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(a) Total flux across the boundary must vanish [in this case: $\int_C u_n = \int_0^{2\pi} (\sin\theta)\rho d\theta = 0$]. Otherwise, gain or loss exists in the region of interest, and physical quantity varies with time (no longer steady-state).

(b) Solutions to Neumann problems are **not unique**. **E.g.** $\{\nabla^2 u=0,\ u_r(r=1,\theta)=\cos(2\theta)\}$ have solutions of the form: $u(r,\theta)=r^2\cos(2\theta)+c$, c is an arbitrary constant. Additional information (such as the value of u at some point) is required.

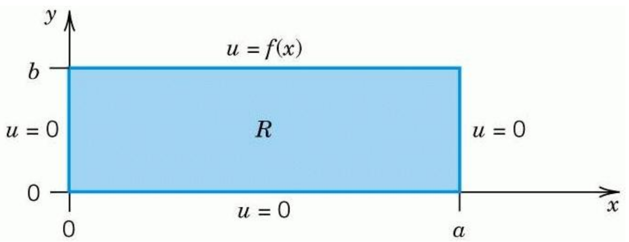
3) Mixed: a mixture of the first two types. **E.g.** $u_n+\gamma(u+g)=0$ (Newton's law of cooling).

Laplace's Equation in Cartesian Coordinate (EK 12.5)

■ Problem: steady state temperature distribution on a rectangular plate.

PDE: $u_t=\alpha^2(u_{xx}+u_{yy})=0 \Rightarrow u_{xx}+u_{yy}=0$, $\{0<x<a,\ 0<y<b\}$

Four Dirichlet BCs: $u(0,y)=0$, $u(a,y)=0$, $u(x,0)=0$, $u(x,b)=f(x)$.



■ Solving Cartesian Laplace's equation by separation of variables:

1) Separation of variables:

Let $u(x,y)=X(x)\cdot Y(y)$, $\Rightarrow$ $X''Y+X\ddot{Y}=0$, divide by XY, $\Rightarrow \dfrac{X''}{X}=-\dfrac{\ddot{Y}}{Y}=-\boldsymbol{k}^2<0$

$\Rightarrow X''+k^2X=0$, $\ddot{Y}-k^2Y=0$ (one PDE → two ODEs)

2) Solving the normal modes by **homogeneous** BCs:

(i) To avoid trivial solution $u(x,y)=0$, homogeneous BCs of $u(x,y)$ → BCs of $X(x)$, $Y(y)$:

$\{u(0,y)=0,\ u(a,y)=0,\ u(x,0)=0\} \to \{X(0)=0,\ X(a)=0,\ Y(0)=0\}$

(ii) $X''+k^2X=0$, $\Rightarrow X(x)=A\cos(kx)+B\sin(kx)$;

By BCs: (i) $X(0)=0 \Rightarrow A=0$; (ii) $X(a)=0 \Rightarrow k=k_n=\dfrac{n\pi}{a}$, $n=\mathbf{1},2,\ldots \Rightarrow X_n(x)=\sin(k_nx)$;

(iii) $\ddot{Y}_n-k_n^2Y_n=0 \Rightarrow Y_n(y)=A_ne^{k_ny}+B_ne^{-k_ny}$;

By BC: $Y(0)=0 \Rightarrow B_n=-A_n$, $\Rightarrow Y_n(y)=A_n\cdot\sinh(k_ny)$

$\Rightarrow$ The n-th normal mode is $u_n(x,y)=X_n(x)\cdot Y_n(y)$:

$$u_n(x,y)=A_n\cdot\sin(k_nx)\cdot\sinh(k_ny) \tag{7.2}$$

We have only one unknown coefficient A_n for each mode. The more homogeneous BCs, the fewer coefficients to be determined.

3) Determining the exact solution by the **nonhomogeneous BC** (similar to the role of ICs in t-dependent PDEs):

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,y)=\sum_{n=1}^{\infty}A_n\sin(k_nx)\cdot\sinh(k_ny) \tag{7.3}$$

Substitute the nonhomogeneous BC into eq. (7.3): $u(x,b)=\sum_{n=1}^{\infty}A_n\sinh(k_nb)\cdot\sin(k_nx)=f(x)$.

By Fourier sine series, $\Rightarrow$

$$A_n=\frac{2}{a\cdot\sinh(n\pi b/a)}\int_0^a f(x)\cdot\sin(k_nx)dx \tag{7.4}$$

Laplace's Equation in Polar Coordinates (EK 12.10, SJF 33, 34)

■ Overview

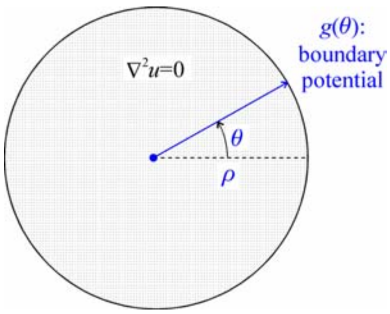
In solving circular membrane problem, we have seen that ∇^2 in polar coordinates leading to different ODEs and normal modes compared to ∇^2 in Cartesian coordinates. In this subsection, we will examine the normal modes of Laplace's equation with circular geometry, including interior, exterior, and annulus problems.

■ (A) Interior problem (SJF 33):

Find the electrostatic potential **within** a circle of radius ρ, given that the potential at boundary is specified.

PDE: $\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ [eq. (6.5)], where ROI = $\{0<r<\rho,\ 0<\theta<2\pi\}$.

BC: $u(\rho,\theta)=g(\theta)$ [implicit BC: $|u(0,\theta)|<\infty$, periodic BC: $u(r,\theta+2n\pi)= u(r,\theta)$].



1) Separation of variables:

   Let $u(r,\theta)=R(r)\cdot\Theta(\theta) \Rightarrow R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\ddot{\Theta} = 0$; divide by $\frac{R\Theta}{r^2}$,

   $\Rightarrow \frac{r^2R'' + rR'}{R} = -\frac{\ddot{\Theta}}{\Theta} = \boldsymbol{k}^2 \geq 0$ (why? Because of BCs)

   $\Rightarrow$ (i) $r^2R'' + rR' - k^2R = 0$ (Euler's eq.); (ii) $\ddot{\Theta} + k^2\Theta = 0$.

2) Solving the normal modes by periodic and implicit BCs:

   (i) $\ddot{\Theta} + k^2\Theta = 0$, $\Rightarrow \Theta(\theta)=c\cdot\cos(k\theta)+d\cdot\sin(k\theta)$;

Transformation of periodic BC: $u(r,\theta+2n\pi)=u(r,\theta) \to \Theta(\theta+2n\pi)=\Theta(\theta)$;

$\Rightarrow k=\boxed{k_n=n}=\mathbf{0}, 1,... \Rightarrow$

$$\Theta_n(\theta) = c\cdot\cos(n\theta)+d\cdot\sin(n\theta) \tag{7.5}$$

(ii) $r^2R''+rR'-n^2R=0$, $\Rightarrow$

$$R_n(r)=\begin{cases} a+b(\ln r), \text{if } n=0; \\ ar^n+b(1/r^n), \text{if } n=1,2,... \end{cases} \tag{7.6}$$

Transformation of implicit BC: $|u(0,\theta)|<\infty \to |R(0)|<\infty$;

$\Rightarrow b=0$ (for arbitrary n), $R_n(r)=a\cdot r^n$; for simplicity, we use $R_n(r)=a(r/\rho)^n$

$\Rightarrow$ The n-th normal mode: $u_n(r,\theta)=R_n(r)\cdot\Theta_n(\theta)$,

$$u_n(r,\theta)=(r/\rho)^n\left[c_n\cdot\cos(n\theta)+d_n\cdot\sin(n\theta)\right] \tag{7.7}$$

3) Determining the exact solution by the nonhomogeneous BC:

$$u(r,\theta)=\sum_{n=0}^{\infty}u_n(r,\theta)=\sum_{n=0}^{\infty}(r/\rho)^n[c_n\cos(n\theta)+d_n\sin(n\theta)] \tag{7.8}$$

Substitute the nonhomogeneous BC into eq. (7.8): $u(\rho,\theta)=\sum_{n=1}^{\infty}1^n\left[c_n\cos(n\theta)+d_n\sin(n\theta)\right]$ $=g(\theta)$, by Fourier sine-cosine series, $\Rightarrow$

$$c_0=\frac{1}{2\pi}\int_0^{2\pi}g(\theta)d\theta,\quad c_n=\frac{1}{\pi}\int_0^{2\pi}g(\theta)\cos(n\theta)d\theta,\quad d_n=\frac{1}{\pi}\int_0^{2\pi}g(\theta)\sin(n\theta)d\theta \tag{7.9}$$

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Solution $u(r,\theta)$ can also be regarded as superposition of "eigen-response":

1) Expand the BC $g(\theta)$ by Fourier series: $g(\theta)=\sum_{n=1}^{\infty}c_n\cos(n\theta)+d_n\sin(n\theta)$

2) Find the solutions of PDE + "eigen-BCs":

$$\begin{cases}\nabla^2u=0\\ u(\rho,\theta)=\sin(n\theta) \text{ or } \cos(n\theta)\end{cases}, \Rightarrow \text{eigen-response is: } u(r,\theta)=\left(\frac{r}{\rho}\right)^n[\sin(n\theta) \text{ or } \cos(n\theta)];$$

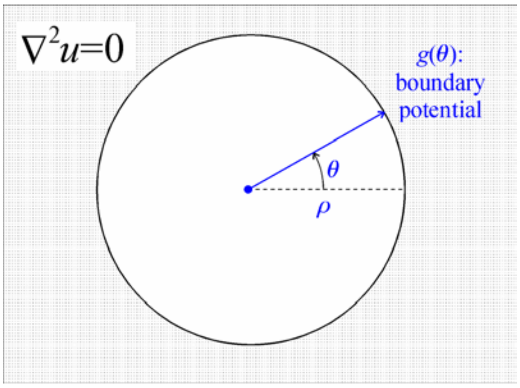
3) Superposition: $u(r,\theta)=\sum_{n=0}^{\infty}(r/\rho)^n\left[c_n\cos(n\theta)+d_n\sin(n\theta)\right]$

■ (B) Exterior problem (SJF 34):

Find the electrostatic potential **outside** a circle of radius ρ with type1 BC.

PDE: $\nabla^2 u=u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$, ROI: $\{\rho<r<\infty,\ 0<\theta<2\pi\}$

BC: $u(\rho,\theta)=g(\theta)$ [implicit BCs: $|u(\infty,\theta)|<\infty$, and $u(r,\theta+2n\pi)=u(r,\theta)$]



As in solving the interior Dirichlet problem, separation of variables leads to eq's (7.5-6):

$\Rightarrow \Theta_n(\theta)=c\cdot\cos(n\theta)+d\cdot\sin(n\theta)$, $\quad R_n(r)=\begin{cases}a+b(\ln r), \text{if } n=0;\\ ar^n+b(1/r^n), \text{if } n=1,2,\ldots\end{cases}$

Transformation of implicit BC: $\{|u(\infty,\theta)|<\infty \rightarrow |R(\infty)|<\infty\}$, $\Rightarrow\{b=0$ if $n=0$; $a=0$, if $n=1, 2, \ldots\}$

$\Rightarrow R_n(r)=\frac{b}{r^n}$, for simplicity, we use $R_n(r)=b(\rho/r)^n$ [$R_n(r)=a(r/\rho)^n$ in the interior problem].

$\Rightarrow$ The n-th normal mode: $u_n(r,\theta)=R_n(r)\cdot\Theta_n(\theta)$,

$$u_n(r,\theta)=(\rho/r)^n\left[c_n\cdot\cos(n\theta)+d_n\cdot\sin(n\theta)\right] \tag{7.10}$$

$$u(r,\theta)=\sum_{n=0}^{\infty}u_n(r,\theta)=\sum_{n=0}^{\infty}(\rho/r)^n\left[c_n\cos(n\theta)+d_n\sin(n\theta)\right] \tag{7.11}$$

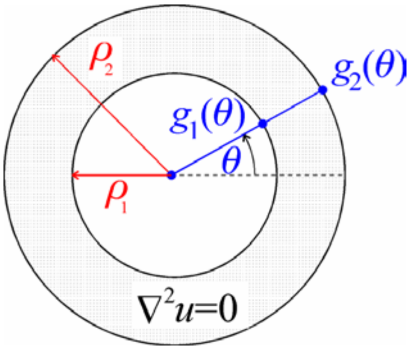
Substitute the nonhomogeneous BC into eq. (7.11): $u(\rho,\theta)=\sum_{n=1}^{\infty}1^n\left[c_n\cos(n\theta)+d_n\sin(n\theta)\right]=$ $g(\theta)$; by Fourier sine-cosine series, $\Rightarrow c_0$, c_n, d_n are determined by eq. (7.9).

■ (*) (C) Annulus problem (SJF 34):

Find the electrostatic potential **between** two circles of radii ρ_1, ρ_2 with type1 BCs.

PDE: $\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$, $\{\rho_1 < r < \rho_2,\ 0<\theta<2\pi\}$

BCs: $u(\rho_1,\theta)=g_1(\theta)$, $u(\rho_2,\theta)=g_2(\theta)$ [periodic BC: $u(r,\theta+2n\pi)= u(r,\theta)$].



As in solving the interior Dirichlet problem, separation of variables leads to eq's (7.5-6):

$$\Rightarrow \Theta_n(\theta) = c\cdot\cos(n\theta)+d\cdot\sin(n\theta), \qquad R_n(r)=\begin{cases} a+b(\ln r), \text{ if } n=0; \\ ar^n + b(1/r^n), \text{ if } n=1,2,\ldots \end{cases}$$

Since the ROI is $\rho_1 < r < \rho_2$, **neither *a* nor *b* should be zero**, the general form of $R_n(r)$ is used, and the general solution $u(r,\theta)$ becomes:

$$u(r,\theta)= a_0 + b_0\ln(r) + \sum_{n=1}^{\infty}\left\{\left[a_n r^n + b_n r^{-n}\right]\cos(n\theta) + \left[c_n r^n + d_n r^{-n}\right]\sin(n\theta)\right\} \tag{7.12}$$

Substitute the nonhomogeneous BCs into eq. (7.12):

$$u(\rho_1,\theta)=[a_0 + b_0\ln(\rho_1)] + \sum_{n=1}^{\infty}\left\{\left[a_n\rho_1^n + b_n\rho_1^{-n}\right]\cos(n\theta) + \left[c_n\rho_1^n + d_n\rho_1^{-n}\right]\sin(n\theta)\right\} = g_1(\theta);$$

$$u(\rho_2,\theta)=[a_0 + b_0\ln(\rho_2)] + \sum_{n=1}^{\infty}\left\{\left[a_n\rho_2^n + b_n\rho_2^{-n}\right]\cos(n\theta) + \left[c_n\rho_2^n + d_n\rho_2^{-n}\right]\sin(n\theta)\right\} = g_2(\theta);$$

$$\Rightarrow \begin{cases} a_0 + b_0\ln(\rho_1) = \frac{1}{2\pi}\int_0^{2\pi} g_1(\phi)d\phi \\ a_0 + b_0\ln(\rho_2) = \frac{1}{2\pi}\int_0^{2\pi} g_2(\phi)d\phi \end{cases}$$; used to solve $\boldsymbol{a_0}$, $\boldsymbol{b_0}$;

$$\Rightarrow \begin{cases} a_n\rho_1^n + b_n\rho_1^{-n} = \frac{1}{\pi}\int_0^{2\pi} g_1(\phi)\cos(n\phi)d\phi \\ a_n\rho_2^n + b_n\rho_2^{-n} = \frac{1}{\pi}\int_0^{2\pi} g_2(\phi)\cos(n\phi)d\phi \end{cases}$$; used to solve $\boldsymbol{a_n}$, $\boldsymbol{b_n}$;

$$\Rightarrow \begin{cases} c_n\rho_1^n + d_n\rho_1^{-n} = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} g_1(\phi)\sin(n\phi)d\phi \\ c_n\rho_2^n + d_n\rho_2^{-n} = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} g_2(\phi)\sin(n\phi)d\phi \end{cases}$$; used to solve $\boldsymbol{c_n}$, $\boldsymbol{d_n}$;

For lack of implicit BCs to simplify the eigenfunctions $R_n(r)$, we have four unknown coefficients $\{a_n, b_n, c_n, d_n\}$ for each mode. The more homogeneous/implicit BCs, the less unknown coefficients to be determined.

E.g. Find the electrostatic potential in the dielectric region of a coaxial cable if the inner and outer conductors have constant potentials V_1 and V_2, respectively.

PDE: $\nabla^2 u = 0$, $\{\rho_1<r<\rho_2, 0<\theta<2\pi\}$

BCs: $u(\rho_1,\theta)=V_1$, $u(\rho_2,\theta)=V_2$

(Method 1) Since the boundary potentials are independent of θ, the governing PDE can be reduced to an ODE: $u_{rr}+(1/r)u_r= 0$.

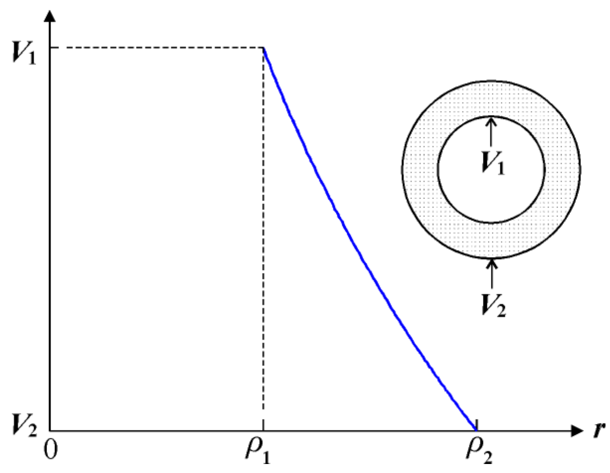
Let $U(r)=u_r$, $\Rightarrow$ $U'(r)+\dfrac{U(r)}{r}=0$, $\Rightarrow$ $U(r)=\dfrac{b}{r}$, $u(r)=a+b\cdot\ln(r)$.

The coefficients a, b are determined by the BCs: $a=\dfrac{V_1\ln(\rho_2)-V_2\ln(\rho_1)}{\ln(\rho_2/\rho_1)}$, $b=\dfrac{V_2-V_1}{\ln(\rho_2/\rho_1)}$.

(Method 2) By the series solution formula eq. (7.12):

$$\begin{cases} a_0 + b_0\ln(\rho_1) = (1/2\pi)\displaystyle\int_0^{2\pi} V_1 d\phi = V_1 \\ a_0 + b_0\ln(\rho_2) = (1/2\pi)\displaystyle\int_0^{2\pi} V_2 d\phi = V_2 \end{cases} \Rightarrow a_0=\frac{V_1\ln(\rho_2)-V_2\ln(\rho_1)}{\ln(\rho_2/\rho_1)},\ b_0=\frac{V_2-V_1}{\ln(\rho_2/\rho_1)};$$

$$\begin{cases} a_n\rho_1^n + b_n\rho_1^{-n} = (1/\pi)\displaystyle\int_0^{2\pi} V_1\cos(n\phi)d\phi = 0 \\ a_n\rho_2^n + b_n\rho_2^{-n} = (1/\pi)\displaystyle\int_0^{2\pi} V_2\cos(n\phi)d\phi = 0 \end{cases} \Rightarrow \{a_n=0,\ b_n=0\};\ \text{similarly, } \{c_n=0,\ d_n=0\}.$$

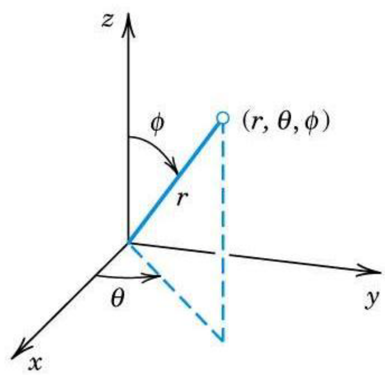


Laplace's Equation in Spherical Coordinates (EK. 12.10)

■ Problem: Find the electrostatic potential of a sphere of radius ρ with prescribed surface potential $f(\phi)$ (assuming **no θ-dependence** for simplicity).

PDE: $\nabla^2 u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\phi\phi} + \frac{\cot\phi}{r^2}u_\phi = (r^2 u_r)_r + \frac{1}{\sin\phi}(\sin\phi \cdot u_\phi)_\phi = 0$;

BC: $u(\rho,\phi)=f(\phi)$ [implicit BCs: $|u(r;\phi=0,\pi)|<\infty$; $|u(r=0,\infty;\phi)|<\infty$ for interior and exterior problems, respectively).



■ Solving spherical Laplace's equation by separation of variables:

1) Separation of variables:

$$u(r,\phi)=R(r)\cdot\Phi(\phi) \Rightarrow \frac{r^2R''+2rR'}{R} = \frac{-(\Phi''+\cot\phi\cdot\Phi')}{\Phi} = k. \Rightarrow$$

(i) $r^2R''+2rR'-kR=0$ (Euler's equation);

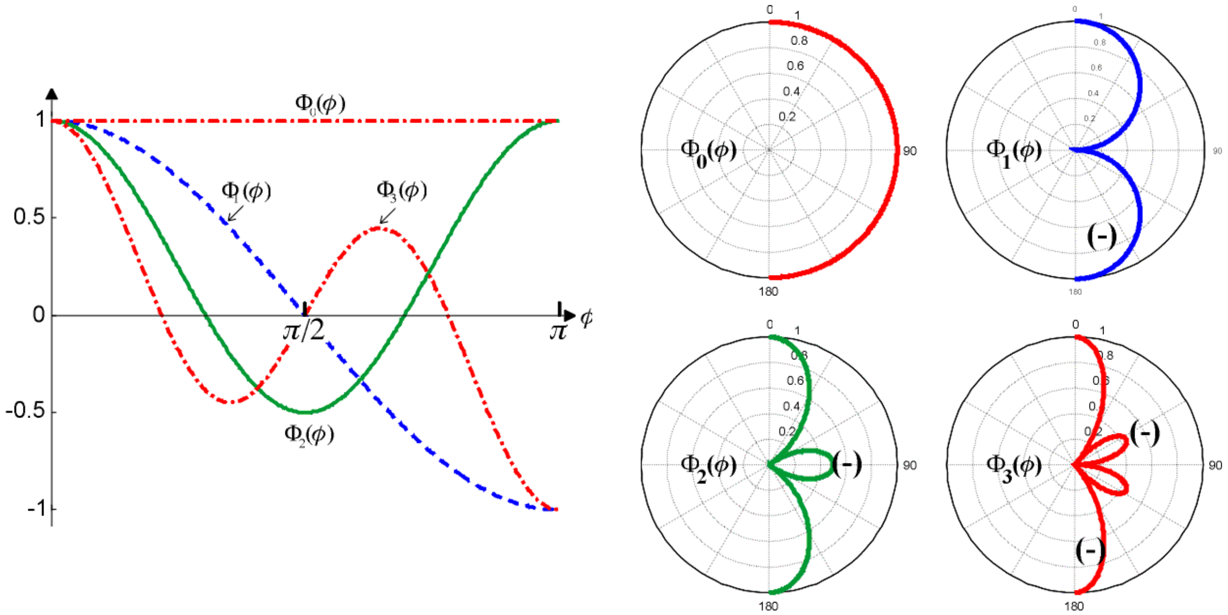
(ii) $\Phi''+\cot\phi\cdot\Phi'+k\Phi=0$ (after change of variable, $\Rightarrow$ Legendre equation)

2) Solving the "modes" by implicit BCs:

(i) Let $\cos\phi = w$, the ODE about $\Phi(\phi)$ becomes: $(1-w^2)\,\Phi''(w) - 2w\,\Phi'(w) + k\Phi(w) = 0$.

By implicit BC: $|\Phi(\phi=0,\ \pi)|<\infty$, i.e. $|\Phi(w=\pm1)|<\infty$, we have discrete eigenvalues $\boxed{k=n(n+1)}$, where n=0, 1, 2,….

Solutions to the Legendre equation: $(1-w^2)\,\Phi''(w) - 2w\,\Phi'(w) + n(n+1)\Phi(w) = 0$ are Legendre polynomials (EK 5.3): $\Phi_n(w)=P_n(w)$, $\Rightarrow$ $\Phi_n(\phi)=P_n(\cos\phi)$.



(ii) The ODE about $R(r)$ becomes: $r^2R'' + 2rR' - n(n+1)R = 0$.

Let $R(r)=r^{\alpha}$, $\Rightarrow$ $\alpha = n,\ -(n+1)$, $R_n(r)=ar^n+br^{-(n+1)}$ [b=0 for interior problems ($r<\rho$), a=0 for exterior problems ($r>\rho$)]. $\Rightarrow$ The n-th normal mode: $u_n(r,\phi)=R_n(r)\cdot\Phi_n(\phi)$,

$$u_n(r,\phi) = [a_n r^n + b_n r^{-(n+1)}]\cdot P_n(\cos\phi) \tag{7.13}$$

3) Solving the entire problem by nonhomogeneous BCs:

(i) $u(r,\phi)=\sum_{n=0}^{\infty} a_n (r/\rho)^n P_n(\cos\phi)$, for **interior** problems ($r<\rho$);

(ii) $u(r,\phi)=\sum_{n=0}^{\infty} b_n (\rho/r)^{n+1} P_n(\cos\phi)$, for **exterior** problems ($r>\rho$);

(iii) $u(r,\phi)=\sum_{n=0}^{\infty} \left(a_n r^n + b_n r^{-(n+1)}\right) P_n(\cos\phi)$, for **annulus** problems ($\rho_1<r<\rho_2$);

In cases (i-ii), substitute nonhomogeneous BC into eq. (7.13):

$u(\rho,\phi)=\sum_{n=0}^{\infty}\begin{pmatrix}a_n\\b_n\end{pmatrix}P_n(\cos\phi)=f(\phi)$. By orthogonality of Legendre's polynomials, $\Rightarrow$

$$\begin{pmatrix}a_n\\b_n\end{pmatrix}=\frac{2n+1}{2}\int_0^{\pi}f(\phi)P_n(\cos\phi)\sin(\phi)d\phi \tag{7.14}$$

In case (iii), a system of equations has to be solved to get $\{a_n,b_n\}$ for each n.

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1) In cases (i-ii), the solution can be derived by: (1) expand the BC $f(\phi)$ by Legendre's polynomials: $f(\phi)=\sum_{n=0}^{\infty}\begin{pmatrix}a_n\\b_n\end{pmatrix}P_n(\cos\phi)$, where coefficients $\begin{pmatrix}a_n\\b_n\end{pmatrix}$ are determined by eq. (7.14). (2) Solve $\begin{cases}\nabla^2u=0\\u(\rho,\phi)=P_n(\cos\phi)\end{cases}$, leading to $u(r,\phi)=\sum_{n=0}^{\infty}\begin{bmatrix}(r/\rho)^n\\(\rho/r)^{n+1}\end{bmatrix}P_n(\cos\phi)$. (3) By superposition, $\Rightarrow u(r,\phi)=\sum_{n=0}^{\infty}\begin{bmatrix}a_n(r/\rho)^n\\b_n(\rho/r)^{n+1}\end{bmatrix}P_n(\cos\phi)$.

2) For interior problem, the solution at the spherical center is: $u(r=0,\phi)=a_0\cdot 1\cdot P_0(\cos\phi)=a_0$, by eq. (7.14), $=\frac{1}{2}\int_0^{\pi}f(\phi)\sin(\phi)d\phi$, which is the average of the boundary function $f(\phi)$ weighted by $\frac{\sin\phi}{2}d\phi$.

3) For exterior problem with constant BC: $f(\phi)=V_0$, eq. (7.14) gives $b_n=0$, except for $b_0=\frac{1}{2}\int_0^{\pi}V_0P_0(\cos\phi)\sin(\phi)d\phi=V_0$, $\Rightarrow$ the solution $u(r,\phi)=\frac{V_0\rho}{r}\propto\frac{1}{r}$, which dies off as $r\to\infty$. This is in opposite to its 2-D polar (or 3-D cylindrical) counterpart, where the exterior solution due to a constant BC is a constant: $u(r,\theta)=V_0$ [eq's (7.9), (7.11)].

4) For exterior problem, the solution in the far-field ($r>>\rho$, i.e. $\frac{\rho}{r}<<1$) is approximated by: $u(r,\phi)\approx\frac{b_0\rho}{r}$. $\Rightarrow$ The spherical BC (source) is approximated by a point source located at the center of strength $b_0=\frac{1}{2}\int_0^{\pi}f(\phi)\sin(\phi)d\phi$.

Appendix 7A – Poisson integral formula for Polar Laplace's Equation

Eq. (7.8) can be simplified as: $u(r,\theta) = c_0+\sum_{n=1}^{\infty}(r/\rho)^n\left[c_n\cos(n\theta)+d_n\sin(n\theta)\right]$, by eq. (7.9),

$$=\left(\frac{1}{2\pi}\int_0^{2\pi}g(\phi)d\phi\right)+\frac{1}{\pi}\sum_{n=1}^{\infty}(r/\rho)^n\left[\int_0^{2\pi}g(\phi)\cos(n\phi)\cos(n\theta)d\phi+\int_0^{2\pi}g(\phi)\sin(n\phi)\sin(n\theta)d\phi\right]$$

$$=\frac{1}{2\pi}\left\{\int_0^{2\pi}\left[1+2\sum_{n=1}^{\infty}\left(\frac{r}{\rho}\right)^n\cos[n(\theta-\phi)]\right]g(\phi)d\phi\right\}=\frac{1}{2\pi}\left\{\int_0^{2\pi}\left[1+\sum_{n=1}^{\infty}\left(\frac{r}{\rho}\right)^n\left(e^{in(\theta-\phi)}+c.c.\right)\right]g(\phi)d\phi\right\}$$

$$=\frac{1}{2\pi}\left\{\int_0^{2\pi}\left[1+\sum_{n=1}^{\infty}\left\{\left(\frac{r}{\rho}e^{i(\theta-\phi)}\right)^n+\left(\frac{r}{\rho}e^{-i(\theta-\phi)}\right)^n\right\}\right]g(\phi)d\phi\right\}$$, by geometric series (等比級數),

$$=\frac{1}{2\pi}\left\{\int_0^{2\pi}\left[1+\frac{re^{i(\theta-\phi)}}{\rho-re^{i(\theta-\phi)}}+\frac{re^{-i(\theta-\phi)}}{\rho-re^{-i(\theta-\phi)}}\right]g(\phi)d\phi\right\}$$, by quotient of complex numbers, $\Rightarrow$

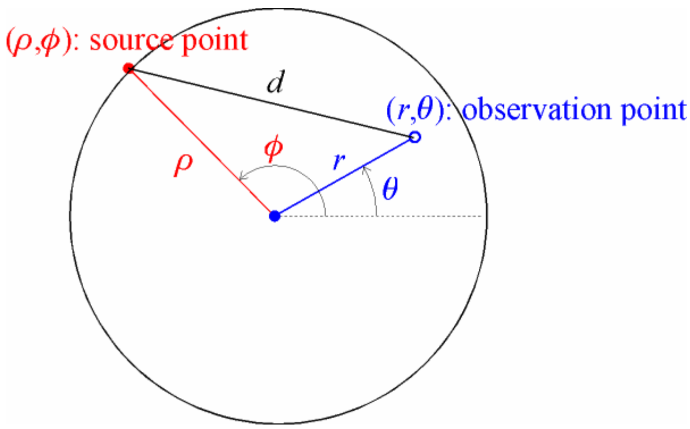
Poisson integral formula:

$$u(r,\theta)=\frac{1}{2\pi}\left\{\int_0^{2\pi}\left[\frac{\rho^2-r^2}{\rho^2-2r\rho\cos(\theta-\phi)+r^2}\right]g(\phi)d\phi\right\} \qquad (7A.1)$$

The potential u at observation point (r,θ) is the weighted average of the boundary potential $g(\theta)$, where the weighting kernel is:

$$K(r,\theta;\phi)=\frac{\rho^2-r^2}{2\pi\, d^2} \qquad (7A.2)$$

$d=[\rho^2-2r\rho\cos(\theta-\phi)+r^2]^{1/2}$ is the distance between observation point (r,θ) and source point (ρ,ϕ).



<Comment>

1) If we observe the circle **center**: r=0, d=ρ, $K(r,\theta;\phi)=\dfrac{1}{2\pi}$, $u(0,\theta)=\dfrac{1}{2\pi}\int_0^{2\pi} g(\phi)d\phi$ $\left[=\int_0^{2\pi} g(\phi)\dfrac{\rho d\phi}{2\pi\rho}\right]$. ⇒ The solution is the average of BC weighted by arc length.

2) If we observe the circular **rim**: r=ρ, d≥0, $K(\rho,\theta;\phi)$=0, except for d=0 ($\theta=\phi$). ⇒ $K(\rho,\theta;\phi)\sim\delta(\phi-\theta)$, $u(\rho,\theta)\sim\int_0^{2\pi} g(\phi)\delta(\phi-\theta)d\phi=g(\theta)$, ⇒ satisfying the specified BC.