Lesson 06 2-D Wave Equations

■ Introduction

When the spatial dimension of the problem is greater than one, we encounter: (i) more BCs; (ii) more complicated normal modes; (iii) non-Cartesian coordinates due to the geometry of the problem. This lesson investigates these issues by 2-D wave equations.

Rectangular Membrane (EK 12.8)

■ Problem: a rectangular membrane defined within $\{0<x<a,\ 0<y<b\}$ with fixed rim and pre-specified initial displacement $\phi(x,y)$, and initial velocity $\gamma(x,y)$.

PDE: $u_{tt} = c^2 (u_{xx} + u_{yy})$, $c^2 = T/\rho$

Type1 BCs: u=0 on the rectangular boundary

Two ICs: $u(x,y,0)=\phi(x,y)$, $u_t(x,y,0)=\gamma(x,y)$

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The number of required BCs is equal to the sum of derivative orders of all spatial variables.

E.g. (i) u_{xx} requires 2 BCs: $u(0,t)=f(t)$, $u(L,t)=g(t)$. (ii) u_{xxxx} requires 4 BCs: $u(0,t)=f(t)$, $u_{xx}(0,t)=g(t)$, $u(L,t)=h(t)$, $u_{xx}(L,t)=k(t)$. (iii) $u_{xx}+ u_{yy}$ requires 4 BCs: $u(0,y,t)=f(t)$, $u(a,y,t)=g(t)$, $u(x,0,t)=h(t)$, $u(x,b,t)=k(t)$.

■ Solving 2-D wave equations by separation of variables

1) Separation of variables:

Let $u(x,y,t)=F(x,y)\cdot T(t)$, substitute it into the PDE, $\Rightarrow$ $F\ddot{T} = c^2(F_{xx}T + F_{yy}T)$; divide by

c^2FT, $\Rightarrow \dfrac{\ddot{T}}{c^2T} = \dfrac{F_{xx}+F_{yy}}{F} = -\lambda^2$, such that (i) both sides must be constant to satisfy the equality for arbitrary x, y, t; (ii) trivial solution $u(x,y,t)=0$ does not occur. $\Rightarrow$

(i) One ODE: $\ddot{T}+\omega^2T=0$, where $\boldsymbol{\omega \equiv c\lambda}$;

(ii) One PDE: $F_{xx}+F_{yy}+\lambda^2F=0$ (2-D Helmholtz equation).

Apply separation of variable further: let $F(x,y)=X(x)\cdot Y(y)$,

$$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} - \lambda^2 = -\boldsymbol{k_x^2}, \Rightarrow \begin{cases} X''+k_x^2X=0 \\ Y''+k_y^2Y=0,\ \boldsymbol{k_y^2 \equiv \lambda^2 - k_x^2} \end{cases}$$

2) Solving the normal modes by homogeneous BCs:

To avoid trivial solution $u(x,y,t)=0$, homogeneous BCs of $u(x,y,t) \to$ BCs of $X(x)$, $Y(y)$:

$\{u(0,y,t)=0,\ u(a,y,t)=0,\ u(x,0,t)=0,\ u(x,b,t)=0\} \to \{X(0)=X(a)=0,\ Y(0)=Y(b)=0\}$;

(i) $X''+k_x^2X=0 \Rightarrow X(x) = A\cdot\cos(k_xx) + B\cdot\sin(k_xx)$;

by BCs: $X(0)=0 \Rightarrow A=0$; $X(a)=0 \Rightarrow k_x=\boxed{k_{x,m}=\dfrac{m\pi}{a}}$, $m=1,2,\ldots \Rightarrow X_m(x)=\sin(k_{x,m}x)$;

(ii) $Y''+k_y^2Y=0 \Rightarrow Y(y) = C\cdot\cos(k_yy)+D\cdot\sin(k_yy)$;

by BCs: $Y(0)=0 \Rightarrow C=0$; $Y(b)=0 \Rightarrow k_y=\boxed{k_{y,n}=\dfrac{n\pi}{b}}$, $n=1,2,\ldots \Rightarrow Y_n(y)=\sin(k_{y,n}y)$;

$F_{mn}(x,y)=X_m(x)\cdot Y_n(y)= \sin(k_{x,m}x)\cdot\sin(k_{y,n}y)$.

(iii) The other ODE becomes: $\ddot{T}+\omega_{mn}^2T=0$, $\omega_{mn}=c\lambda_{mn}=\boxed{c\sqrt{k_{x,m}^2+k_{y,n}^2}}=c\pi\sqrt{\dfrac{m^2}{a^2}+\dfrac{n^2}{b^2}}$;

$\Rightarrow T_{mn}(t)=A_{mn}\cdot\cos(\omega_{mn}t)+ B_{mn}\cdot\sin(\omega_{mn}t)$,

$\Rightarrow$ the (m,n) normal mode is formulated as: $u_{mn}(x,y,t)= X_m(x)\cdot Y_n(y)\cdot T_{mn}(t)$,

$$u_{mn}(x,y,t) = [A_{mn}\cos(\omega_{mn}t)+B_{mn}\sin(\omega_{mn}t)]\cdot\sin(k_{x,m}x)\sin(k_{y,n}y) \qquad (6.1)$$

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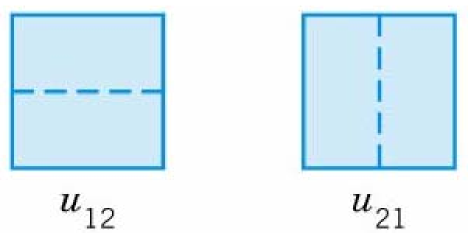
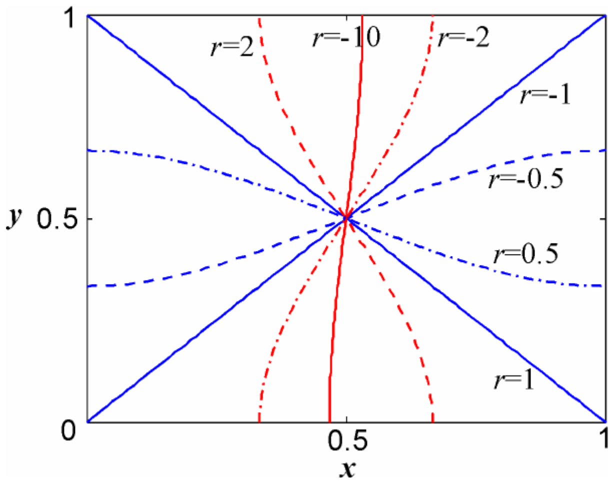
(a) $u_{mn}(x,y,t)$ are eigenfunctions, $k_{x,m}$, $k_{y,n}$, ω_{mn} are eigenvalues of the vibrating membrane.

Each mode vibrates with frequency $\nu_{mn}=\frac{\omega_{mn}}{2\pi}=\frac{c}{2}\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}$, which is NOT integral times of the fundamental frequency ν_{11}. ⇒ Drums sound differently with violins (Lesson 2).

(b) (∗) Unlike 1-D vibrating string, different modes may have the same frequency (**degenerate** modes). As a result, the spatial distribution corresponding to some resonant frequency ν_{mn} may change with relative coefficients A_{mn}, B_{mn}.

E.g. If $a=b=1$ ⇒ $F_{12}(x,y)=\sin(\pi x)\sin(2\pi y)$ and $F_{21}(x,y)=\sin(2\pi x)\sin(\pi y)$ have identical resonant frequency: $\nu_{12}=\nu_{21}=c\sqrt{5}/2$ but different spatial profiles (see below left).





The superposition of these two degenerate modes still vibrates at frequency of $c\sqrt{5}/2$: $u=u_{12}+u_{21}=(A_{12}\cos\omega t+B_{12}\sin\omega t)F_{12}(x,y)+(A_{21}\cos\omega t+B_{21}\sin\omega t)F_{21}(x,y)$,

where $\omega\equiv c\pi\sqrt{5}$. Assume $B_{12}=B_{21}=0$, $r\equiv A_{21}/A_{12}$ ⇒ $u\propto(\cos\omega t)(F_{12}+rF_{21})$. The nodal line (set of points where displacements are always zero) is the solution to: $(F_{12}+rF_{21})=0$, i.e. $(\sin\pi x\cdot\sin\pi y)(\cos\pi y+r\cdot\cos\pi x)=0$;

$$\Rightarrow \begin{cases} y=\pm\cos^{-1}(-r\cdot\cos\pi x)/\pi,\ \text{for}\ |r|<1 \\ x=\pm\cos^{-1}(-(r^{-1})\cdot\cos\pi y)/\pi,\ \text{for}\ |r|>1 \end{cases}$$ (see right figure ↑)

3) Determining the exact solution by ICs:

Since 2-D wave equation is linear and homogeneous, we can expand the exact solution

$u(x,y,t)$ by a **double series**:

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}u_{mn}(x,y,t)$$

$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[A_{mn}\cos(\omega_{mn}t)+B_{mn}(\sin\omega_{mn}t)]\sin(k_{x,m}x)\sin(k_{y,n}y) \qquad (6.2)$$

Substitute the IC into eq. (6.2):

(i) $u(x,y,0)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin(k_{x,m}x)\sin(k_{y,n}y)=\phi(x,y)$, by double Fourier sine series, $\Rightarrow$

$$A_{mn}=\frac{4}{ab}\int_0^b\int_0^a\phi(x,y)\cdot\sin(k_{x,m}x)\sin(k_{y,n}y)dxdy \qquad (6.3)$$

(ii) $u_t(x,y,0)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\omega_{mn}\sin(k_{x,m}x)\sin(k_{y,n}y)=\gamma(x,y)$, $\Rightarrow$

$$B_{mn}=\frac{4}{ab\omega_{mn}}\int_0^b\int_0^a\gamma(x,y)\cdot\sin(k_{x,m}x)\sin(k_{y,n}y)dxdy \qquad (6.4)$$

Laplacian in Polar Coordinates (EK 12.9)

■ Concept

In solving PDE problems, it is preferred to choose a coordinate system that can describe the physical boundaries in a simple way. **E.g**. Choose polar coordinates to solve circular membrane problem, where the boundary is easily described as: $\{r=\text{constant}\}$.

■ ∇^2 in polar coordinates

∇^2 operator can be transformed from Cartesian into polar coordinates using chain rule. We denote both $u(x,y)$ and $u(r,\theta)$ as u for simplicity, though the function forms are different.

By (i) $r=\sqrt{x^2+y^2}$, $\Rightarrow$ $\boldsymbol{r_x}=\frac{x}{\sqrt{x^2+y^2}}=\frac{x}{r}=\cos\theta$; $\boldsymbol{r_{xx}}=\frac{y^2}{r^3}=\frac{\sin\theta}{r}$; (ii) $\tan\theta=\frac{y}{x}$, $\Rightarrow$

$(\tan\theta)_x=(\sec^2\theta)\theta_x=\frac{-y}{x^2}$, $\Rightarrow$ $\boldsymbol{\theta_x}=\frac{-y\cos^2\theta}{x^2}=\frac{-y}{r^2}=\frac{-\sin\theta}{r}$; $\boldsymbol{\theta_{xx}}=\frac{2xy}{r^4}=\frac{\sin 2\theta}{r^2}$;

$u_x = u_r r_x + u_\theta \theta_x$;

$\boldsymbol{u_{xx}} = (u_r r_x)_x + (u_\theta \theta_x)_x = [(u_r)_x r_x + u_r r_{xx}] + [(u_\theta)_x \theta_x + u_\theta \theta_{xx}]$

$= [(u_{rr} r_x + u_{r\theta} \theta_x) r_x + u_r r_{xx}] + [(u_{\theta r} r_x + u_{\theta\theta} \theta_x) \theta_x + u_\theta \theta_{xx}] = \dfrac{x^2}{r^2} u_{rr} - \dfrac{2xy}{r^3} u_{r\theta} + \dfrac{y^2}{r^4} u_{\theta\theta} + \dfrac{y^2}{r^3} u_r + \dfrac{2xy}{r^4} u_\theta$,

Similarly, $\boldsymbol{u_{yy}} = \dfrac{y^2}{r^2} u_{rr} + \dfrac{2xy}{r^3} u_{r\theta} + \dfrac{x^2}{r^4} u_{\theta\theta} - \dfrac{y^2}{r^3} u_r - \dfrac{2xy}{r^4} u_\theta$. $\nabla^2 u = u_{xx} + u_{yy}$, $\Rightarrow$

$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \tag{6.5}$$

(∗) Circular Membrane (EK 12.9)

■ Problem: consider a circular membrane of radius ρ with fixed rim and pre-specified initial displacement & velocity.

PDE: $u_{tt} = c^2 \left(u_{rr} + \dfrac{1}{r} u_r + \dfrac{1}{r^2} u_{\theta\theta} \right)$, $\{0<r<\rho,\ t>0\}$

BC: $u(r=\rho, \theta, t) = 0$

Two ICs: $u(r,\theta,0) = f(r)$, $u_t(r,\theta,0) = g(r)$ (independent of θ);

■ Solving 2-D wave equation by separation of variables

1) Separation of variables:

Let $u(r,\theta,t) = U(r,\theta)\cdot T(t) \Rightarrow U\ddot{T} = c^2 (\nabla^2_{r\theta} U) T$; divide by $c^2 UT$, $\Rightarrow \dfrac{\ddot{T}}{c^2 T} = \dfrac{\nabla^2_{r\theta} U}{U} = -\lambda^2$, such that (i) both sides must be constant to satisfy the equality for arbitrary r, θ, t; (ii) trivial solution $u(r,\theta,t)=0$ does not occur. $\Rightarrow$

(i) one ODE: $\ddot{T} + \omega^2 T = 0$, $\boldsymbol{\omega \equiv c\lambda}$; (ii) one PDE: $\nabla^2_{r\theta} U + \lambda^2 U = 0$.

Apply separation of variable further: let $U(r,\theta) = R(r)\cdot\Theta(\theta)$, and $\nabla^2_{r\theta}$ is substituted by eq.

(6.5), $\Rightarrow \left(R''+\frac{1}{r}R'\right)\Theta+\frac{1}{r^2}R\Theta''+\lambda^2 R\Theta=0$, divide by $\frac{R\Theta}{r^2}$,

$$\Rightarrow \left(\frac{r^2R''+rR'}{R}+\lambda^2 r^2\right)=-\frac{\Theta''}{\Theta}=\boldsymbol{k_\theta^2} \Rightarrow \begin{cases} r^2R''+rR'+(\lambda^2r^2-k_\theta^2)R=0, (\textbf{Bessel's eq.}) \\ \Theta''+k_\theta^2\Theta=0 \end{cases}$$

2) Solving the normal modes by **periodic** & homogeneous BCs:

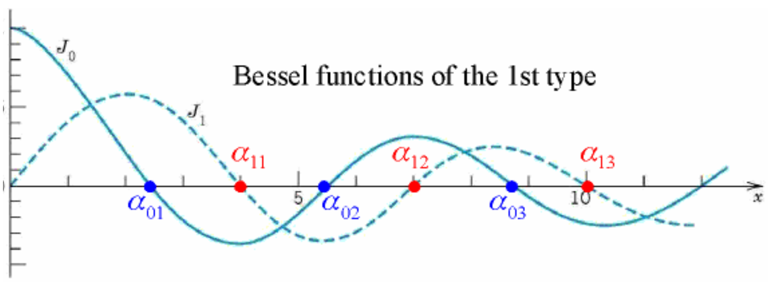
(i) $\Theta''+k_\theta^2\Theta=0$, $\Rightarrow \Theta(\theta)=A\cdot\sin(k_\theta\cdot\theta)+B\cdot\cos(k_\theta\cdot\theta)$

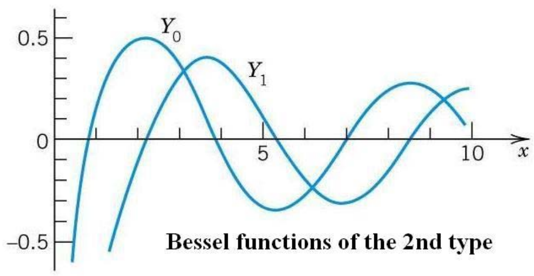
Transformation of the periodic BC: $u(r,\theta+2n\pi,t)=u(r,\theta,t) \rightarrow \Theta(\theta+2n\pi)=\Theta(\theta)$,

$\Rightarrow \boxed{k_\theta=m}=\mathbf{0}$, 1, 2, …, $\Theta_m(\theta)=A\cdot\sin(m\theta)+B\cdot\cos(m\theta)\propto\cos(m\theta+\phi)\rightarrow\cos(m\theta)$, if the position of θ=0 is properly defined.

(ii) ODE about $R(r)$ becomes: $r^2R''+rR'+(\lambda^2r^2-\boldsymbol{m^2})R=0$, which can be solved by the Frobenius (series) method (EK 5.4-6).

$\Rightarrow R(r)=C\cdot J_m(\lambda r)+D\cdot Y_m(\lambda r)$, where $J_m(x)$, $Y_m(x)$ are Bessel functions of 1st, 2nd types.





By the implicit BC: $|u(0,\theta,t)|<\infty \rightarrow |R(0)|<\infty$, $\Rightarrow D=0$ [for $Y_m(0)\rightarrow-\infty$], $R(r)=J_m(\lambda r)$.

By the homogeneous BC: $u(\rho,\theta,t)=0 \rightarrow R(\rho)=0$, $\Rightarrow J_m(\lambda\rho)=0$, $\lambda=\boxed{\lambda_{mn}=\frac{\alpha_{mn}}{\rho}}$, where α_{mn} is the n-th node of $J_m(x)$, n=**1**, 2, ……$\Rightarrow R_{mn}(r)=J_m(\lambda_{mn}r)$;

(iii) ODE about $T(t)$ becomes: $\ddot{T}+\omega_{mn}^2T=0$, $\omega_{mn}=c\lambda_{mn}=\boxed{\frac{c\alpha_{mn}}{\rho}}$;

$\Rightarrow T_{mn}(t)=A_{mn}\cdot\cos(\omega_{mn}t)+B_{mn}\cdot\sin(\omega_{mn}t)$,

$\Rightarrow$ the (m,n) normal mode is formulated as: $u_{mn}(r,\theta,t)=R_{mn}(r)\cdot\Theta_m(\theta)\cdot T_{mn}(t)$,

$$u_{mn}(r,\theta,t)=\left[A_{mn}\cos(\omega_{mn}t)+B_{mn}\sin(\omega_{mn}t)\right]\cdot J_m(\lambda_{mn}r)\cos(m\theta) \qquad (6.6)$$

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(a) $R_{mn}(r)$ has $n-1$ concentric nodal circles in $0<r<\rho$, which are determined by:

$$\lambda_{mn}r=\frac{\alpha_{mn}}{\rho}r=\alpha_{mn'},\quad n'=1,2,\ldots,(n-1);\Rightarrow\ r_{n'}=\left(\frac{\alpha_{mn'}}{\alpha_{mn}}\right)\rho<\rho.$$

(b) $u_{mn}(r,\theta,t)$ are eigenfunctions, λ_{mn} ω_{mn} are eigenvalues of the vibrating membrane. Each mode vibrates at frequency:

$$\nu_{mn}=\frac{\omega_{mn}}{2\pi}=\frac{c\alpha_{mn}}{2\pi\rho} \tag{6.7}$$

The fundamental frequency $\nu_{01}=\dfrac{c\alpha_{01}}{2\pi\rho}\approx 0.38\dfrac{c}{\rho}$, $\Rightarrow$ The larger dimension (ρ), the lower frequency.

(c) Since the nodes α_{mn} of $J_m(r)$ are irregularly spaced, $\Rightarrow$ ν_{mn} are not integral times of ν_{01}, the sound of a drum is different from that of a violin [eq. (2.2)].

3) Determining the exact solution by ICs:

Expand the exact solution $u(r,\theta,t)$ by a double series:

$$u(r,\theta,t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}u_{mn}(r,\theta,t)$$
$$=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(A_{mn}\cos\omega_{mn}t+B_{mn}\sin\omega_{mn}t\right)\cdot J_m\left(\lambda_{mn}r\right)\cos\left(m\theta\right) \tag{6.8}$$

Consider θ–independent vibrations $\Rightarrow m=0$,

$$u(r,t)=\sum_{n=1}^{\infty}\left(A_{0n}\cos\omega_{0n}t+B_{0n}\sin\omega_{0n}t\right)\cdot J_0\left(\lambda_{0n}r\right) \tag{6.9}$$

where the displacement peak always occurs at the circle center $r=0$.

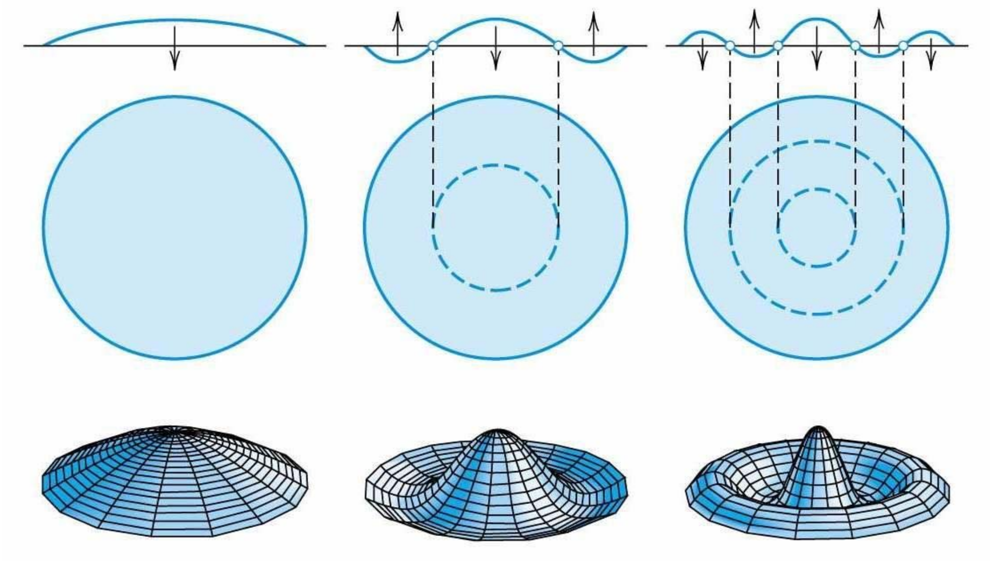
Substitute ICs into eq. (6.9):

(i) $u(r,0)=\sum_{n=1}^{\infty}A_{0n}J_0\left(\dfrac{\alpha_{0n}}{\rho}r\right)=f(r)$, by Fourier-Bessel series (EK 5.8), $\Rightarrow$

$$A_{0n}=\frac{2}{\rho^2J_1^2(\alpha_{0n})}\int_0^{\rho}rf(r)J_0\left(\frac{\alpha_{0n}}{\rho}r\right)dr \tag{6.10}$$

(ii) $u_t(r,0)=\sum_{n=1}^{\infty}B_{0n}\lambda_{0n}J_0\left(\dfrac{\alpha_{0n}}{\rho}r\right)=g(r)$, $\Rightarrow$

$$B_{0n} = \frac{2}{c\alpha_{0n}\rho J_1^2(\alpha_{0n})} \int_0^{\rho} r g(r) J_0\left(\frac{\alpha_{0n}}{\rho} r\right) dr \tag{6.11}$$



$n = 1$ $n = 2$ $n = 3$