Lesson 03 Heat Equation with Different BCs

■ (*) Physical meaning (SJF 2)

Let $u(x,t)$ represent the temperature of a thin rod governed by the (conduction) heat equation:

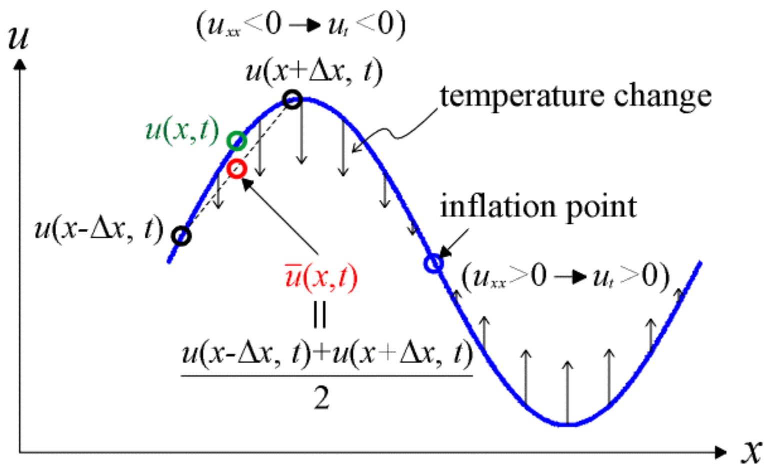
$$u_t = \alpha^2 u_{xx} \tag{3.1}$$

where α^2 is the thermal diffusivity (derived by conservation of energy, Appendix 3A). Eq. (3.1) means that the time-rate of change in temperature (u_t) is proportional to the concavity (u_{xx}) of the temperature distribution. By eq. (1.2), we have:

$$u_t \propto -[u(x,t) - \bar{u}(x,t)] \tag{3.2}$$

Therefore, eq. (3.1) models two facts: (1) heat always flows from high- to low-temperature regions; (2) the flow rate is proportional to the temperature gradient.

E.g. if the temperature at x is higher than its surrounding average, $\Rightarrow$ $u > \bar{u}$, $u_{xx} < 0$, $u_t < 0$, temperature is decreasing at a rate proportional to $[u(x,t) - \bar{u}(x,t)]$.



In steady state, $u_t=0$, $\Rightarrow$ $u_{xx}=0$, i.e. heat flow tends to neutralize the curvature of the temperature distribution, leading to a **linear** temperature spatial profile.

<Comment>

In wave equation [eq. (1.1)], $u_{xx}<0$ does not guarantee $u_t<0$ (but $u_{tt}<0$). The solutions to wave equation and heat equation behave very differently (vibration vs. diffusion).

Heat Equation with Various BCs (EK 12.5, SJF 3)

■ Type1 (Dirichlet) BC: temperature is specified on the boundary, i.e. u=$g(t)$.

Consider a thin rod of length L with two ends fixed at zero temperature.

PDE: $u_t = \alpha^2 u_{xx}$

Two homogeneous BCs: $\{u(0,t)=0,\ u(L,t)=0\}$

One IC: $u(x,0)=\phi(x)$, for only first-order partial derivative with respect to t is involved.

Solving heat equation by separation of variables:

1) Separation of variables:

   Let $u(x,t)=X(x)\cdot T(t) \Rightarrow\ X'' + k^2 X = 0,\ \dot{T} + \alpha^2 k^2 T = 0$ (Lesson 2)

2) Solving the normal modes by homogeneous BCs:

   (1) To avoid trivial solution $u(x,t)=0$, homogeneous BCs of $u(x,t) \rightarrow$ BCs of $X(x)$:

   $\{u(0,t)=0,\ u(L,t)=0\} \rightarrow \{X(0)=0,\ X(L)=0\}$

   (2) The spatial ODE: $X'' + k^2 X = 0,\ \Rightarrow X(x)=A\cos(kx)+B\sin(kx)$;

   By BCs, $\Rightarrow$ (i) $X(0)=0 \Rightarrow A=0$; (ii) $X(L)=0 \Rightarrow k = k_n = \dfrac{n\pi}{L}$, $n=\mathbf{1},2,\ldots \Rightarrow X_n(x)=\sin(k_n x)$;

   The temporal ODE: $\dot{T} + \lambda_n^2 T = 0$, $\lambda_n = \alpha k_n = \dfrac{n\pi\alpha}{L}$; $\Rightarrow T_n(t)=\exp(-\lambda_n^2 t)$;

   $\Rightarrow$ n-th normal mode is $u_n(x,t)= X_n(x)\cdot T_n(t)$:

$$u_n(x,t) = A_n \cdot \sin(k_n x)\cdot \exp(-\lambda_n^2 t) \tag{3.3}$$

<Comment>

(a) Eq's (2.1) and (3.3) have the same spatial shape (due to same spatial ODE and BCs) but different forms of temporal evolution (oscillation vs. decaying), for the orders of temporal ODEs are two and one, respectively.

(b) Higher order modes [$u_n(x,t)$ with larger n] will decay faster, for the decay constant

$\lambda_n^2 \propto n^2$. $\Rightarrow$ Temperature profile will be smoother as time elapses (low-pass filtering).

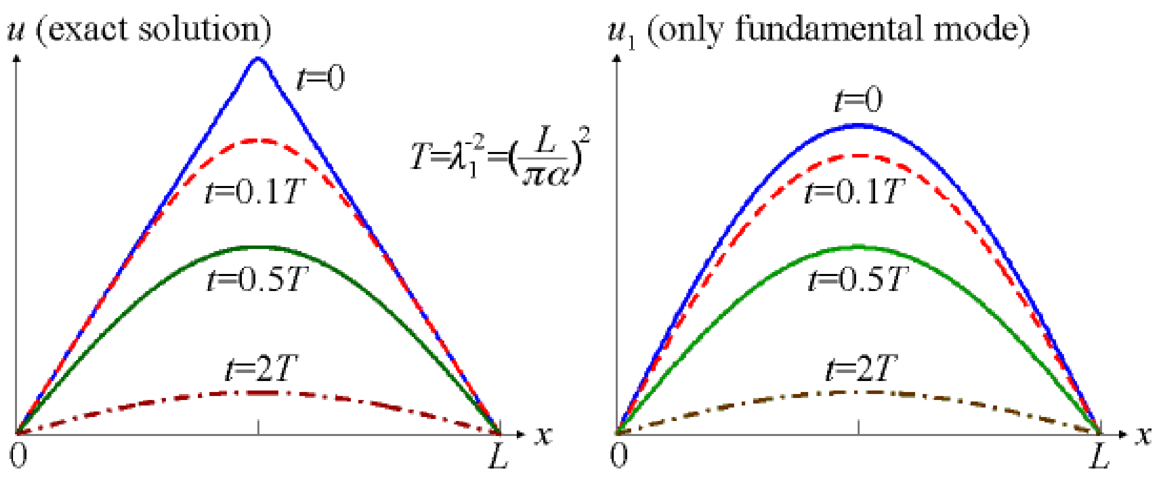
3) Determining the exact solution by IC:

$$u(x,t)=\sum_{n=1}^{\infty} u_n(x,t)=\sum_{n=1}^{\infty} A_n \sin(k_n x)\cdot \exp(-\lambda_n^2 t) \tag{3.4}$$

Substitute the IC into eq. (3.4): $u(x,0)=\sum_{n=0}^{\infty} A_n \sin(k_n x)=\phi(x)$. By Fourier sine series, $\Rightarrow$

$$A_n=\frac{2}{L}\int_0^L \phi(x)\cdot \sin(k_n x)dx \tag{3.5}$$

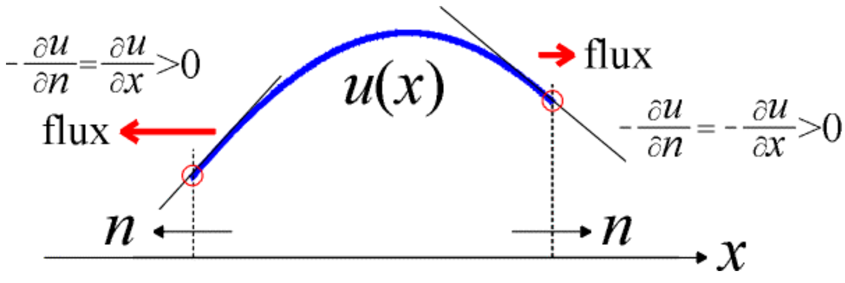
E.g. Let initial temperature distribution is triangular: $u(x,0)=\phi(x)=\begin{cases} x, \text{ for } 0<x<L/2 \\ L-x, \text{ for } L/2<x<L \end{cases}$. By eq. (3.5), $A_n=\begin{cases} (-1)^{(n-1)/2}\left(\dfrac{4L}{n^2\pi^2}\right), \text{ if } n \text{ is odd} \\ 0, \text{ if } n \text{ is even} \end{cases}$. As shown below, exact solution (left) will converge to the fundamental mode (right) as time elapses. [$T=\lambda_1^{-2}$ represents decay time constant of fundamental mode: $u_1(x,T)=e^{-1}u_1(x,0)$.]



■ Type2 (Neumann) BC:

By (experimental) Fourier's law of cooling, heat flows in the direction where temperature $u(\vec{r},t)$ decreases most rapidly, and the flow rate is proportional to the rate of temperature change in that direction. $\Rightarrow$ Heat flux $\vec{q}(\vec{r},t)$ (W/m^2)$=-\kappa$ (W/mK)$\cdot\nabla u(\vec{r},t)$ (Appendix 3A).

On the boundary, heat flux is in parallel with the outward normal direction $\vec{n}$: $\vec{q}=q\,\vec{n}=-\kappa(\nabla u)$. Taking inner product with $\vec{n}$, $\Rightarrow q=-\kappa\dfrac{\partial u}{\partial n}$, i.e. $\dfrac{\partial u}{\partial n}=-\dfrac{q(t)}{\kappa}=g(t)$.



Consider a thin rod of length L with two "insulated" ends.

PDE: $u_t=\alpha^2 u_{xx}$

Two homogeneous BCs: $\{u_x(0,t)=0,\ u_x(L,t)=0\}$

One IC: $u(x,0)=\phi(x)$.

1) Separation of variables: $X''+k^2X=0$, $\dot{T}+\alpha^2k^2T=0$;

2) Solving the normal modes by homogeneous BCs:

Spatial ODE: $X''+k^2X=0$, $\Rightarrow X=A\cos(kx)+B\sin(kx)$, $X'(x)=-k[A\sin(kx)-B\cos(kx)]$;

Transformation of BCs: $\{u_x(0,t)=u_x(L,t)=0\}\rightarrow\{X'(0)=X'(L)=0\}$;

By BCs: (i) $X'(0)=0\Rightarrow B=0$, (ii) $X'(L)=0\Rightarrow k=k_n=\dfrac{n\pi}{L}$, $n=\mathbf{0},1,2,\ldots$

$\Rightarrow X_n(x)=\cos(k_nx)$, **different BCs produce different modes!**

Temporal ODE: $\dot{T}+\lambda_n^2T=0$, $\lambda_n=\alpha k_n=\dfrac{n\pi\alpha}{L}$; $\Rightarrow T_n(t)=\exp\left(-\lambda_n^2t\right)$;

$\Rightarrow$ n-th normal mode is $u_n(x,t)=X_n(x)\cdot T_n(t)$:

$$u_n(x,t)=A_n\cos(k_nx)\cdot\exp\left(-\lambda_n^2t\right) \tag{3.6}$$

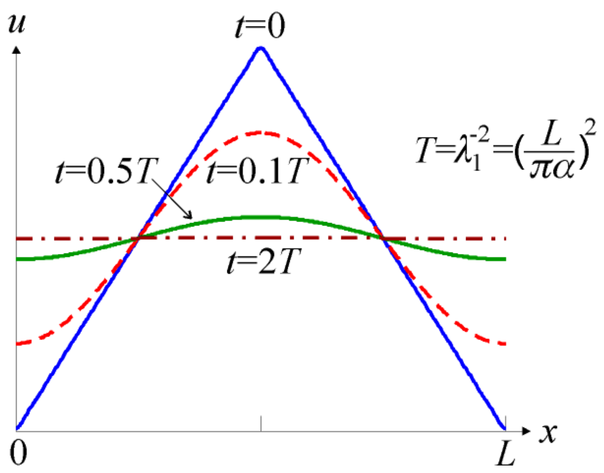
3) Determining the exact solution by IC:

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}A_n\cos(k_nx)\cdot\exp\left(-\lambda_n^2t\right) \tag{3.7}$$

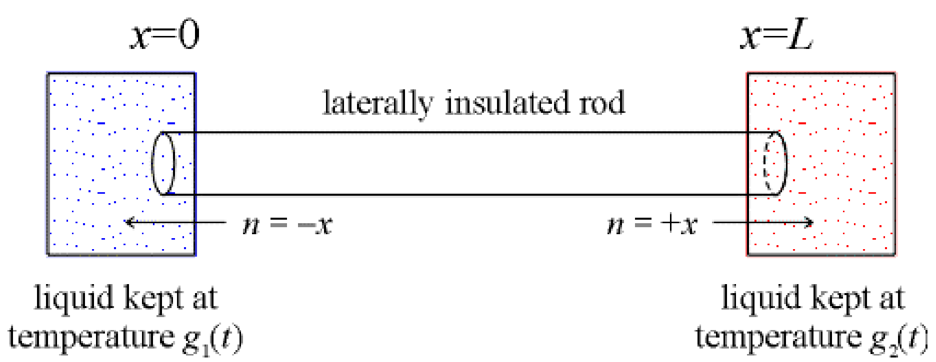
Substitute the IC into eq. (3.7): $u(x,0)=\sum_{n=0}^{\infty}A_n\cos(k_nx)=\phi(x)$. By Fourier sine series, $\Rightarrow$

$$A_0 = \frac{1}{L}\int_0^L \phi(x)dx\,, \quad A_n = \frac{2}{L}\int_0^L \phi(x)\cdot\cos(k_n x)dx \tag{3.8}$$

E.g. Let initial temperature distribution is triangular: $u(x,0)=\phi(x)=\begin{cases} x, \text{ for } 0<x<L/2 \\ L-x, \text{ for } L/2<x<L \end{cases}$. By eq's (3.7–8), exact solution will converge to A_0 (average temperature of initial distribution) as time elapses. This makes sense, for no energy escapes from the rod, and heat flow tends to remove any temperature curvature.



■ **(*)** Type3 (mixed) BC: temperature of the surrounding medium is specified, i.e. $\frac{\partial u}{\partial n}$ $+\gamma[u+g(t)]=0$, where $\gamma=\frac{h}{\kappa}$, h is heat-exchange coefficient, κ is thermal conductivity.



1) By the Newton's law of cooling: heat flux is in the direction from high- to low-

temperature T, and is proportional to its difference ΔT "across" the boundary.

Outward heat flux at x=0: $h[u(0,t)-g_1(t)]$, outward heat flux at x=L: $h[u(L,t)-g_2(t)]$.

2) By the Fourier's law of cooling (experimental): $q=-\kappa\dfrac{\partial u}{\partial n}$.

Outward heat flux at x=0: $-\kappa\dfrac{\partial u(0,t)}{\partial(-x)}=\dfrac{\partial u(0,t)}{\partial x}$, outward heat flux at x=L: $-\kappa\dfrac{\partial u(L,t)}{\partial x}$.

3) Equating 1 & 2, we have:

$$\begin{cases} u_x(0,t)=\gamma[u(0,t)-g_1(t)] \\ u_x(L,t)=-\gamma[u(L,t)-g_2(t)] \end{cases}, \text{ where } \gamma=\frac{h}{\kappa} \tag{3.9}$$

Consider a thin rod of length L with one end fixed at zero temperature, the other end immersed in a liquid of zero temperature (SJF 7).

PDE: $u_t=\alpha^2 u_{xx}$

Two homogeneous BCs: $\{u(0,t)=0,\ u_x(L,t)+\gamma u(L,t)=0\}$

One IC: $u(x,0)=\phi(x)$.

1) Separation of variables: $u(x,t)=X(x)\cdot T(t) \Rightarrow\ X''+k^2X=0,\ \dot{T}+\alpha^2k^2T=0$

2) Solving the normal modes by homogeneous BCs:

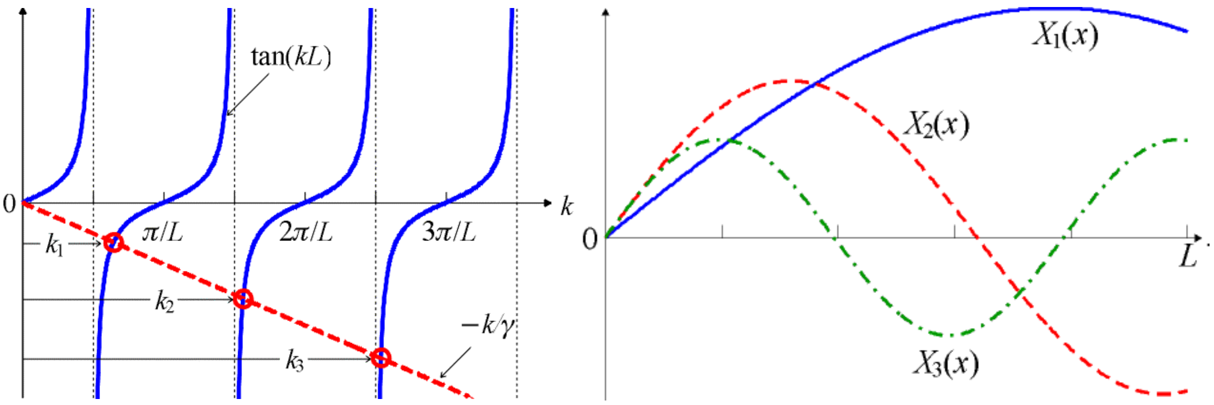
Spatial ODE: $X''+k^2X=0$, $\Rightarrow X(x)=A\cos(kx)+B\sin(kx)$;

Transformation of BCs: $\{u(0,t)=0,\ u_x(L,t)+\gamma u(L,t)=0\} \rightarrow \{X(0)=0,\ X'(L)+\gamma X(L)=0\}$

By BCs: (i) A=0, (ii) $k=k_n$, where $\{k_n\}$ are irregularly-spaced **discrete roots** of a nonlinear equation:

$$\tan(kL)=-\frac{k}{\gamma} \tag{3.10}$$

$\Rightarrow X_n(x)=\sin(k_n x)$ [different from that of eq. (3.3)].



Temporal ODE: $\dot{T}+\lambda_n^2 T=0$, $\lambda_n=\alpha k_n$; $\Rightarrow T_n(t)=\exp\left(-\lambda_n^2 t\right)$;

$\Rightarrow$ n-th normal mode:

$$u_n(x,t)=A_n\sin(k_n x)\cdot\exp\left(-\lambda_n^2 t\right) \tag{3.11}$$

3) Determining the exact solution by IC:

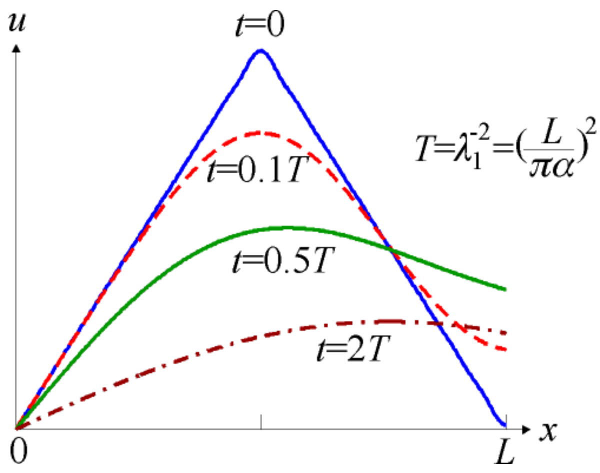
$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}A_n\sin(k_n x)\cdot\exp\left(-\lambda_n^2 t\right) \tag{3.12}$$

Substitute the IC into eq. (3.12): $u(x,0)=\sum_{n=0}^{\infty}A_n\sin(k_n x)=\phi(x)$;

By orthogonality of $X_n(x)$ in $[0,L]$ (for the spatial ODE is a Sturm-Liouville problem): $\int_0^L\sin(k_n x)\cdot\sin(k_m x)dx=\dfrac{L}{2}-\delta_{mn}\dfrac{\sin(2k_n L)}{4k_n}=\dfrac{L}{2}[1-\delta_{mn}\mathrm{sinc}(2k_n L)]$, where $\delta_{mn}=\begin{cases}1, \text{if } m=n\\ 0, \text{otherwise}\end{cases}$, $\mathrm{sinc}(x)\equiv\dfrac{\sin x}{x}$, $\Rightarrow$

$$A_n=\frac{2}{L[1-\mathrm{sinc}(2k_n L)]}\int_0^L\phi(x)\cdot\sin(k_n x)dx \tag{3.13}$$

E.g. Let initial temperature distribution is triangular: $u(x,0)=\phi(x)=\begin{cases}x, \text{ for } 0<x<L/2\\ L-x, \text{ for } L/2<x<L\end{cases}$. By eq's (3.12–13), exact solution behaves differently with those in type-1 and type-2 BCs. Can you justify why $u(L,t)\neq 0$ even the right end is immersed in a liquid of zero-temperature and $u(L,0)=0$?



Heat Equation without Boundary (EK 12.6)

■ Problem: heat diffusion along a thin rod of infinite length.

PDE: $u_t = \alpha^2 u_{xx}$

No BC

One IC: $u(x,0)=\phi(x)$

■ Solving heat equations by separation of variables & Fourier integrals:

1) Separation of variables: $u(x,t)=X(x)T(t) \Rightarrow \;\; X''+k^2X=0\,,\;\; \dot{T}+\alpha^2k^2T=0$

2) No BC $\Rightarrow$ eigenvalue k is NOT quantized, but continuous:

Spatial ODE: $X''+k^2X=0 \Rightarrow X_k(x)=A(k)\cos(kx)+B(k)\sin(kx)$ [$A(k)$, $B(k)$ are justified by substitution]; temporal ODE: $\dot{T}+\alpha^2k^2T=0 \;\; \Rightarrow T_k(t)=\exp\left(-\alpha^2k^2t\right)$,

$\Rightarrow$ n-th normal mode:

$$u_k(x,t) = [A(k)\cdot\cos(kx)+B(k)\cdot\sin(kx)]\cdot\exp\left(-\alpha^2k^2t\right) \tag{3.14}$$

3) Determining the exact solution by IC:

$$u(x,t)=\int_0^\infty u_k(x,t;k)dk=\int_0^\infty [A(k)\cdot\cos(kx)+B(k)\cdot\sin(kx)]\cdot e^{-\alpha^2k^2t}dk \tag{3.15}$$

Substitute the IC into eq. (3.15): $u(x,0)=\int_0^\infty [A(k)\cdot\cos(kx)+B(k)\cdot\sin(kx)]dk=\phi(x)$; by

Fourier integrals (EK 11.7), ⇒

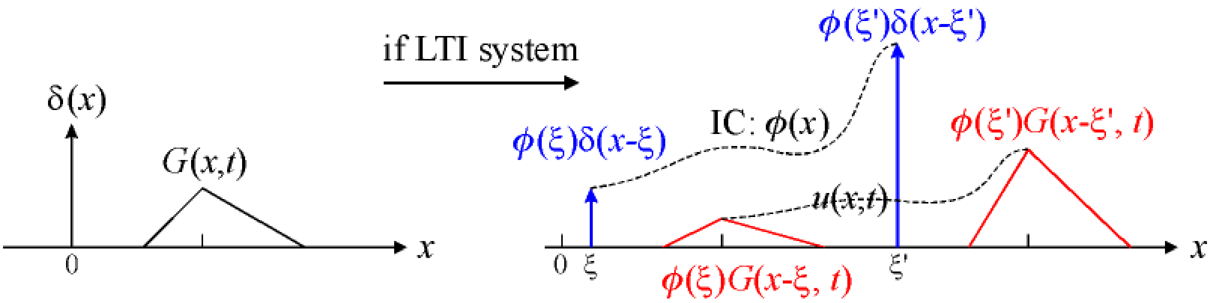
$$A(k)=\frac{1}{\pi}\int_{-\infty}^{\infty}\phi(\xi)\cdot\cos(k\xi)d\xi\ ,\ B(k)=\frac{1}{\pi}\int_{-\infty}^{\infty}\phi(\xi)\cdot\sin(k\xi)d\xi \tag{3.16}$$

■ **(*)** Green's function

Integral solution of the form of eq. (3.15) can be simplified as (EK 12.6):

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi\cdot t}}\int_{-\infty}^{\infty}\phi(\xi)\exp\left[-\frac{(x-\xi)^2}{4\alpha^2 t}\right]d\xi=\phi(x)\otimes G(x,t) \tag{3.17}$$

where $G(x,t)=\frac{1}{2\alpha\sqrt{\pi\cdot t}}\exp\left(-\frac{x^2}{4\alpha^2 t}\right)$ is called the Green's function of the system, which can be interpreted as the temperature response to an initial temperature impulse: $\phi(x)=\delta(x)$.



Since linear homogeneous PDE + linear homogeneous BCs (like this particular problem) describe linear and time invariant (LTI) systems, ⇒ an input of $c_1\delta(x-x_1)+c_2\delta(x-x_2)$ results in an output of $c_1G(x-x_1, t)+c_2G(x-x_2, t)$.

We can decompose the initial distribution $\phi(x)$ as a continuum of impulses: $\phi(x)=\int_{-\infty}^{\infty}\phi(\xi)\cdot\delta(x-\xi)d\xi$. Each impulse $\phi(\xi)\cdot\delta(x-\xi)$ [located at $x=\xi$ and with magnitude $\phi(\xi)$] has an output $\phi(\xi)\cdot G(x-\xi,t)$, and their superposition gives the overall response $u(x,t)$.

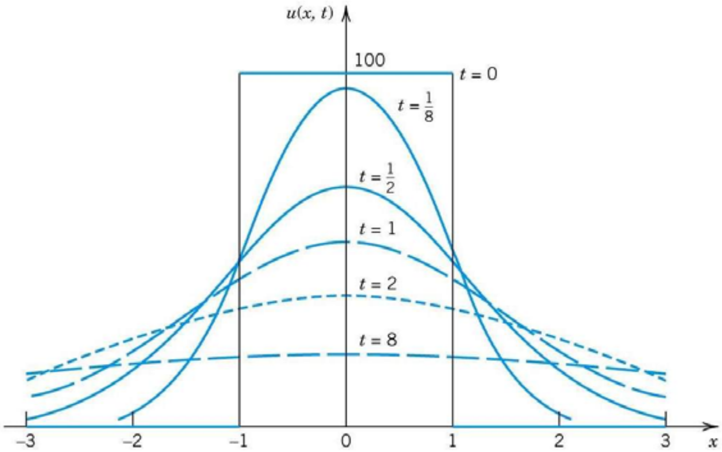
You may find that another equivalent formula is useful in some practical evaluations:

$$u(x,t)=\left(1/\sqrt{\pi}\right)\int_{-\infty}^{\infty}\phi(x+2\alpha z\sqrt{t})e^{-z^2}dz \tag{3.18}$$

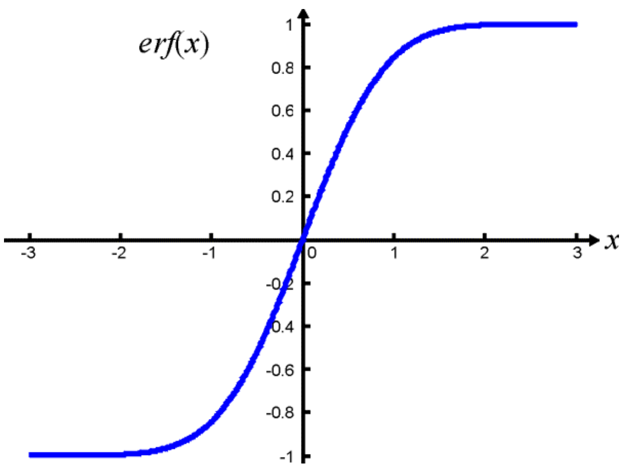
E.g. Find $u(x,t)$ if initial temperature distribution $\phi(x)=\begin{cases} U_0, \text{ for } -1<x<1 \\ 0, \text{ otherwise} \end{cases}$

Ans: By eq. (3.18): $u(x,t)=\dfrac{1}{\sqrt{\pi}}\displaystyle\int_{-(1+x)/2\alpha\sqrt{t}}^{(1-x)/2\alpha\sqrt{t}} e^{-z^2}dz = \dfrac{1}{2}\left\{erf\left(\dfrac{1-x}{2\alpha\sqrt{t}}\right)-erf\left(\dfrac{-1-x}{2\alpha\sqrt{t}}\right)\right\}$, $\Rightarrow$

temperature profile $u(x)$ gets smoother as t increases.



Error function is defined as: $erf(x)\equiv\dfrac{2}{\pi}\displaystyle\int_0^x e^{-t^2}dt$.



Appendix 3A − Derivation of conductive heat equation (by D. W. Trim)

To describe conductive heat flow in a medium, we define a 3-D heat flux vector $\vec{q}(\vec{r},t)$ (W/m^2). Its direction and magnitude represent the direction of heat flow and the amount of heat per unit time crossing unit area normal to the direction of $\vec{q}$. By the (experimental) **Fourier's law of cooling**, heat flows in the direction where temperature $u(\vec{r},t)$ decreases most rapidly, and the flow rate is proportional to the rate of temperature change in that direction:

$$\vec{q}(\vec{r},t) = -\kappa \cdot \nabla u(\vec{r},t) \tag{3A.1}$$

where κ (W/mK)>0 is the thermal conductivity.

Heat is accumulated in a volume V enclosed by surface S because of: (i) flux across S, (ii) internal heat source $g(\vec{r},t)$ (W/m^3):

$$\iint_S \vec{q} \cdot (-d\vec{s}) + \iiint_V g \cdot dv \quad \text{(W)}$$

The heat accumulation causes temperature increase to maintain energy conservation:

$$\iiint_V \frac{\partial u}{\partial t} s\rho dv = \iint_S \vec{q} \cdot (-d\vec{s}) + \iiint_V g \cdot dv$$

where s (J/kgK) and ρ (Kg/m^3) are specific heat and density of the medium. By eq. (3A.1) and divergence theorem, $\iint_S \vec{q} \cdot (-d\vec{s}) = \iiint_V \nabla \cdot (\kappa \nabla u) dv$, $\Rightarrow$

$$\iiint_V [\rho s u_t - g - \nabla \cdot (\kappa \nabla u)] dv = 0 \tag{3A.2}$$

If the integrand is continuous throughout an arbitrarily small volume V, eq. (3A.2) becomes:

$$u_t = \alpha^2 \left(\nabla^2 u + \frac{g}{\kappa} \right) \tag{3A.3}$$

where α^2 (m^2/sec) is the thermal diffusivity.