Lesson 02 Separation of Variables & D'Alembert's Solutions

Solving PDEs by Separation of Variables

■ When to use?

1) PDE is linear and homogeneous (variable coefficients are OK).
2) BCs are also linear and homogeneous. **E.g.** $\{\alpha\cdot u_x(0,t)+\beta\cdot u(0,t)=0,\ \gamma\cdot u_x(L,t)+\delta\cdot u(L,t)=0\}$.

■ How to use?

The basic idea lies on **superposition** of solutions to linear homogeneous PDEs. It consists of three steps:

1) Separation of variables: a **PDE** of n variables $\Rightarrow$ n **ODEs** (usually Sturm-Liouville problems, EK 5.7, see Appendix 2A).
2) Solving the ODEs by **BCs** to get normal **modes** (solutions satisfying PDE and BCs).
3) Determining exact solution (expansion **coefficients** of modes) by **ICs**

■ Initial-boundary-value problem (IBVP): standing wave

A string of length L with two fixed ends, initial displacement $\phi(x)$, and initial velocity $\gamma(x)$ can be modeled as:

PDE: $u_{tt} = c^2 u_{xx}$

Two BCs: $u(0,t)=0$, $u(L,t)=0$

Two ICs: $u(x,0)=\phi(x)$, $u_t(x,0)=\gamma(x)$

1) Separation of variables:

   Let $u(x,t)=X(x)T(t)$, substitute it into the PDE, $\Rightarrow$ $X\ddot{T} = c^2 X'' T$; divide by c^2XT,

   $\Rightarrow \dfrac{\ddot{T}}{c^2 T} = \dfrac{X''}{X} = \boldsymbol{a}$ (both sides must be **constant** to maintain the equality for arbitrary x, t);

$\Rightarrow X'' - aX = 0$, $\ddot{T} - c^2 aT = 0$ (one PDE → two ODEs)

2) Solving the normal modes by BCs $\{u(0,t)=X(0)T(t)=0$, and $u(L,t)=X(L)T(t)=0\}$:

(1) If $T(t)=0$, ⇒ $u(x,t)=0$ becomes a trivial solution. As a result, $\{X(0)=0, X(L)=0\}$, i.e. BCs of $u(x,t)$ → BCs of $X(x)$. used in solving.

(2) If $a=0$, the ODE $X'' - aX = 0$ is reduced to $X''=0$, ⇒ $X(x)=Ax+B$. By BCs in (1), ⇒ $X(x)=0$, $u(x,t)=0$ becomes a trivial solution. ⇒ $a\neq 0$.

(3) If $a=\mu^2>0$, the ODE becomes $X'' - \mu^2 X = 0$, ⇒ $X(x)=Ae^{\mu x}+Be^{-\mu x}$. By BCs in (1), ⇒ $X(x)=0$, $u(x,t)=0$ becomes a trivial solution. ⇒ a must be negative.

(4) Let $\boldsymbol{a=-k^2}<0$, the ODE becomes $X'' + k^2 X = 0$, ⇒ $X(x)=A\cos(kx)+B\sin(kx)$.

By (1), ⇒ $A=0$, $k = \boxed{k_n=\frac{n\pi}{L}}$, $n=1, 2, \ldots$(a and k are quantized); ⇒ $X_n(x)=\sin(k_n x)$;

The other ODE becomes $\ddot{T} + \omega_n^2 T = 0$, $\boxed{\omega_n=\frac{n\pi c}{L}}$; ⇒ $T_n(t)=A_n\cos(\omega_n t)+ B_n\sin(\omega_n t)$;

⇒ the n-th normal **mode** (a function satisfying PDE and BCs) is $u_n(x,t)= X_n(x)\cdot T_n(t)$,

$$u_n(x,t) = [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)]\cdot \sin(k_n x) \tag{2.1}$$

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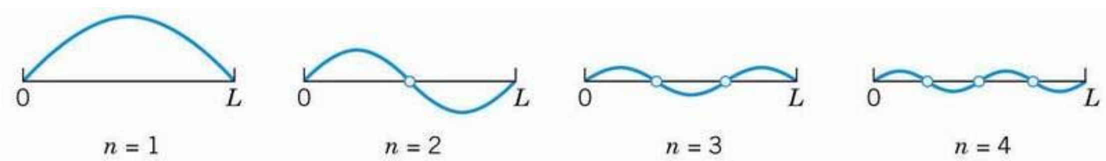
(a) $u_n(x,t)$ are called eigenfunctions, and $\{k_n, \omega_n\}$ are eigenvalues of the vibrating string.

(b) $u_1(x,t)$ is called fundamental mode; other modes with $n>1$ are overtones (泛音). Each mode $u_n(x,t)$ vibrates with a unique frequency:

$$\nu_n=\frac{\omega_n}{2\pi}=n\nu_1,\ \nu_1=\frac{\sqrt{T/\rho}}{2L} \tag{2.2}$$

where ν_1 is the fundamental frequency. The spatial shape of mode remains unchanged (but amplitude varies) with time.



(c) The relation $\nu_n = n\nu_1$ implies that overtone frequencies of violin string are always integral times of fundamental frequency (harmonic resonance). However, this is not true in the case of drumhead (EK 12.8).

(d) By eq. (2.2), frequency tuning can be done by changing tension T, mass density ρ, or string length L.

3) Determining the exact solution by ICs:

Since the PDE and BCs are linear and homogeneous, superposition of normal modes $u_n(x,t)$ still satisfies the same PDE and BCs. We can represent the exact solution $u(x,t)$ by an infinite series:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right] \cdot \sin(k_n x) \tag{2.3}$$

Substitute the two ICs into eq. (2.3): $u(x,0) = \sum_{n=0}^{\infty} A_n \sin(k_n x) = \phi(x)$, $u_t(x,0) = \sum_{n=0}^{\infty} B_n \omega_n \sin(k_n x) = \gamma(x)$. By Fourier sine series (EK 11.3), ⇒

$$A_n = \frac{2}{L}\int_0^L \phi(x)\cdot\sin(k_n x)dx, \quad B_n = \frac{2}{L\omega_n}\int_0^L \gamma(x)\cdot\sin(k_n x)dx \tag{2.4}$$

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In addition to deriving the exact solution, we solve the normal modes $\{u_n(x,t)\}$ because of:

1) $\{u_n(x,t)\}$ forms a **complete**, and **orthogonal** set within the interval $x=[0,L]$ (Appendix 2A). The completeness ensures that any solution $u(x,t)$ can always be represented, and the orthogonality simplifies the determination of expanding coefficients $\{A_n, B_n\}$.

2) PDE and BCs (normal modes) fully describe the system characteristics, while ICs simply determine how the system is excited (excited modes and their relative weighting).

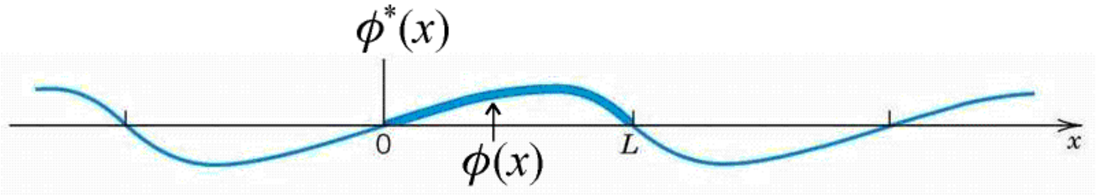
3) Knowledge about normal modes helps to determine initial excitation. **E.g.** If we want the string only vibrating with fundamental frequency ν_1, the initial displacement and velocity should be of the shape $X_1(x)$, leaving $A_n = B_n = 0$ for all $n>1$.

■ **(*)** Why $u_{tt} = c^2 u_{xx}$ is called "wave" equation?

For simplicity, let initial velocity $\gamma(x)=0$, $\Rightarrow$ $\{B_n\}=0$, $u(x,t)=\sum_{n=1}^{\infty} A_n \cos(\omega_n t)\cdot\sin(k_n x)$. By the trigonometric formula $\cos\alpha\cdot\sin\beta = \dfrac{\sin(\beta-\alpha)+\sin(\beta+\alpha)}{2}$ and $\omega_n = ck_n$, we have:

$$u(x,t)=\frac{1}{2}\left\{\sum_{n=1}^{\infty} A_n \sin[k_n(x-ct)] + A_n \sin[k_n(x+ct)]\right\}=\frac{1}{2}\left[\phi^*(x-ct)+\phi^*(x+ct)\right] \tag{2.5}$$

where ϕ^* is the "odd periodic expansion" of initial displacement $u(x,0)=\phi(x)$ with period $2L$. (Since $\phi(x)$ is only defined for $[0,L]$, $\phi(x\pm ct)$ could be undefined for $t\neq 0$.)



Eq. (2.5) means the initial displacement function $\phi(x)$ is equally decomposed into two parts, each propagates with velocity c but in opposite directions (for they are functions of $x\pm ct$). Their superposition determines the displacement at arbitrary time t, $\Rightarrow$ wave behavior!

D'Alembert's Solution of Wave Equation

■ Initial value problem (IVP): traveling wave

Eq. (2.5) implies that the solutions to $u_{tt} = c^2 u_{xx}$ behave like a wave. This concept is more evident and complete when considering an infinite string (no "reflection" due to boundary) with nonzero initial velocity.

PDE: $u_{tt} = c^2 u_{xx}$

No BC

Two ICs: $u(x,0)=\phi(x)$, $u_t(x,0)=\gamma(x)$

Solving the IVP (SJF 17)

1) Changing to canonical coordinates: $(x,t) \rightarrow (\xi,\eta)$ (Appendix 1A). Let $\boldsymbol{\xi=x+ct}$, $\boldsymbol{\eta=x-ct}$; $u_{tt} = c^2 u_{xx}$ is transformed into $\boldsymbol{u_{\xi\eta}=0}$ by chain rule:

$$u_t = \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial t} = c\left(u_\xi - u_\eta\right),$$

$$u_{tt} = c\left[\left(\frac{\partial u_\xi}{\partial \xi}\cdot\frac{\partial \xi}{\partial t} + \frac{\partial u_\xi}{\partial \eta}\cdot\frac{\partial \eta}{\partial t}\right) - \left(\frac{\partial u_\eta}{\partial \xi}\cdot\frac{\partial \xi}{\partial t} + \frac{\partial u_\eta}{\partial \eta}\cdot\frac{\partial \eta}{\partial t}\right)\right] = c^2\left(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}\right),$$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = u_\xi + u_\eta,$$

$$u_{xx} = \left(\frac{\partial u_\xi}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u_\xi}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}\right) + \left(\frac{\partial u_\eta}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u_\eta}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}\right) = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta},$$

$u_{tt} = c^2 u_{xx} \Rightarrow c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) = c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}), \Rightarrow u_{\xi\eta} = 0.$

2) Solving the equation in the $\xi\eta$-domain by two integrations: (i) $u_\eta(\xi,\eta)=\delta(\eta)$, where $\delta(\eta)$ is an arbitrary function of η. (ii) $\boldsymbol{u(\xi,\eta)=\Delta(\eta)+\psi(\xi)}$, where $\Delta(\eta)=\int \delta(\eta)d\eta$; $\Delta(\eta)$ and $\psi(\xi)$ can be arbitrary functions of η and ξ, respectively.

3) Transforming back to the xt-domain to get general solution: $\boldsymbol{u(x,t)=\Delta(x-ct)+\psi(x+ct)}$. This result means the solution must be the superposition of two moving waves with identical velocity c but in opposite directions.

4) Applying ICs to get the exact solution: (i) $u(x,0)=\phi(x)$, $\Rightarrow \Delta(x)+\psi(x)=\phi(x)$; (ii) $u_t(x,0)=\gamma(x)$: by $u_t(x,t)=\left.\frac{d\Delta}{dx'}\frac{\partial x'}{\partial t}\right|_{x'=x-ct} + \left.\frac{d\psi}{dx'}\frac{\partial x'}{\partial t}\right|_{x'=x+ct} = -c\Delta'(x-ct) + c\psi'(x+ct)$, $\Rightarrow -\Delta'(x) + \psi'(x) = \frac{\gamma(x)}{c}$. By integration from x_0 to x, $\Rightarrow -\Delta(x)+\psi(x)=\frac{1}{c}\left[\int_{xo}^{x}\gamma(x')dx'\right]+K$. Solve (i-ii), $\Rightarrow \Delta(x)=\frac{\phi(x)}{2} - \frac{1}{2c}\left[\int_{xo}^{x}\gamma(x')dx'\right] - \frac{K}{2}$, $\psi(x)=\frac{\phi(x)}{2} + \frac{1}{2c}\left[\int_{xo}^{x}\gamma(x')dx'\right] + \frac{K}{2}$. The exact solution is of the form (D'Alembert solution):

$$u(x,t)=\frac{\phi(x-ct)+\phi(x+ct)}{2} + \frac{1}{2c}\left[\int_{x-ct}^{x+ct}\gamma(x')dx'\right] \qquad (2.6)$$

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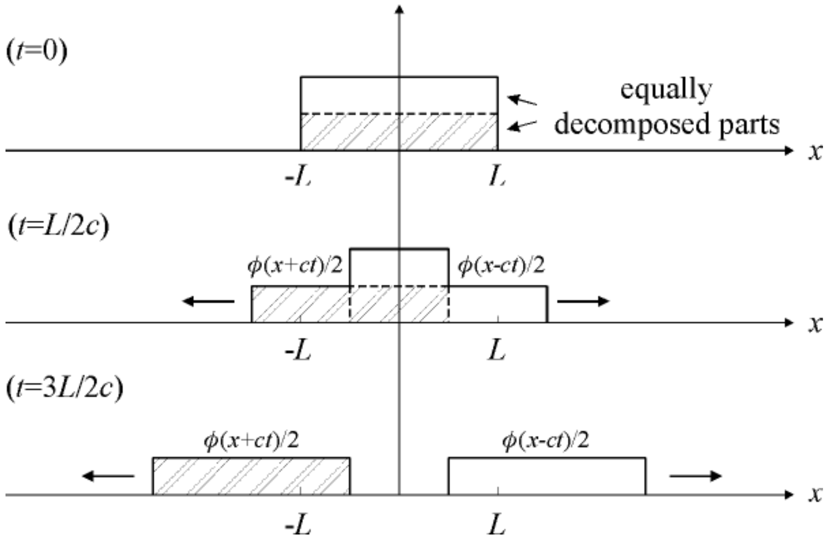
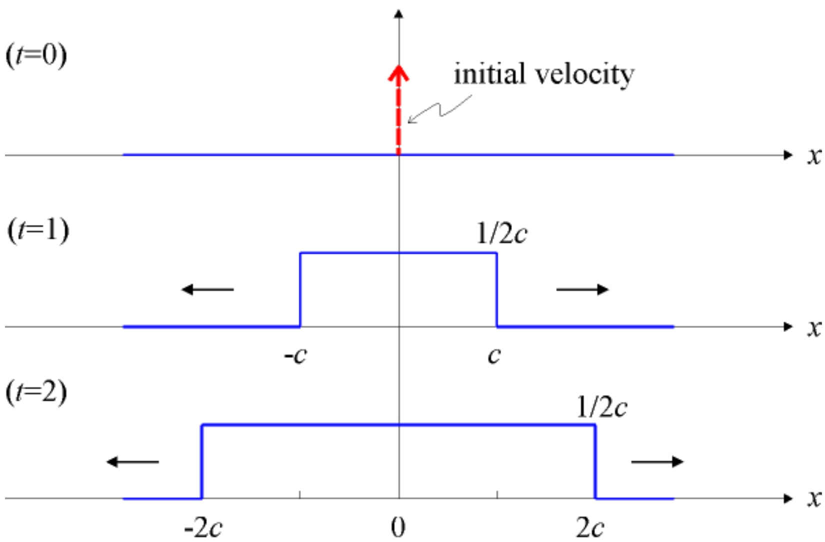
1) The first term of eq. (2.6) is the same as eq. (2.5) (decomposed traveling waves).

E.g. $u(x,0)=\phi(x)=\begin{cases}1, \text{ for } |x|<L \\ 0, \text{ otherwise}\end{cases}$, $u_t(x,0)=\gamma(x)=0$.



2) The second term of eq. (2.6) indicates that displacement $u(x_0,t_0)$ is contributed by the velocity distribution of string particles within a finite range $x_0-ct_0 \le x \le x_0+ct_0$ at t=0. In other words, string particle velocity will expand its "range of influence" with wave velocity c along the string omni-directionally.

E.g. $u(x,0)=\phi(x)=0$, $u_t(x,0)=\gamma(x)=\delta(x)$. By eq. (2.6), $u(x,t)=\begin{cases}1, \text{ for } -ct<x<ct \\ 0, \text{ otherwise}\end{cases}$.



3) Since it is usually very difficult to find general solutions, the above procedure is rarely used in solving PDEs.

Appendix 2A – Sturm-Liouville (SL) Problem (EK 5.7)

■ Definition

Many important functions in engineering, such as Legendre polynomials, Bessel functions, are solutions to a type of linear, homogeneous, 2nd-order ODE:

$$[p(x)y'(x)]' + [q(x) + \lambda r(x)]y(x) = 0 \tag{2A.1}$$

with (linear, homogeneous) BCs:

$$k_1 y(a) + k_2 y'(a) = 0 \tag{2A.2}$$

$$l_1 y(b) + l_2 y'(b) = 0 \tag{2A.3}$$

in the region of interest (ROI): $a \le x \le b$, where $r(x) > 0$, and λ used to be unspecified (need to be solved). Eq's (2A.1-2) describe an eigenvalue problem, whose solutions are eigenfunctions $\{y_i(x)\}$ and eigenvalues $\{\lambda_i\}$.

Singular problem: if $p(a)=0$, eq. (2A.2) is replaced by: $|y(a)|, \ |y'(a)| < \infty$. If $p(b)=0$, eq. (2A.3) is replaced by: $|y(b)|, \ |y'(b)| < \infty$.

■ Orthogonality of eigenfunctions

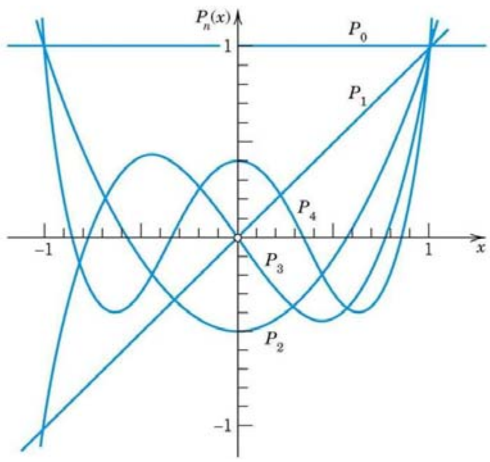
If $p(x)$, $q(x)$, $r(x)$, $p'(x)$ of eq. (2A.1) are real-valued and continuous within the ROI, and $y_m(x)$, $y_n(x)$ are eigenfunctions of the problem corresponding to different eigenvalues λ_m, λ_n; $\Rightarrow$ (1) all eigenvalues are real, (2) $y_m(x)$, $y_n(x)$ must be **orthogonal** on the ROI with respect to the weight function $r(x)$, i.e.

$$\int_a^b y_m(x) y_n(x) r(x) dx = 0 \tag{2A.3}$$

E.g. $y''(x) + \lambda y(x) = 0$, BCs: $\{y(0)=0,\ y(\pi)=0\}$. $\Rightarrow$ $p(x)=1$, $q(x)=0$, $r(x)=1$. $\Rightarrow$ $y_n(x)=\sin(nx)$

are eigenfunctions with eigenvalues $\lambda=n$. $\sin(mx)$, $\sin(nx)$ are orthogonal in the interval $0\le x\le\pi$ with respect to the weight function $r(x)=1$, i.e. $\int_0^\pi \sin(mx)\sin(nx)dx = 0$.

E.g. Legendre's equation: $\left[(1-x^2)y'(x)\right]' + \lambda y(x) = 0$, $\Rightarrow$ $p(x)=1-x^2$, $q(x)=0$, $r(x)=1$. For the ROI $-1\le x\le 1$, $p(1)= p(-1)=0$, $\Rightarrow$ singular problem, BCs are replaced by $|y(\pm1)|<\infty$. By the Frobenius method, we derive Legendre polynomials $P_n(x)$ as eigenfunctions $y_n(x)$ with eigenvalues $\lambda=n(n+1)$. $\Rightarrow$ $P_m(x)$, $P_n(x)$ are orthogonal in the interval $-1\le x\le 1$ with respect to the weight function $r(x)=1$, i.e. $\int_{-1}^1 P_m(x)P_n(x)dx = 0$.



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The importance of SL problem (arising from performing separation of variables for the PDE) lies on: (1) each eigenfunction satisfies the separated ODE and corresponding BCs, thus only ICs need to be taken account afterwards; (2) eigenfunctions form a "complete" set, and any function in the ROI can be represented by their superposition; (3) eigenfunctions are "orthogonal", facilitating the determination of expansion coefficients.