Lesson 01 Introduction to PDEs & Modeling of 1-D Wave Equations

Introduction to PDEs

■ Equation and differential equation

Equation: $f(x)=0$, ⇒ unknown x is a **number** or set of numbers.

Differential equation: $f(u, u', u'',\ldots)=g(x)$, ⇒ unknown u is a **function**. Note that $u', u'',\ldots$are not "extra" unknowns for they can be derived by differentiation once u is determined.

■ What're ODEs?

An equation containing derivative(s) of an unknown function (dependent variable) u with **single** independent variable. **E.g.** $u''(x)+u(x)=0$.

■ What're PDEs?

An equation containing partial derivative(s) of an unknown function u with **two or more** independent variables. **E.g.** $u_t = u_{xx}$.

■ Why PDEs?

People sense the real world via four dimensions (x, y, z, t), therefore, physical quantities (e.g. electrical field, temperature, electron distribution in an atom) are fully described by four variables. ⇒ Most physical laws are described in terms of PDE's, where the derivatives represent physical quantities.

E.g. Electrostatics (potential theory), EM waves (Maxwell's equations), quantum mechanics (Schrödinger's equation), heat transfer (heat equation), fluid mechanics.

■ Classification of PDEs

1) Order of PDE: the order of the highest partial derivative. **E.g.** $u_t = u_{xx}$ (2nd order); $u_t = u_x$

(1st order).

2) Linearity: (i) linear algebraic equation: $a_0+a_1x=b$, where coefficients $\{a_i\}$ and b are constants independent of unknown x. (ii) linear ODE: $L\cdot U^T=b(x)$, where $L=[a_0(x), a_1(x),\ldots]$, $U=[u, u', u'',\ldots]$, coefficients $\{a_i(x)\}$ and $b(x)$ are functions of x independent of unknown $u(x)$. (iii) linear PDE: $L\cdot U^T=b$. **E.g.** A 1st-order linear PDE with two variables is of the form: $L=[a_0, a_1, a_2]$, $U=[u, u_x, u_y]$, coefficients $\{a_i(x,y)\}$ and $b(x,y)$ are functions of x and y independent of unknown $u(x,y)$. **E.g.** An equation containing some nonlinear operation about the unknown u or its derivative(s), such as: u^2, u_xu_y, $\sin(u_{xy})$, is nonlinear.

3) Homogeneity: An equation only containing unknown function u and its derivative(s) is homogeneous. **E.g.** $u_x+xu_y=e^xu$ is homogeneous; $u_x+xu_y=e^x$ is non-homogeneous.

4) **(*)** Linear 2nd-order PDEs in two variables $Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=G$ (A~G are functions of x, y) are categorized as (Appendix 1A):

   (i) Hyperbolic: if $B^2-4AC>0$. **E.g.** wave equation $u_{tt}= u_{xx}$

   (ii) Parabolic: if $B^2-4AC=0$. **E.g.** heat flow and diffusion $u_t= u_{xx}$

   (iii) Elliptic: if $B^2-4AC<0$. **E.g.** steady-state phenomena $u_{xx}+ u_{yy}=0$

   The solutions of these three types of PDEs behave very differently (wave propagation, diffusion, steady-state).

■ Solution to PDEs

A **function** satisfying the equation everywhere in the region of interest (ROI).

In general, infinite number of functions can satisfy the same PDE. Additional constraints, such as boundary conditions (BC, designate function values or its derivatives on the spatial boarders of ROI), and initial conditions (IC, designate function values or its derivatives at the temporal start) are necessary to get unique solution. **E.g.** $u= x^2-y^2$ and $e^x\cos(y)$ satisfy $u_{xx}+ u_{yy}= 0$. However, only x^2-y^2 satisfies both the PDE and BC: $u(0,0)=0$.

■ Fundamental theorem (superposition)

If u_1, u_2 are solutions of a **linear** and **homogeneous** (齊次, source-free) PDE in some region R, then $u=c_1u_1+c_2u_2$ remains a solution to that equation in R.

This theorem is useful in deriving the response of a linear system with various ICs or external sources. One can represent the final solution as superposition of **modes** (set of orthogonal functions satisfying the PDE and BCs) $\{u_1, u_2, \ldots\}$, then find the expansion coefficients $\{c_1, c_2, \ldots\}$ using specified ICs or source functions (methods of separation of variables, eigenfunction expansion, see Lessons 2 and 4).

■ Methods of solving PDEs

1) **(*)** Solving undetermined exponents: valid for linear PDEs of **constant** coefficients, homogeneous or nonhomogeneous, only get general solutions. (Appendix 1B)
2) Separation of variables: reduce a PDE with n variables into n ODEs (EK 12.3)
3) Eigenfunction expansion: expand the solution of a nonhomogeneous PDE as superposition of "spatial eigenfunctions" (satisfying homogeneous PDE and BCs), find the coefficients (functions of t) by solving a sequence of ODEs (SJF 9).
4) Integral transforms: reduce the number of variables one at a time ($n \to n-1$), repeat the process until n=1 (ODE). (EK 12.11).
5) **(*)** Numerical methods: transform a PDE into a system of difference equations (差分方程式, with discrete independent variables), then solve it by iterative techniques. In many cases, this is the only method that will work (EK 21).
6) **(*)** Others: variational(變分) methods (solution of PDE→ minimum of functional, SJF 45), perturbation methods (a nonlinear PDE→ sequence of linear PDEs, SJF 46)….

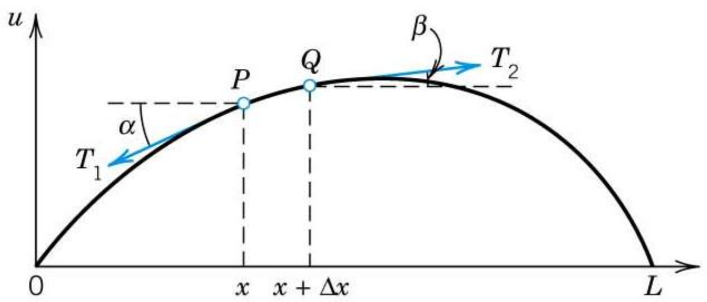
Modeling of 1-D Wave Equation

■ Problem: formulate a PDE governing the motion of a vibrating string.

■ Assumptions

1) Mass density (mass per unit length) ρ is uniform
2) The string does not offer resistance to bending $\Rightarrow$ no frictional nor restoring force
3) The string is light $\Rightarrow$ gravitational force is negligible
4) Every particle of the string moves vertically $\Rightarrow$ transverse wave
5) Displacement and slope of the string are small

■ Modeling



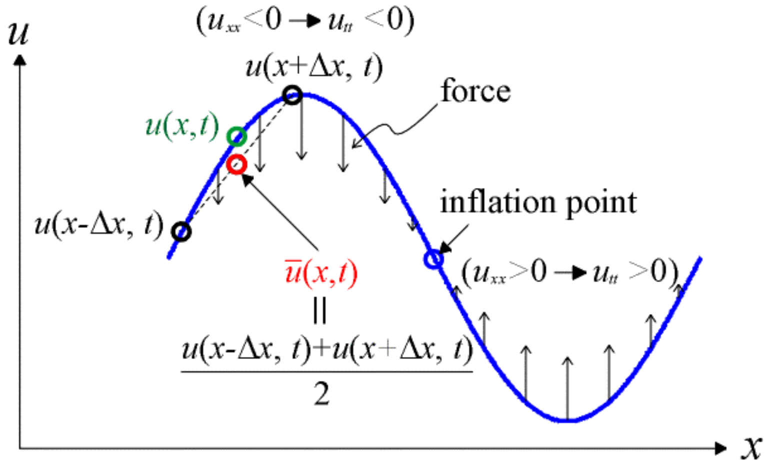
1) No horizontal motion $\Rightarrow$ net horizontal force is zero $\Rightarrow$ $T_2\cos\beta = T_1\cos\alpha = T$…(1)
2) Newton's second law: vertical force=$m\times a$ $\Rightarrow$ $T_2\sin\beta - T_1\sin\alpha = (\rho\,\Delta x)\times u_{tt}$…(2)
3) $\dfrac{(2)}{(1)} = \tan\beta - \tan\alpha = \left(\dfrac{\rho\Delta x}{T}\right)u_{tt}$;
4) $\tan\beta$ means tangential slope of string at point Q, $\Rightarrow$ $\tan\beta = u_x(x+\Delta x)$, similarly, $\tan\alpha =$ $u_x(x)$, $\Rightarrow$ $\dfrac{u_x(x+\Delta x)-u_x(x)}{\Delta x} = \left(\dfrac{\rho}{T}\right)u_{tt}$, $u_{xx} = \left(\dfrac{\rho}{T}\right)u_{tt}$, $\Rightarrow$

$$u_{tt} = c^2 u_{xx},\quad c^2 = \frac{T}{\rho} \qquad (1.1)$$

(∗) Wave equation can be generalized to: $u_{tt} = c^2 u_{xx} - \beta u_t - \gamma u + F(x,t)$, where β, γ, F describe friction, restoring, and external forces (F=g for gravitation force), respectively (SJF 16).

■ **(*)** Intuitive interpretation (SJF 16)

Eq. (1.1) implies that force ($\propto u_{tt}$) is proportional to the concavity (凹性 $\propto u_{xx}$) of the string. ⇒ We can directly depict the force acting on the string by its shape. **E.g.** $u_{xx}<0$, ⇒ $u_{tt}<0$, force acts downward (however, the particle at x can still move upward if $u_t>0$).



To have an insight, we approximate the derivative by the limit of difference quotient:

$$u_{xx}(x,t) = \lim_{\Delta x\to 0}\frac{u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)}{(\Delta x)^2} = -\frac{2}{\Delta x^2}[u(x,t)-\bar{u}(x,t)] \quad (1.2)$$

where $\bar{u}(x,t) \equiv \frac{u(x+\Delta x,t)+u(x-\Delta x,t)}{2}$ represents the average displacement of neighboring points.

By comparing: (1) wave equation: $u_{tt} = c^2 u_{xx} = -\frac{2c^2}{\Delta x^2}[u(x,t)-\bar{u}(x,t)]$; and (2) **Hook's law**: $F= -k\cdot(y-y_0)$, y_0 denotes the equilibrium position of a spring, we conclude that the motion of a particle of a string at x is like that of an elastic spring, where (a) the equilibrium position is represented by the average position of the neighboring particles; (b) the elastic constant $k \propto c^2$. ⇒ The entire **string** is therefore a combination of **coupled** small **springs**, and will behave "oscillation".

Appendix 1A − Canonical form of linear 2nd-order PDEs with 2 variables

An arbitrary linear 2nd-order PDE with two variables can be represented as:

$$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y + F(x,y)u = G(x,y) \qquad (1A.1)$$

By change of variables:

$$\begin{cases} \xi = \xi(x,y) \\ \eta = \eta(x,y) \end{cases} \qquad (1A.2)$$

we may get a different PDE with new variables ξ, η:

$\boldsymbol{u_x} = u_\xi\, \xi_x + u_\eta\, \eta_x$, $\boldsymbol{u_y} = u_\xi\, \xi_y + u_\eta\, \eta_y$;

$\boldsymbol{u_{xx}} = \frac{\partial}{\partial \xi}\left(u_\xi \xi_x + u_\eta \eta_x\right) \xi_x + \frac{\partial}{\partial \eta}\left(u_\xi \xi_x + u_\eta \eta_x\right) \eta_x + \frac{\partial}{\partial x}\left(u_\xi \xi_x + u_\eta \eta_x\right)$

$= (\xi_x)^2 u_{\xi\xi} + 2(\xi_x \eta_x) u_{\xi\eta} + (\eta_x)^2 u_{\eta\eta} + (\xi_{xx}) u_\xi + (\eta_{xx}) u_\eta$;

$\boldsymbol{u_{xy}} = (\xi_x \xi_y) u_{\xi\xi} + (\xi_x \eta_y + \eta_x \xi_y) u_{\xi\eta} + (\eta_x \eta_y) u_{\eta\eta} + (\xi_{xy}) u_\xi + (\eta_{xy}) u_\eta$;

$\boldsymbol{u_{yy}} = (\xi_y)^2 u_{\xi\xi} + 2(\xi_y \eta_y) u_{\xi\eta} + (\eta_y)^2 u_{\eta\eta} + (\xi_{yy}) u_\xi + (\eta_{yy}) u_\eta$. $\Rightarrow$ Eq. (1A.1) becomes:

$$\overline{A}\, u_{\xi\xi} + \overline{B}\, u_{\xi\eta} + \overline{C}\, u_{\eta\eta} + \overline{D}\, u_\xi + \overline{E}\, u_\eta + \overline{F}\, u = \overline{G} \qquad (1A.3)$$

where $\overline{A} = (\xi_x)^2 A + (\xi_x \xi_y) B + (\xi_y)^2 C$, $\overline{B} = 2(\xi_x \eta_x) A + (\xi_x \eta_y + \eta_x \xi_y) B + 2(\xi_y \eta_y) C$, $\overline{C} = (\eta_x)^2 A + (\eta_x \eta_y) B + (\eta_y)^2 C$, $\overline{D} = (\xi_{xx}) A + (\xi_{xy}) B + (\xi_{yy}) C + (\xi_x) D + (\xi_y) E$, $\overline{E} = (\eta_{xx}) A + (\eta_{xy}) B + (\eta_{yy}) C + (\eta_x) D + (\eta_y) E$, $\overline{F} = F$, $\overline{G} = G$.

By choosing proper transformations $\xi(x,y)$, $\eta(x,y)$, we can make $\overline{A} = \overline{C} = 0$ to simplify the PDE, $\Rightarrow (\mu_x)^2 A + (\mu_x \mu_y) B + (\mu_y)^2 C = 0$, where μ can be either ξ or η; $\Rightarrow$

$$A(x,y) \cdot \left(\frac{\mu_x}{\mu_y}\right)^2 + B(x,y) \cdot \left(\frac{\mu_x}{\mu_y}\right) + C(x,y) = 0 \qquad (1A.4)$$

A vertical/horizontal line in the $\xi\eta$-plane, i.e. μ=constant, corresponds to a characteristic curve in the xy-plane, on which $d\mu = \mu_x \cdot dx + \mu_y \cdot dy = 0$, $\Rightarrow \frac{dy}{dx} = -\left(\frac{\mu_x}{\mu_y}\right)$. Eq. (1A.4) becomes:

$$A(x,y) \cdot \left(\frac{dy}{dx}\right)^2 - B(x,y) \cdot \left(\frac{dy}{dx}\right) + C(x,y) = 0 \qquad (1A.5)$$

which leads two 1st-order nonhomogeneous ODEs:

$$\frac{dy}{dx}=\frac{B\pm\sqrt{B^2-4AC}}{2A} \tag{1A.6}$$

Solutions to eq. (1A.6) are of the form: $\boldsymbol{f(x,y)}$**=constant**, $\Rightarrow \mu(x,y)=f(x,y)$, from which $\{\xi,\ \eta\}$ are determined.

Canonical forms:

1) Hyperbolic ($B^2-4AC>0$): eq. (1A.6) leads to two distinct solutions, from which $\xi(x,y)$, $\eta(x,y)$ are determined. $\Rightarrow$ Eq. (1A.3) is simplified as: $u_{\xi\eta}=H(\xi,\eta,u,u_\xi,u_\eta)$. Another change of variables: $\begin{cases}\alpha=\xi+\eta\\ \beta=\xi-\eta\end{cases}$ gives rise to the canonical form:

$$u_{\alpha\alpha}-u_{\beta\beta}=H(\alpha,\beta,u,u_\alpha,u_\beta) \tag{1A.7}$$

E.g. $y^2u_{xx}-x^2u_{yy}=0$, $\Rightarrow A=y^2$, $C=-x^2$, $\{B,\ D\sim G\}=0$. By eq. (1A.6), $\frac{dy}{dx}=\pm\frac{x}{y}$, $\Rightarrow y^2\pm x^2$ =constant, $\Rightarrow \begin{cases}\xi=y^2-x^2\\ \eta=y^2+x^2\end{cases}$, $4(\xi^2-\eta^2)u_{\xi\eta}-2\eta u_\xi+-2\xi u_\eta=0$. By $\begin{cases}\alpha=\xi+\eta\\ \beta=\xi-\eta\end{cases}$,

$\Rightarrow u_{\alpha\alpha}-u_{\beta\beta}=-(\beta\cdot u_\alpha+\alpha\cdot u_\beta)/2\alpha\beta$.

2) Parabolic: if $B^2-4AC=0$: Eq. (1A.6) only leads to one ODE: $\frac{dy}{dx}=\frac{B}{2A}$, whose solution $f(x,y)$=constant is used to determine $\xi(x,y)$, while $\eta(x,y)$ can be arbitrarily chosen (normally just $\eta=y$). $\Rightarrow$ Eq. (1A.3) is simplified as:

$$u_{\eta\eta}=H(\xi,\eta,u,u_\xi,u_\eta) \tag{1A.8}$$

E.g. $u_{xx}+2u_{xy}+u_{yy}=0$, $\Rightarrow A=1$, $B=2$, $C=1$, $\{D-G\}=0$. By eq. (1A.6), $\frac{dy}{dx}=1$, $\Rightarrow y-x=$ constant, $\Rightarrow \begin{cases}\xi=y-x\\ \eta=y\end{cases}$, $u_{\eta\eta}=0$.

3) Elliptic: if $B^2-4AC<0$: eq. (1A.6) leads to two distinct complex solutions, from which $\xi(x,y)$, $\eta(x,y)$ are determined. Another change of variables: $\begin{cases}\alpha=(\xi+\eta)/2\\ \beta=(\xi-\eta)/2i\end{cases}$ gives rise to the canonical form:

$$u_{\alpha\alpha} + u_{\beta\beta} = H(\alpha,\beta,u,u_\alpha,u_\beta) \qquad (1A.9)$$

E.g. $y^2u_{xx} + x^2u_{yy} = 0$, $\Rightarrow A=y^2$, $C=x^2$, $\{B, D\text{-}G\}=0$. By eq. (1A.6), $\frac{dy}{dx} = \pm i\frac{x}{y}$, $\Rightarrow y^2 \pm ix^2$ =constant, $\Rightarrow \begin{cases}\xi = y^2 + ix^2 \\ \eta = y^2 - ix^2\end{cases}$. By $\begin{cases}\alpha = (\xi+\eta)/2 = y^2 \\ \beta = (\xi-\eta)/2i = x^2\end{cases}$,

$\Rightarrow u_{\alpha\alpha} + u_{\beta\beta} = -(\beta \cdot u_\alpha + \alpha \cdot u_\beta)/2\alpha\beta$.

Eq's (1A.7−9) represent **generalized** wave equation, heat equation, and Laplace's equation, respectively. By transforming a linear 2nd-order PDE into its canonical form, we can predict the behavior of its solution (wave propagation, diffusion, or steady-state).

Appendix 1B − General solutions to linear PDEs of constant coefficients

An arbitrary 2nd-order linear PDE of constant coefficients can be represented as:

$$Au_{xx}+ Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G(x,y) \tag{1B.1}$$

where $A\sim F$ are constants, and $A\sim C$ cannot be zeros simultaneously. The general solution of eq. (1B.1) consists of homogeneous and particular solutions: $u(x,y)= u_h(x,y)+ u_p(x,y)$, which can be solved by the same procedures used in solving linear ODEs of constant coefficients.

E.g. Solve: $u_{xx}+ u_{yy} = 20e^{2x+y}$.

Ans: Substitute $\boldsymbol{u_h(x,y)=e^{\alpha x+\beta y}}$ into $u_{xx}+u_{yy}=0$, we have: $\alpha^2+\beta^2=0$, $\alpha=\pm i\beta$. $\Rightarrow u_h(x,y)= e^{\pm\beta(ix+y)}$ $= e^{i\beta(x\pm iy)}$. Since β is arbitrary, $\Rightarrow u_h(x,y)=f(x+iy)+g(x-iy)$, where f, g are arbitrary functions (check the consistency).

By the method of undetermined coefficients, substitute $\boldsymbol{u_p(x,y)}=a\cdot e^{2x+y}$ into $u_{xx}+u_{yy}=20e^{2x+y}$, we have: $(4a+a)e^{2x+y}=20e^{2x+y}$, $\Rightarrow a=4$. $u(x,y)= f(x+iy)+g(x-iy)+4e^{2x+y}$.

Note: the exact forms of f, g should be determined by BCs and ICs.

E.g. Solve: $u_{xx}- u_{yy} - u_x+ u_y = 2\cos(3x+2y)$.

Ans: Substitute $\boldsymbol{u_h(x,y)=e^{\alpha x+\beta y}}$ into $u_{xx}-u_{yy}-u_x+u_y=0$, we have: $\alpha^2-\beta^2-\alpha+\beta =0$, $\alpha=\{\beta, 1-\beta\}$. $\Rightarrow u_h(x,y)=\begin{cases} e^{\beta(x+y)} \\ e^x e^{-\beta(x-y)} \end{cases} = f(x+y)+e^x\cdot g(x-y)$, where f, g are arbitrary functions.

Substitute $\boldsymbol{u_p(x,y)}= a\cdot\cos(3x+2y)+ b\cdot\sin(3x+2y)$ into $u_{xx}-u_{yy}-u_x+u_y = 2\cos(3x+2y)$, we have: $-(5a+b)\cos(3x+2y)+ (a-5b)\sin(3x+2y)= 2\cos(3x+2y)$, $\Rightarrow a=-\frac{5}{13},\ -\frac{1}{13}$.

$u(x,y)= f(x+y)+e^x\cdot g(x-y)-\frac{5}{13}\cos(3x+2y)-\frac{1}{13}\sin(3x+2y)$.