Edited by Sheng-Ju Ho

Homework Solutions # 6

1) $\sum a_n z^n$ has radius of convergence R：

$$\lim_{n\to\infty}\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right|<1 \Rightarrow \lim_{n\to\infty}\frac{a_{n+1}}{a_n}|z|<1 \Rightarrow |z|<\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=R$$

Similarly,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}z^{2(n+1)}}{a_n z^{2n}}\right|<1 \Rightarrow \lim_{n\to\infty}\frac{a_{n+1}}{a_n}|z|^2<1 \Rightarrow |z|^2<\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=R$$

So we obtain $|z|<\sqrt{R}$, i.e. $\sum a_n z^{2n}$ has radius of convergence $\sqrt{R}$.

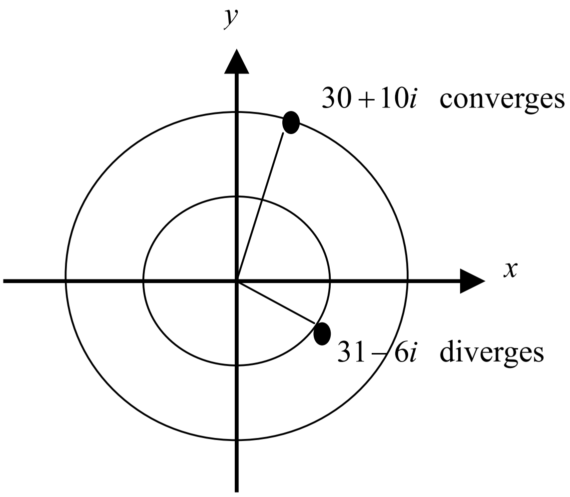
2) $$\lim_{n\to\infty}\left|\frac{\frac{(n+1)^{n+1}}{3^{n+1}\cdot(n+1)!}(z-i)^{n+1}}{\frac{n^n}{3^n\cdot n!}(z-i)^n}\right|<1$$

$$\Rightarrow \lim_{n\to\infty}\frac{\frac{(n+1)^{n+1}}{3^{n+1}\cdot(n+1)!}}{\frac{n^n}{3^n\cdot n!}}|z-i|<1$$

$$\Rightarrow |z-i|<\lim_{n\to\infty}\frac{3n^n}{(n+1)^n}=3\cdot\frac{\infty}{\infty}=3$$

So the region of convergence is $|z-i|<3$

3) No, if this is true, we would have convergence at a point from point ($0+0i$) than ($30-10i$), but it diverges at point ($31-6i$). [From Theorem 1 at P674]



4) (A) Using Cauchy-Hadamard formula:

$$\lim_{n\to\infty}\frac{\binom{n+1+m}{m}}{\binom{n+m}{m}}|z|<1$$

$$\Rightarrow \lim_{n\to\infty}\frac{(n+1+m)(n+m)...(n+3)(n+2)}{(n+m)...(n+2)(n+1)}|z|<1$$

$$\Rightarrow \lim_{n\to\infty}\frac{(n+1+m)}{(n+1)}|z|<1 \Rightarrow |z|<1$$

The radius of convergence R is 1.

(B) Using Theorem 3 at P680: Termwise Integration of a Power Series.

$$\sum_{n=0}^{\infty}\binom{n+m}{m}z^n = 1+\frac{(m+1)}{1!}z+\frac{(m+2)(m+1)}{2!}z^2+\frac{(m+3)(m+2)(m+1)}{3!}z^3+\cdots$$

Do integration term by term:

$$\Rightarrow z+\frac{(m+1)}{2!}z^2+\frac{(m+2)(m+1)}{3!}z^3+\frac{(m+3)(m+2)(m+1)}{4!}z^4+\cdots$$

To find R: [$\dfrac{(n+1)_{th}\text{ term}}{(n)_{th}\text{ term}}$ to find the rule]

$$\Rightarrow \lim_{n\to\infty} \frac{m+n+1}{n+2}|z|<1 \Rightarrow |z|<1$$

The radius of convergence R is 1.

5) $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} f(n)$, $\ln[f(n)] = \frac{\ln n}{n} \to 0$, as $n\to\infty$ (n grows faster than ln n). $\Rightarrow$ $\lim_{n\to\infty} f(n) = 1$.

6) (a,c) [See Ref 1] For Fibonacci numbers, we get the following sequence of numbers:

$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \cdots$

The sequence of ratios of consecutive Fibonacci numbers:

$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \cdots$

This sequence converges, that is, there is a single real number which the terms of this sequence approach more and more closely, eventually arbitrarily closely.

We may discover this number by exploiting the recursive definition of the Fibonacci sequence in the following way. Let us denote the n^{th} term of the sequence of ratios by x_n, that is,

$$x_1 = \frac{1}{1} \quad , \quad x_2 = \frac{2}{1} \quad , \quad \cdots \quad , \quad x_n = \frac{F(n+1)}{F(n)}$$

Then using the recursive definition of F(n) given above, we have:

$$x_n = \frac{F(n+1)}{F(n)} = \frac{F(n)+F(n-1)}{F(n)} = 1 + \frac{F(n-1)}{F(n)} = 1 + \frac{1}{\frac{F(n)}{F(n-1)}} = 1 + \frac{1}{x_{n-1}}$$

Now supposing for the moment that the sequence converges to a real number x (a fact which requires proof, but we'll leave that aside), we may observe that both x_n and x_{n-1} have the same limit, that is,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n-1} = x$$

Consequently, the real number x to which the sequence of ratios converges must satisfy the following equation:

$$x = 1 + \frac{1}{x} \Rightarrow x^2 - x - 1 = 0$$

This is a simple equation to solve for x: it is really a quadratic equation, and its positive root is the value we are looking for:

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{F(n+1)}{F(n)} = \frac{1+\sqrt{5}}{2}$$

7) $$\frac{\sin(t)}{t} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} \pm \cdots$$

Perform integration term by term, we can obtain the series of $\int_0^z \frac{\sin(t)}{t} dt$:

$$\int_0^z \frac{\sin(t)}{t} dt = \left[t - \frac{1}{3}\frac{t^3}{3!} + \frac{1}{5}\frac{t^5}{5!} - \frac{1}{7}\frac{t^7}{7!} \pm \cdots \right]\Bigg|_0^z = z - \frac{1}{3}\frac{z^3}{3!} + \frac{1}{5}\frac{z^5}{5!} - \frac{1}{7}\frac{z^7}{7!} \pm \cdots$$

8) (a) Let $t = z - (-1+i) \Rightarrow z = t + (-1+i)$

$$f(z) = \ln(z) = \ln[t + (-1+i)] = \ln(-1+i) + \ln\left[1 + \frac{t}{-1+i}\right]$$

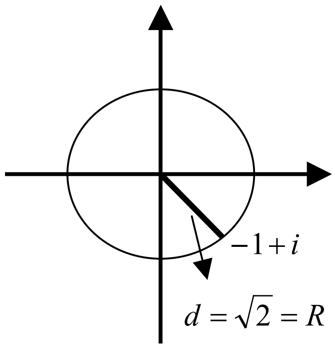
$$\Rightarrow f(t) = \ln(-1+i) + \left[\left(\frac{t}{-1+i}\right) - \frac{\left(\frac{t}{-1+i}\right)^2}{2} + \frac{\left(\frac{t}{-1+i}\right)^3}{3} - \frac{\left(\frac{t}{-1+i}\right)^4}{4} \pm \cdots \right]$$

$$\Rightarrow f(z)=\ln(-1+i)+\left\{\left[\frac{z}{-1+i}-1\right]-\frac{\left[\frac{z}{-1+i}-1\right]^2}{2}+\frac{\left[\frac{z}{-1+i}-1\right]^3}{3}\mp\cdots\right\}$$

$$=\ln(-1+i)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(-1+i)^n}(z-z_0)^n\ ,\ z_0=-1+i,\ \text{for}\ \ 0<\left|\frac{z}{-1+i}-1\right|<1\,.$$

(b) The nearest singularity is 0. So the distance d is the length between 0 and -1+i.

$d=\sqrt{2}$



(c) Yes, the distance between expanded center and the nearest singularity stands for radius of convergence.

9) Let $t=z-1\Rightarrow z=t+1$

$$f(z)=\frac{e^z}{z-1}=\frac{e^{t+1}}{t}$$

$$\Rightarrow f(t)=\frac{1}{t}\left[1+(t+1)+\frac{(t+1)^2}{2!}+\frac{(t+1)^3}{3!}+\cdots\right]$$

$$\Rightarrow f(z)=\frac{1}{z-1}\left[1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right],\ \ 0<|z-1|<R$$

10) $f(z)=\dfrac{\frac{1}{z^3}}{1-\frac{1}{z}}=\dfrac{1}{z^3-z^2}=\dfrac{1}{z^2(z-1)}$

So, $z=0$ is 2nd-order pole and $z=1$ is 1st-order pole.

Properties of singularity z_0 are verified by the Laurent series centered at z_0 and having an ROC "nearest" to z_0: $-\sum_{n=2}^{\infty}\frac{1}{z^n}$ ($z=0$ is 2nd-order pole). However, the given series $\frac{1}{z^3}+\frac{1}{z^4}+\frac{1}{z^5}+\ldots$ has an ROC of $|z|>1$ (not the nearest).

11) $I=f(z)=\oint_C \dfrac{e^z}{z(z+1)}dz=2\pi i\cdot\left\{\operatorname*{Res}_{z\to 0}\{f(z)\}+\operatorname*{Res}_{z\to -1}\{f(z)\}\right\}$

[$z=0$ and $z=-1$ are 1st-order pole]

$\operatorname*{Res}_{z\to 0}\{f(z)\}=\lim_{z\to 0} z\cdot f(z)=\lim_{z\to 0}\dfrac{e^z}{z+1}=1$

$\operatorname*{Res}_{z\to -1}\{f(z)\}=\lim_{z\to -1}(z+1)\cdot f(z)=\lim_{z\to -1}\dfrac{e^z}{z}=-e^{-1}$

So, $I=2\pi i\cdot\left\{\operatorname*{Res}_{z\to 0}\{f(z)\}+\operatorname*{Res}_{z\to -1}\{f(z)\}\right\}=2\pi i\left[1-\dfrac{1}{e}\right]$

12) $f(z)=\oint_C \dfrac{\cosh(z)}{z(z-3i)}=2\pi i\cdot\left\{\operatorname*{Res}_{z\to 0}\{f(z)\}\right\}$

[$z=0$ is 1st-order pole]

Do not consider $z=3i$, it isn't located in C

$\operatorname*{Res}_{z\to 0}\{f(z)\}=\lim_{z\to 0} z\dfrac{\cosh(z)}{z(z-3i)}=\dfrac{i}{3}$

So, $f(z)=2\pi i\cdot\left\{\operatorname*{Res}_{z\to 0}\{f(z)\}\right\}=-\frac{2\pi}{3}$

Reference

[1] http://www.mathacademy.com/pr/prime/articles/fibonac/index.asp