Homework Solutions #5

1) 根據尤拉公式: $\cos(nx)=\dfrac{e^{inx}+e^{-inx}}{2}$, $\sin(nx)=\dfrac{e^{inx}-e^{-inx}}{2i}$

1a) $$\sum_{n=0}^{\infty}P^n(\frac{e^{inx}+e^{-inx}}{2})=\frac{1}{2}\left[\sum_{n=0}^{\infty}P^n e^{inx}+\sum_{n=0}^{\infty}P^n e^{-inx}\right]=\frac{1}{2}\left[\frac{1}{1-Pe^{ix}}+\frac{1}{1-Pe^{-ix}}\right]$$

$$=\frac{1}{2}\left[\frac{2-Pe^{ix}-Pe^{-ix}}{1-2P(\frac{e^{ix}+e^{-ix}}{2})+P^2}\right]=\frac{1-P\cos x}{1-2P\cos x+P^2}$$

1b) $$\sum_{n=0}^{\infty}P^n(\frac{e^{inx}-e^{-inx}}{2i})=\frac{1}{2i}\left[\sum_{n=0}^{\infty}P^n e^{inx}-\sum_{n=0}^{\infty}P^n e^{-inx}\right]=\frac{1}{2i}\left[\frac{1}{1-Pe^{ix}}-\frac{1}{1-Pe^{-ix}}\right]$$

$$=\frac{1}{2i}\left[\frac{Pe^{ix}-Pe^{-ix}}{1-2P(\frac{e^{ix}+e^{-ix}}{2})+P^2}\right]=\frac{\sin x}{1-2P\cos x+P^2}$$

2) $8z^2-(36-6i)z+4z-11i=0$

$$\left({}^{*}\ az^2+bz+c=0,\ z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right)$$

$$z=\frac{1}{16}\left[(36-6i)\pm\sqrt{(-84-80i)}\right]=\frac{1}{16}\left[(36-6i)\pm(-4+10i)\right]=2+4i,\ \frac{5}{2}-i$$

不是共軛複數

3)

3e) 根據 Cauchy-Riemann 條件: $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$

$$f(z)=u(x,y)+iv(x,y)=x+i\cdot 0$$

$$\frac{\partial u}{\partial x}=1\neq\frac{\partial v}{\partial y}=0$$

無論任意 z 皆不可微分(不存在)

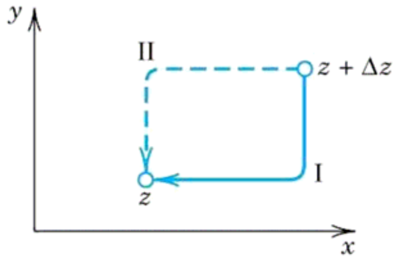
其它例子: $f(z)=iy$

3f) $f(z)=|z|^2$

$$f'(z)=\frac{|z+\Delta z|^2-|z|^2}{\Delta z}=\frac{(z+\Delta z)(\bar{z}+\Delta\bar{z})-z\bar{z}}{\Delta z}=\frac{z\bar{z}+z\Delta\bar{z}+\Delta z\bar{z}-z\bar{z}}{\Delta z}$$

$$=z\frac{\Delta\bar{z}}{\Delta z}+\bar{z}$$

1. 當 $z=0\Rightarrow f'(z)=0$ 可微分
2. 當 $z\neq 0$ $(z=x+iy)$



For path I: $\Delta y=0\Rightarrow \Delta z=\Delta x,\ \Delta\bar{z}=\Delta x$

$$f'(z)=z\frac{\Delta x}{\Delta x}+\bar{z}=z+\bar{z}$$

For path II: $\Delta x=0\Rightarrow \Delta z=\Delta y,\ \Delta\bar{z}=-\Delta y$

$$f'(z)=z\frac{-\Delta y}{\Delta y}+\bar{z}=-z+\bar{z}$$

不可微分

4)

Cauchy-Riemann equations:

直角座標系: $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

極座標系: $r=\sqrt{x^2+y^2}$, $\theta=\tan^{-1}\frac{y}{x}$

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}-\frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y}=0\ \Rightarrow\frac{\partial u}{\partial r}\frac{x}{\sqrt{x^2+y^2}}-\frac{\partial v}{\partial \theta}\frac{1}{r}\frac{x}{\sqrt{x^2+y^2}}=0$$

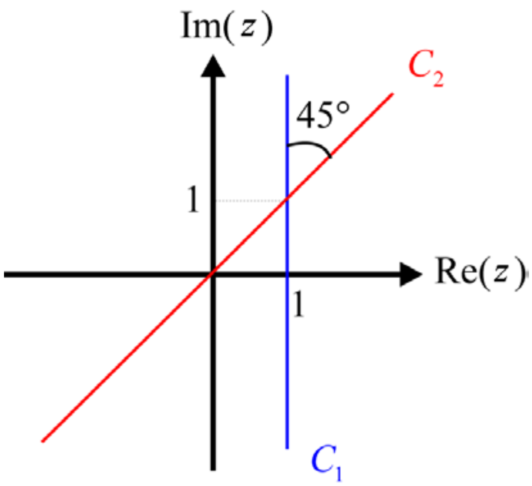
$$\Rightarrow\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}+\frac{\partial v}{\partial r}\frac{\partial r}{\partial x}=0 \quad \Rightarrow \frac{\partial u}{\partial \theta}\frac{1}{r}\frac{y}{\sqrt{x^2+y^2}}+\frac{\partial v}{\partial r}\frac{y}{\sqrt{x^2+y^2}}=0$$

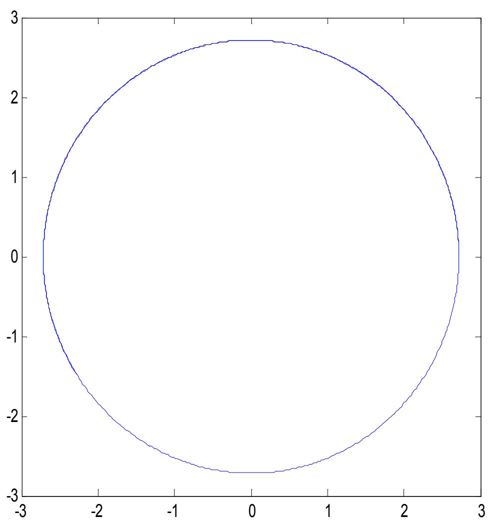
$$\Rightarrow \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

5)

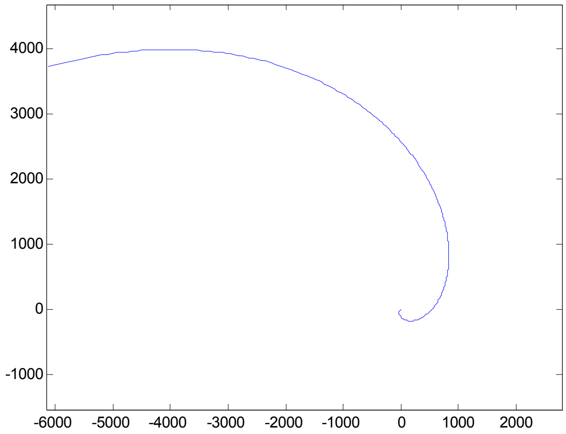
5a)



5b) $C_1^*: e^{1+it}=e\cdot e^{it}=e(\cos t+i\sin t)$



$C_2^*: e^{t+it} = e^t \cdot e^{it} = e^t(\cos t + i\sin t)$



5c) 由講義公式(9A.2) 可知:

$$\phi_{C_1} = \theta_{C_1} + \delta = 90° + \delta,\ \ \delta = \text{Arg}\{f'(z_0)\},\ \ z_0 = 1+i$$

$$\phi_{C_2} = \theta_{C_2} + \delta = 45° + \delta$$

$$\Rightarrow \phi_{C_1} - \phi_{C_2} = 45°$$

6) $w = \cos^{-1} z,\ \ z = \cos w = \dfrac{e^{iw}+e^{-iw}}{2}$

$$\begin{cases} e^{iw}+e^{-iw}-2z=0 \\ e^{2iw}-2ze^{iw}+1=0 \end{cases} \Rightarrow e^{iw} = \frac{2z \pm \sqrt{4z^2-4}}{2} = z \pm \sqrt{z^2-1}$$

$$\Rightarrow w = -i \cdot \ln(z+\sqrt{z^2-1}) = \cos^{-1} z$$

7)

7a) 令 $z = re^{i\theta}$, $z^* = re^{-i\theta}$, $dz = \dfrac{\partial z}{\partial r}dr + \dfrac{\partial z}{\partial \theta}d\theta = e^{i\theta}dr + ire^{i\theta}d\theta$

$$\int_{|z|=1} z^* dz = \int_{|z|=1} e^{-i\theta} d(e^{i\theta}) = i\int_{|z|=1} e^{-i\theta}e^{i\theta}d\theta = i2\pi = 2\pi i$$

7b) $\int_C z^* dz = \int_{C_1} re^{-i\theta} d(re^{i\theta}) + \int_{C_2} re^{-i\theta} d(re^{i\theta}) + \int_{C_3} re^{-i\theta} d(re^{i\theta})$

$= \int_0^1 re^{-i\theta} e^{i\theta} dr + \int_0^{\frac{\pi}{4}} re^{-i\theta} e^{i\theta} d\theta + \int_1^0 re^{-i\theta} e^{i\theta} dr$

$= \frac{1}{2} + i\frac{\pi}{4} - \frac{1}{2} = i\frac{\pi}{4}$

8)

8a) 令 $z = e^{i\theta}$

$I = \int_C e^{-i\theta} d(e^{i\theta}) = i\int_C e^{-i\theta} e^{i\theta} d\theta = i\int_{-\frac{\pi}{2}}^{\frac{5\pi}{4}} d\theta = i\frac{7}{4}\pi$

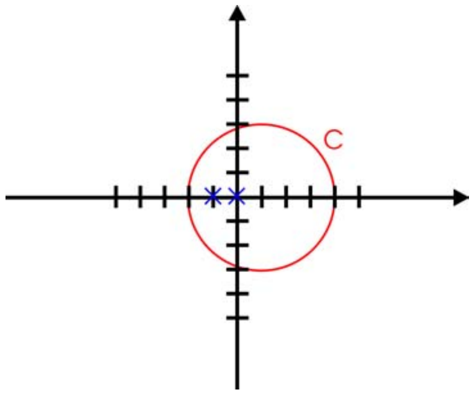
8b) Yes.

分支切割取正實軸，利用 $\frac{1}{z}$ 本身是除了 $z = 0$ 之外，皆為解析函數

$I = \int_C \frac{1}{z} dz = \ln z \Big|_{-i}^{\frac{-1}{\sqrt{2}}(1+i)} = (\ln 1 + i\frac{5\pi}{4}) - (\ln 1 - i\frac{\pi}{2}) = i\frac{7}{4}\pi$

9)

9a) $I = \oint_C \frac{e^z}{z(z+1)} dz$



$= \oint_C e^z \left(\frac{1}{z} - \frac{1}{z+1}\right) dz$

$= \oint_C \frac{e^z}{e} dz - \oint_C \frac{e^z}{z+1} dz$

$f(z) = e^z$ 為全函數(entire function)

$I = 2\pi i \cdot f(0) - 2\pi i \cdot f(-1)$

$= 2\pi i - 2\pi i e^{-1} = 2\pi i(1 - e^{-1})$

9b) $I = \oint_C \frac{5z^2 - 3z + 2}{(z-1)^3} dz$

$f(z) = 5z^2 - 3z + 2$ 為全函數(entire function)

$I = \frac{2\pi i}{2!} f''(1) = \pi i \cdot 10 = 10\pi i$