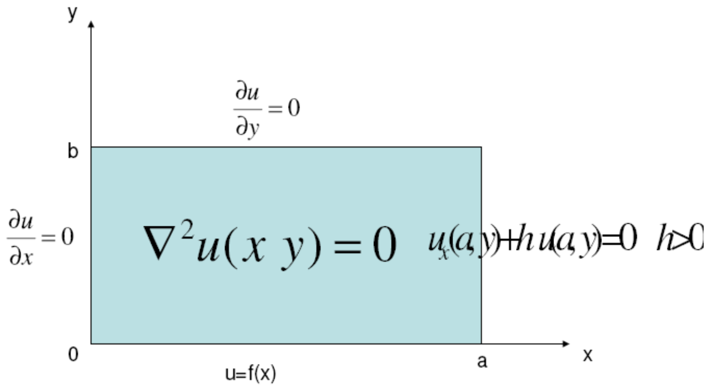
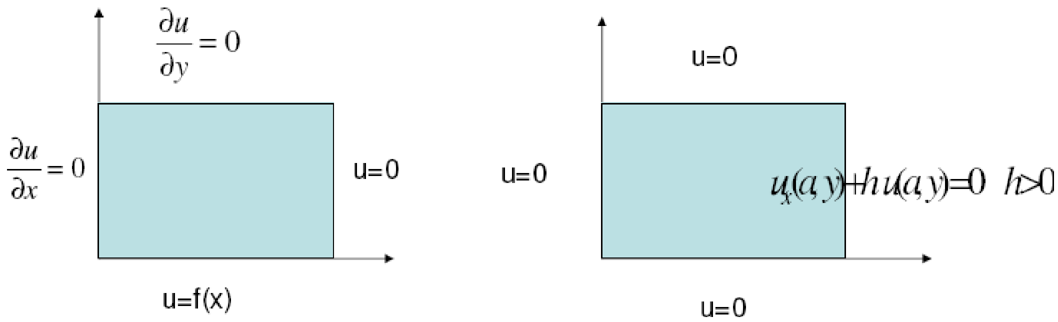
Homework Solutions #4

1) 依題意，畫成如下圖

可將此圖化簡成兩個圖相加(其實隨便拆，好算、高興算得出來就好)



$$\Downarrow$$



(a) (b)

(a) 令 $u_a(x,y)=X(x)Y(y)$，代入 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$，$\Rightarrow \overbrace{\frac{X''}{X}}^{=-k^2}+\overbrace{\frac{Y''}{Y}}^{=k^2}=0$

$X''+k^2X=0$, BCs: $X'(0)=X(a)=0$, $\Rightarrow$

$X=c_1\cos(k_n x)$, $k_n=\frac{(2n-1)\pi}{2a}$, n=1, 2, 3, ….

$Y''-k_n^2X=0$ & $\overbrace{Y'(0)=0}^{\text{平移法(本來在b處移0處)}}$ $\Rightarrow$

$Y=c_2\cosh\left[\frac{(2n-1)\pi}{2a}(b-y)\right]$ 因為平移法所以y變成y-b

So, $u_a=\sum_{n=1}^{\infty}A_n\cosh\left[\frac{(2n-1)\pi}{2a}(b-y)\right]\cos\left[\frac{(2n-1)\pi}{2a}x\right]$

$$u_a(x,0)=\sum A_n \cosh\left[\frac{(2n-1)\pi b}{2a}\right]\cos\left[\frac{(2n-1)\pi}{2a}x\right]=f(x)$$

where , $A_n \cosh\left(\frac{(2n-1)\pi b}{2a}\right)=\frac{2}{b}\int_0^b f(x)\cos\left(\frac{(2n-1)\pi}{2a}x\right)dx$

so , $u_a=\sum A_n \cosh\left[\frac{(2n-1)\pi}{2a}(b-y)\right]\cos\left[\frac{(2n-1)\pi}{2a}x\right]$

$$A_n=\frac{2}{b\cdot\cosh\left[\frac{(2n-1)\pi b}{2a}\right]}\int_0^b f(x)\cos\left[\frac{(2n-1)\pi}{2a}x\right]dx$$

(b) 照著(a)的作法，先解決y方向(因為homogeneous BCs)，這與(a)為什麼要先解x方向道理是一樣的。

$Y''+k^2Y=0$, BCs: $Y(0)=Y(b)=0$, $\Rightarrow$

$Y=c_1\sin(k_n x)$, $k_n=\frac{n\pi}{b}$, n=1, 2, 3, ….

$X''-k_n^2X=0$, two BCs: $X(0)$=0, $X'(a)+hX'(a)$=0.

利用 $X(0)$=0 條件，可以得到： $X=c_2\sinh\left(\frac{n\pi}{b}x\right)$

利用 $X'(a)+hX'(a)$=0 條件，可以得到： X=0. $\Rightarrow u_b$=0

所以

$$u=u_a=\sum_{n=1}^{\infty} A_n \cosh\left[\frac{(2n-1)\pi}{2a}(b-y)\right]\cos\left[\frac{(2n-1)\pi}{2a}x\right]$$
$$A_n=\frac{2}{b\cdot\cosh\left[\frac{(2n-1)\pi b}{2a}\right]}\int_0^b f(x)\cos\left[\frac{(2n-1)\pi}{2a}x\right]dx$$

2) According to the ODE: $-\frac{\ddot{\Theta}}{\Theta}=\lambda$, $\Rightarrow \ddot{\Theta}+\lambda\Theta=0$

If $\lambda=-k^2<0$, $\Theta=c_1e^{k\theta}+c_2e^{-k\theta}$, which is in violation of periodic BC: $\Theta(0)=\Theta(2\pi)$. So λ cannot be negative.

3) $\nabla^2u=\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right]+\frac{1}{r^2}\frac{\partial^2u}{\partial\phi^2}+\frac{\partial^2u}{\partial z^2}\xrightarrow{\text{dependenting on r}}\nabla^2u=\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right]$

$$\nabla^2u=\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right]=\frac{1}{r}\frac{du}{dr}+\frac{d^2u}{dr^2}=0$$

Let $\frac{du}{dr}=u',\Rightarrow\ \frac{du'}{dr}+\frac{1}{r}u'=0$

$\frac{du'}{u'}=-\frac{dr}{r},\Rightarrow\ \ln u'=-\ln r+c_1,\ \ u'=\frac{c}{r},\ \boxed{u=c\ln r+k}$

For interior problem, $|u(r=0)|<\infty$, $\Rightarrow c=0$, $\boxed{u(r)=k}$.

For exterior problem, $|u(r=\infty)|<\infty$, $\Rightarrow c=0$, $\boxed{u(r)=k}$.

4) Laplace equation in spherical coordinates:

$$\nabla^2 u=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r})+\frac{1}{r^2\sin\phi}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\frac{\partial u}{\partial\phi})$$

Because Laplace equation only depends on r

$$\nabla^2 u=\frac{1}{r^2}\frac{d}{dr}(r^2\frac{du}{dr})=0\Rightarrow\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r})=0\Rightarrow r^2\frac{d^2u}{dr^2}+2r\frac{du}{dr}=0 \qquad (1)$$

We substitute $u(r)=r^m$ into (1). This gives

$m(m-1)r^m+2mr^m=0,\Rightarrow m=\ -1,0$

$u(r)=C_1r^{-1}+C_2r^0=\frac{c}{r}+k$ (c and k are constants)

For interior problem, $|u(r=0)|<\infty$, $\Rightarrow c=0$, $\boxed{u(r)=k}$.

For exterior problem, $|u(r=\infty)|<\infty$, $\Rightarrow$ $\boxed{u(r)=c/r+k}$

討論問題三與問題四的情況，在邊界內部 $u(r)$ 皆為常數分布;而在邊界外部 $u(r)$ 在球座標分布情況下隨著 r 的增加而減少，在柱座標分布情況下為常數。

5) PDE: $u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$, ROI$=\left\{1<r<2, 0<\theta<\frac{\pi}{2}\right\}$

BCs: $u(1,\theta)=\sin(2\theta)$, $u(2,\theta)=0$, $u(r,0)=0$, $u\left(r,\frac{\pi}{2}\right)=0$

Separation of variables

Let $u(r,\theta)=R(r)\Theta(\theta)\Rightarrow R''\Theta+\frac{1}{r}R'\Theta+\frac{1}{r^2}R\ddot{\Theta}=0$

Assume $\frac{r^2R''+rR'}{R}=-\frac{\ddot{\Theta}}{\Theta}=k^2\ge 0\Rightarrow\begin{cases}r^2R''+rR'-k^2R=0\\ \ddot{\Theta}+k^2\Theta=0\end{cases}$

<step1> $\Theta(\theta)=a\cos(k\theta)+b\sin(k\theta)$

By BC: $u\left(r,\frac{\pi}{2}\right)=\Theta\left(\frac{\pi}{2}\right)=a\cos\left(k\frac{\pi}{2}\right)+b\sin\left(k\frac{\pi}{2}\right)=0$

$$\Rightarrow a = 0, \quad k = 2n, \quad n = 0,1,2...$$

$$\Rightarrow \Theta_n(\theta) = b\sin(2n\theta), \quad n = 0,1,2...$$

<step2> $r^2 R'' + rR' - (2n)^2 R = 0 \Rightarrow R_n(r) = \begin{cases} c + d\ln r, n = 0 \\ cr^{2n} + dr^{-2n}, n = 1,2,3... \end{cases}$

By BC: $u(2,\theta) = R(2) = \begin{cases} c + d\ln 2 = 0 \Rightarrow c = -d\ln 2 \\ 2^{2n}c + 2^{-2n}d = 0 \Rightarrow c = -2^{-4n}d \end{cases}$

$$\Rightarrow R_n(r) = \begin{cases} d\left(r^{-2n} - \ln 2 \cdot r^{2n}\right), n = 0 \\ d\left(r^{-2n} - 2^{-4n} r^{2n}\right), n = 1,2,3... \end{cases}$$

<step3> $u_n(r,\theta) = \sum_1^{\infty} d_n\left(r^{-2n} - 2^{-4n} r^{2n}\right)\sin(2n\theta)$ [sin(2$n\theta$)=0 when n=0]

By BC: $u(1,\theta) = \sum_1^{\infty} d_n\left(1 - 2^{-4n}\right)\sin(2n\theta) = \sin(2\theta) \Rightarrow n = 1$

$$\Rightarrow d_1\left(1 - \frac{1}{16}\right) = 1 \Rightarrow d_1 = \frac{16}{15}$$

$$\Rightarrow \boxed{u(r,\theta) = \left(\frac{16}{15}r^{-2} - r^2\right)\sin(2\theta)}$$